Maximal Revenue with Multiple Goods: Nonmonotonicity and Other Observations*

Sergiu Hart† Philip J. Reny‡

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Abstract

Consider the problem of maximizing the revenue from selling a number of goods to a single buyer. We show that, unlike the case of one good, when the buyer's values for the goods increase the seller's maximal revenue may well *decrease*. We also provide a characterization of revenue-maximizing mechanisms (more generally, of "seller-favorable" mechanisms) that circumvents nondifferentiability issues. Finally, through simple and transparent examples, we clarify the need for and the use of randomization when maximizing revenue in the multiple-good versus the one-good case.

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†Department of Economics, Institute of Mathematics, and Center for the Study of Rationality, The Hebrew University of Jerusalem. *E-mail*: hart@huji.ac.il *Web site*: http://www.ma.huji.ac.il/hart

‡Department of Economics, University of Chicago. *E-mail*: preny@uchicago.edu *Web site*: http://home.uchicago.edu/~preny

1 Introduction

Consider the problem of a seller who wishes to maximize the revenue from selling multiple goods to a single buyer with private information about his value for the goods. In contrast to the one-good case where a complete solution has been known for years,[1] a general solution in the case of multiple goods remains elusive and, except under special circumstances,[2] very little is known even about the form of the solution or its properties. The purpose of the present note is to provide simple examples that highlight some important differences between the one-good and the multiple-good cases, and to provide a treatment of incentive compatibility that avoids nondifferentiability issues which, while ultimately harmless, are often a distracting nuisance within the analysis.

In Section 3, we exhibit the surprising phenomenon that the seller's maximal revenue may well *decrease* when the buyer's values for the goods *increase.* This revenue nonmonotonicity can occur only when there is *more than one good*: revenue is easily shown to be nondecreasing in the buyer's value when there is only a single good.

In Section 2, we restrict attention to "seller-favorable" mechanisms and characterize their revenue using directional derivatives, which exist everywhere (and therefore circumvent nondifferentiability issues arising from incentive compatibility).

In Section 4, we present a simple example where randomization is necessary for revenue maximization, and clarify why randomization is needed *only* when there are multiple goods.

Since the maximal revenue problem appears significantly less well behaved when the values of the goods are correlated (cf.[3] Hart and Nisan 2012a, 2012b), it is important to obtain examples with independent, and even independent and identically distributed, values. We do so both for

[1]See Myerson (1981), who also allows for multiple buyers.

[2]See, e.g., Armstrong (1996), Thanassoulis (2004), Pycia (2006), Manelli and Vincent (2006, 2007), and Pavlov (2011).

[3]For instance, deterministic mechanisms always ensure at least one half of the maximal revenue in the independent case, versus an arbitrarily small fraction in the general (correlated) case.

revenue-nonmonotonicity and for randomization.

1.1 Preliminaries

The seller possesses $k \geq 1$ goods (or "items"), which are worth nothing to him (and there are no costs). The valuation of the goods to the buyer is given by a vector[4] $x = (x_1, x_2, ..., x_k) \in \mathbb{R}^k$, where x_i is his value for good i. The valuation is assumed to be additive over the goods: the value of a set $L \subset \{1, 2, ..., k\}$ of goods is $\sum_{i \in L} x_i$. The buyer knows the valuation vector x, whereas the seller knows only that x is drawn from a given probability distribution $\mathcal{F}$ on $\mathbb{R}^k$ with support D. We make no further assumptions on $\mathcal{F}$. In particular, $\mathcal{F}$ may possess atoms and its support D may be finite or infinite and hence need not be convex or even connected. The seller and the buyer are each risk-neutral and have quasilinear utilities.

A *(direct) mechanism* for selling the k goods is given by a pair of functions (q, s), where $q : D \to [0, 1]^k$ and $s : D \to \mathbb{R}$. If the buyer reports that his valuation is x, then $q_i(x) \in [0, 1]$ is the probability that he receives good i (for $i = 1, ..., k$), and $s(x)$ is the payment that the seller receives from the buyer. When the buyer reports his valuation x truthfully, his payoff is $b(x) = \sum_{i=1}^k q_i(x)x_i - s(x) = q(x) \cdot x - s(x)$, where $q(x) \equiv (q_1(x), ..., q_k(x))$, and the seller's payoff is[5] $s(x)$. A mechanism (q, s) is *individually rational* (*IR*) if $b(x) \geq 0$ for all $x \in D$ and it is *incentive compatible* (*IC*) if $b(x) \geq q(y) \cdot x - s(y)$ for all $x, y \in D$. By the Revelation Principle, the *maximal revenue* from the distribution $\mathcal{F}$ is $\textsc{Rev}(\mathcal{F}) := \sup \mathbb{E}_{\mathcal{F}} [s(x)]$, where x is distributed according to $\mathcal{F}$, and the supremum is over all IC and IR mechanisms (q, s).

If (q, s) is IC, then it is useful to extend the buyer's payoff function b from D to all $\mathbb{R}^k$ by $b(x) := \sup_{(p,t) \in R}(p \cdot x - t)$, where $R := \{(q(x), s(x)) : x \in D\}$ is the range of (q, s). So defined, b is a convex function, being the pointwise supremum of affine functions. The IC property of (q, s) ensures that the

[4] $\mathbb{R}$ denotes the real line, $\mathbb{R}^k$ the k-dimensional Euclidean space, and $\mathbb{R}^k_+ = \{x \in \mathbb{R}^k : x \geq 0\}$ its nonnegative orthant. Negative valuations are not ruled out.

[5] In the literature this is called transfer, cost, price, or revenue, and denoted by t, c, p, and so on. We hope that using the mnemonic s for the *s*eller's final payoff and b for the *b*uyer's final payoff will avoid confusion.

values of b remain unchanged on D, and also ensures that b is finite for every $x \in \mathbb{R}^k$. Henceforth, "the buyer's payoff function b" will mean the above extension of b to all of[6] $\mathbb{R}^k$.

Let f be a real convex function defined on $\mathbb{R}^k$. The *directional derivative at $x \in \mathbb{R}^k$ in the direction $y \in \mathbb{R}^k$* is $f'(x;y) := \lim_{\delta \to 0^+}(f(x+\delta y) - f(x))/\delta$. Since f is convex, $f'(x;y)$ always exists. If $0 \leq f(x+z) - f(x) \leq \sum_{i=1}^k z_i$ holds for every $x, z \in \mathbb{R}^k$ with $z \geq 0$ then the function f is *nondecreasing* and *nonexpansive*.[7]

Let $\mathcal{B}^k$ be the collection of all real functions on $\mathbb{R}^k$ that are nondecreasing, nonexpansive, and convex.

2 Seller-Favorable Mechanisms

When maximizing revenue one may without loss of generality consider only mechanisms that are "seller-favorable," which means that whenever the buyer is indifferent he chooses an outcome that maximizes the seller's revenue (i.e., ties are broken by the buyer in favor of the seller). Formally, an incentive-compatible mechanism (q, s) with buyer's payoff function b is *seller-favorable* if there is no other incentive-compatible mechanism $(\bar{q}, \bar{s})$ having the same payoff function b for the buyer (i.e., $\bar{q}(x) \cdot x - \bar{s}(x) = b(x)$ for all x in D) and such that $\bar{s}(x) \geq s(x)$ for every $x \in D$, with strict inequality for some x. In this section we will see that the restriction to seller-favorable mechanisms simplifies the analysis (it circumvents nondifferentiability issues); moreover, seller-favorable mechanisms arise not only from revenue-maximization considerations, but also from strict implementation.[8]

The characterization of IC mechanisms (q, s) as being those whose assignment function, q, is a subgradient of the buyer's convex payoff function is well known (starting with Rochet 1985). It is an inconvenient and often technically annoying fact that the buyer's convex payoff function, while dif-

[6] The domain D is irrelevant, as any IC mechanism can be extended to the whole space $\mathbb{R}^k$ (see footnote 12 below).

[7] For convex f, this is equivalent to $0 \leq \partial f(x)/\partial x_i \leq 1$ for all i and all x where the derivative exists (i.e., a.e.).

[8] See the last paragraph of Remark *(a)* at the end of this section.

ferentiable almost everywhere, need not be differentiable everywhere. Proofs that are otherwise simple and elegant often require detours through subgradient selection arguments.[9]

Such detours can be avoided when one restricts attention to seller-favorable mechanisms. The reason is that the buyer's payoff function is not differentiable only when he is indifferent between a number of reports. But if the mechanism (q, s) is seller-favorable, the buyer's truthful report must maximize the seller's payoff among all of the buyer's optimal reports. As we show, this implies that $q(x) \cdot x = b'(x; x)$ for every buyer valuation[10] $x \in D$. Consequently, in a seller-favorable mechanism the buyer's payoff function, b, completely determines the seller's payoff function s at *every* $x \in D$, whether a point of differentiability of b or not, and $s(x) = b'(x; x) - b(x)$ for all $x \in D$.

Lemma 1 *If (q, s) is IC then the buyer's payoff function b belongs to $\mathcal{B}^k$ and $s(x) \leq b'(x; x) - b(x)$ for every $x \in D$.*

Proof. Recall (Section 1.1) that b is a convex function and $b(x) = \sup_{(p,t)\in R}(p \cdot x - t)$, where $R = \{(q(x), s(x)) : x \in D\}$ is the range of (q, s). IC also implies that the range of $s(\cdot)$ is bounded. Thus, $\bar{R}$, the closure of R, is a compact subset of $[0, 1]^k \times \mathbb{R}$, and so for every $x \in \mathbb{R}^k$ there is $(p^*(x), t^*(x)) \in \bar{R}$ such that $b(x) = p^*(x) \cdot x - t^*(x)$ (for $x \in D$ take $(p^*(x), t^*(x)) = (q(x), s(x))$. Therefore, for every $x, y \in \mathbb{R}^k$ we get

$$b(y) - b(x) \geq (p^*(x) \cdot y - t^*(x)) - (p^*(x) \cdot x - t^*(x)) = p^*(x) \cdot (y - x), \quad (1)$$

which says that $p^*(x)$ is a subgradient of[11] b at x. Thus, $p^*(x) \cdot x \leq \sup\{p \cdot x : p \in \partial b(x)\} = b'(x; x)$ and so $s(x) = q(x) \cdot x - b(x) \leq b'(x; x) - b(x)$ for every[12]

[9] E.g., Lemma A.4 in Manelli and Vincent (2007).

[10] This formula holds even when D is a finite set since b is a convex function defined on all of $\mathbb{R}^k$. Hence $b'(x; x)$ is well defined for every $x \in \mathbb{R}^k$, and in particular for $x \in D$.

[11] For a convex function f on $\mathbb{R}^k$, a vector $p \in \mathbb{R}^k$ is a *subgradient* of f at $x \in \mathbb{R}^k$ if $f(y) - f(x) \geq p \cdot (y - x)$ for all $y \in \mathbb{R}^k$. Letting $\partial f(x)$ denote the set of subgradients of f at x (which is always a nonempty closed set), we have $f'(x; y) = \sup\{p \cdot y : p \in \partial f(x)\}$ (see Rockafellar 1970).

[12] Note that (p^*, t^*) is an IC mechanism on all of $\mathbb{R}^k$ that extends the given IC mechanism (q, s) on D. This shows that it is without loss of generality to require the incentive constraints to hold on all of $\mathbb{R}^k$, and not merely on D.

$x \in D$.

Taking $y = x + z$ with $z \geq 0$ in (1) implies that $0 \leq p^*(x) \cdot z \leq b(x+z) - b(x) \leq p^*(x+z) \cdot z \leq \sum_{i=1}^k z_i$, and so b is nondecreasing and nonexpansive. ■

Lemma 2 *Let $b \in \mathcal{B}^k$. Then there is an IC mechanism $(\bar{q}, \bar{s})$ such that the buyer's payoff function is b and the seller's payoff is $\bar{s}(x) = b'(x; x) - b(x)$ for all x.*

Proof. Being nondecreasing, nonexpansive, and convex on $\mathbb{R}^k$, the function b satisfies $0 \leq b(x) - b(x - z) \leq p \cdot z \leq b(x + z) - b(x) \leq \sum_{i=1}^k z_i$ for every $x \in \mathbb{R}^k$, every $p \in \partial b(x)$, and every $z \in \mathbb{R}^k_+$. In particular, $\partial b(x) \subset [0, 1]^k$ and so $b'(x; x) = \sup_{p \in \partial b(x)} p \cdot x$ is attained at some $\bar{q}(x) \in [0, 1]^k$, i.e., $b'(x; x) = \bar{q}(x) \cdot x$. Define $\bar{s}(x) := b'(x; x) - b(x) = \bar{q}(x) \cdot x - b(x)$. Then $\bar{q}(y) \cdot y - \bar{s}(y) = b(y) \geq \bar{q}(x) \cdot (y - x) + b(x) = \bar{q}(x) \cdot y - \bar{s}(x)$ (using the definitions of $\bar{s}(y)$ and $\bar{s}(x)$, and $\bar{q}(x) \in \partial b(x)$), and so $(\bar{q}, \bar{s})$ is IC. ■

Together, Lemmas 1 and 2 imply the following.

Corollary 3 *Let (q, s) be an IC mechanism with buyer's payoff function b. Then (q, s) is seller-favorable if and only if $q(x) \cdot x = b'(x; x)$ and $s(x) = b'(x; x) - b(x)$ for every x.*

Consider now the problem of maximizing the seller's expected revenue subject to individual rationality (IR) for the buyer (i.e., $b \geq 0$). Since it is without loss of generality to restrict attention to seller-favorable mechanisms, a consequence of Corollary 3 is the following.

Corollary 4 *The seller's maximal expected revenue is*

$$\textsc{Rev}(\mathcal{F}) = \sup_{b \in \mathcal{B}^k, b \geq 0} \mathbb{E}_{\mathcal{F}} \left[b'(x; x) - b(x) \right]. \tag{2}$$

Remarks. *(a) Strict implementation.* Given any IC mechanism (q, s), there are numerous ways to eliminate, at arbitrarily small cost, the problem of the buyer having only weak incentives to report truthfully. For example, one can introduce an arbitrarily small positive probability that, after the buyer reports his valuation, the given (IC) mechanism is replaced by a random reserve price on each good.

Another alternative is to choose any arbitrarily small $\varepsilon > 0$, and use instead the mechanism $(q, (1-\varepsilon)s)$ (it need not be IC), which amounts to giving a constant discount (fraction ε) on all prices. This mechanism *guarantees* to the seller, for *any* optimal choices of the buyer, a payoff of at least $(1-\varepsilon)s(x)$ for every valuation x of the buyer.[13] Thus, the seller is guaranteed at least $1-\varepsilon$ times his payoff in the original mechanism, regardless of which optimal report the buyer makes.[14]

In fact, the $(q, (1-\varepsilon)s)$ mechanism guarantees to the seller not merely $(1-\varepsilon)s(x)$ for every x, but $(1-\varepsilon)\bar{s}(x) = (1-\varepsilon)(b'(x;x) - b(x))$, the maximal seller-favorable payoffs, for every[15] x (indeed, in the argument of footnote 13 replace $(q(x), s(x))$ with a $(q(z), s(z))$ that satisfies $b'(x;x) = q(z)\cdot x$ and $s(z) = q(z)\cdot x - b(x)$).

(b) Boundary points. Let $C \subset \mathbb{R}^k$ be a convex set that includes D, the support of $\mathcal{F}$, let $\bar{x}$ be a boundary point of C, and let $\lambda \neq 0$ belong to the normal cone to C at $\bar{x}$, i.e., $\lambda\cdot\bar{x} \geq \lambda\cdot x$ for every $x \in C$. If $\lambda\cdot\bar{x} \geq 0$ and (q,s) is seller-favorable, then we can assume w.l.o.g. that $\bar{q} := q(\bar{x})$ is maximal in the direction λ, i.e., $\tilde{q} := \bar{q} + \varepsilon\lambda \notin [0,1]^k$ for every $\varepsilon > 0$. Indeed, $\tilde{q} \in \partial b(\bar{x})$ (since $\lambda\cdot(x-\bar{x}) \leq 0$ and $\bar{q} \in \partial b(\bar{x})$), and so $b'(\bar{x};\bar{x}) \geq \tilde{q}\cdot\bar{x}$; but $\tilde{q}\cdot\bar{x} \geq \bar{q}\cdot\bar{x} = b'(\bar{x};\bar{x})$ (by Corollary 3) and so $b'(\bar{x};\bar{x}) = \tilde{q}\cdot\bar{x}$ and we can replace $\bar{q}$ by $\tilde{q}$. Moreover,

[13]If y is an optimal report of a buyer with valuation x (as in the proof of Lemma 1, one may need to consider the closure of the range of $(q,(1-\varepsilon)s)$), then

$$\begin{aligned} q(y)\cdot x - (1-\varepsilon)s(y) &\geq q(x)\cdot x - (1-\varepsilon)s(x) \\ &= [q(x)\cdot x - s(x)] + \varepsilon s(x) \\ &\geq [q(y)\cdot x - s(y)] + \varepsilon s(x) \end{aligned}$$

(by IC of (q,s)). Hence $s(y) \geq s(x)$ (subtract and divide by ε), and so the seller's payoff, $(1-\varepsilon)s(y)$, is at least $(1-\varepsilon)s(x)$.

[14]Thus the possibility of multiple optimal reports for the buyer, which is sometimes described as problematic (see for instance footnote 3 in Manelli and Vincent 2007), in fact isn't.

[15]Thus the tie-breaking rule in favor of the seller is obtained as the limit of *any* optimal behavior of the buyer in the perturbed mechanisms.

if $\lambda \cdot \bar{x} > 0$ then $\tilde{q} \cdot \bar{x} > \bar{q} \cdot \bar{x} = b'(\bar{x}; \bar{x})$, a contradiction, and so $\bar{q}$ *must* be maximal in the direction λ.

In particular, we have:

- w.l.o.g. $q_i(\bar{x}) = 0$ when $\bar{x}_i = 0$ (take $\lambda = -e^{(i)}$, where $e^{(i)} \in \mathbb{R}^k_+$ is the i-th unit vector);
- $q_i(\bar{x}) = 1$ when $\bar{x}_i = \max\{x_i : x \in C\} > 0$ (for instance, if $C = [0,1]^k$, then $q_i(\bar{x}) = 1$ when $\bar{x}_i = 1$; take $\lambda = e^{(i)}$);
- $\max_i q_i(\bar{x}) = 1$ when $\sum_i \bar{x}_i = \max\{\sum_i x_i : x \in C\} > 0$ (for instance, if C is the unit simplex in $\mathbb{R}^k_+$; take $\lambda = (1, 1, ..., 1)$).

(c) $b'(x; x) = \lim_{\delta \to 0^+}(b((1+\delta)x) - b(x))/\delta$ is the right-derivative of the function $t \to b(tx)$ at[16] $t = 1$, and $s(x) = b'(x; x) - b(x)$ is the right-derivative of the function $t \to b(tx) - tb(x)$ at $t = 1$ (these functions relate to the local returns to scale of b). If $b(0) = 0$ (which, when maximizing revenue, can always be assumed when[17] $C \subset \mathbb{R}^k_+$), then $b'(x; x) \geq b(x)$ and[18] $s(x) \geq 0$ (i.e., there are no positive transfers from seller to buyer).

3 Nonmonotonicity: Increasing Values May Decrease Revenue

When the buyer's values for the goods increase, what happens to the seller's maximal revenue? It stands to reason that the revenue should also increase, as there is now more value for the seller to "extract."[19] While this can easily be seen to be true when there is one good,[20] it is perhaps a surprise that

[16] In the one-dimensional case ($k = 1$) we have $b'(x; x) = xb'_+(x)$. A useful property is $\int_{t_1}^{t_2} b'(tx; tx)dt = b(t_2 x) - b(t_1 x)$ (cf. Rockafellar 1970, Corollary 24.2.1).

[17] If $b(0) > 0$ then the revenue from $\tilde{b}(x) = b(x) - b(0)$ is higher by the amount $b(0)$ than the revenue from b.

[18] Since $0 = b(0) \geq b(x) + q(x) \cdot (0 - x) = -s(x)$.

[19] What we compare is the maximal revenue from two given distributions, one having higher values than the other (formally, this means first-order stochastic dominance).

[20] Another case where the revenue is easily seen to increase is when all valuations increase *uniformly* by the same amount (i.e., each x is replaced by $x + z$ for a fixed vector $z \gneqq 0$).

it no longer holds when there are multiple goods. As a consequence, if the distribution $\mathcal{F}$ of the buyer's valuation is not precisely known, but a certain lower bound $\mathcal{F}_0$ to $\mathcal{F}$ is given (in the first-order stochastic dominance sense), then computing the optimal revenue for $\mathcal{F}_0$ does *not* necessarily yield a lower bound on the optimal revenue for $\mathcal{F}$.

3.1 Monotonicity for one good

When there is only one good, i.e., $k = 1$, incentive compatibility (IC) implies that a buyer with a higher valuation pays no less than a buyer with a lower valuation. Thus increasing the valuation of the buyer can only increase the revenue.

Proposition 5 *When there is one good, i.e., $k = 1$, if F_2 fist-order stochastically dominates F_1 then* $\text{Rev}(F_2) \geq \text{Rev}(F_1)$.

Proof. First, we claim that every IC mechanism is monotonic in the sense that the seller's payoff increases weakly with the buyer's value: if $x > y$ then $s(x) \geq s(y)$. Indeed, for all x, y, the IC inequalities at x and at y imply $(q(x) - q(y))x \geq s(x) - s(y) \geq (q(x) - q(y))y$; when $x > y$ it follows that $q(x) - q(y) \geq 0$ and thus $s(x) - s(y) \geq 0$.

Second, the first-order stochastic dominance implies that $\mathbb{E}_{F_1}[s(x)] \leq \mathbb{E}_{F_2}[s(x)]$ for every IC mechanism, since s is a nondecreasing function. ∎

Remark. Note that Proposition 5 also follows easily from Myerson's (1981) characterization of the optimal revenue when there is one good as $\text{Rev}(F) = \sup_{p \geq 0} p \cdot (1 - F(p))$; however, the proof above shows that the monotonicity of the revenue holds not only for optimal mechanisms, but also for *any* incentive-compatible mechanism.

3.2 Nonmonotonicity for multiple goods

Now, does the above hold when there are more goods? That is, does increasing the buyer's valuations yield higher revenue to the seller? The surprising answer is that this is no longer true when there is more than one good.

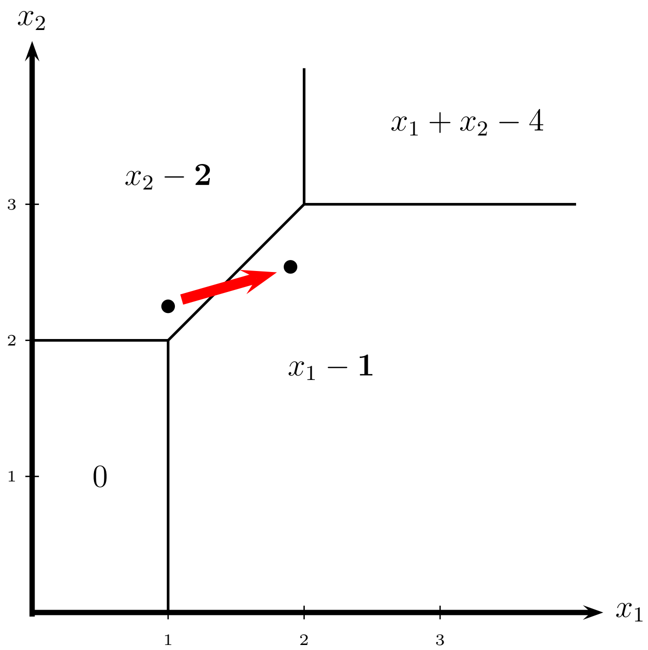

Figure 1: The nonmonotonic mechanism (3)

When there are multiple goods one can construct examples of IR and IC mechanisms that are not monotonic.[21] Take for instance the mechanism where the buyer is offered a choice among the following four outcomes: get nothing and pay nothing (with payoff $= 0$); or get good 1 for price 1 (with payoff $= x_1 - 1$); or get good 2 for price 2 (with payoff $= x_2 - 2$); or get both goods for price 4 (with payoff $= x_1 + x_2 - 4$); thus,

$$b(x_1, x_2) = \max\{0,\ x_1 - 1,\ x_2 - 2,\ x_1 + x_2 - 4\}. \tag{3}$$

See Figure 1 for the regions in the buyer's valuation space where each outcome is chosen. If the valuation of the buyer is, say, $(1.3, 2.4)$, then his optimal choice is to pay 2 for good 2, whereas if his values increase to, say, $(1.7, 2.6)$, then his optimal choice is to pay 1 for good 1. Thus the seller receives a lower payment (1 instead of 2) when the buyer's values increase.

The more difficult question is whether this nonmonotonicity can also occur for the *maximal* revenue. The two examples below, a simpler one where

[21]The first such example was constructed with Noam Nisan.

the unique optimal mechanism is precisely the above deterministic mechanism[22] (3) but the valuations of the two goods are correlated, and a more complicated one where the valuations of the two goods are independent and identically distributed, show that the maximal revenue can indeed be non-monotonic.

Example E1. For every $0 \le \alpha \le 1/4$, let $\mathcal{F}_\alpha$ be the following distribution on $\mathbb{R}^2$:

$$\mathcal{F}_\alpha = \begin{cases} (1,1), & \text{with probability } 1/4, \\ (1,2), & \text{with probability } 1/4 - \alpha, \\ (2,2), & \text{with probability } \alpha, \\ (2,3), & \text{with probability } 1/2. \end{cases}$$

As α increases, probability mass is moved from $(1,2)$ to $(2,2)$, and so $\mathcal{F}_\alpha$ *first-order stochastically dominates* $\mathcal{F}_{\alpha'}$ when $\alpha > \alpha'$. Nevertheless, the maximal revenue $\textsc{Rev}(\mathcal{F}_\alpha)$ *decreases* with α (in the region $0 \le \alpha \le 1/12$).

Proposition 6 *In Example E1: for every* $0 \le \alpha \le 1/12$,

$$\textsc{Rev}(\mathcal{F}_\alpha) = 11/4 - \alpha.$$

Proof. First, the revenue of $11/4 - \alpha$ is achieved by the mechanism with b given by (3): $(1/4) \cdot 1 + (1/4 - \alpha) \cdot 2 + \alpha \cdot 1 + (1/2) \cdot 4 = 11/4 - \alpha$.

Second, we show that a higher revenue cannot be obtained. Consider the following inequalities:

$$\begin{array}{rcl|l} q_1^{11} + q_2^{11} - s^{11} & \ge & 0 & 1 \\ q_1^{12} + 2q_2^{12} - s^{12} & \ge & q_1^{11} + 2q_2^{11} - s^{11} & 1/2 \\ 2q_1^{22} + 2q_2^{22} - s^{22} & \ge & 2q_1^{11} + 2q_2^{11} - s^{11} & 3\alpha \\ 2q_1^{23} + 3q_2^{23} - s^{23} & \ge & 2q_1^{11} + 3q_2^{11} - s^{11} & 1/4 - 3\alpha \\ 2q_1^{23} + 3q_2^{23} - s^{23} & \ge & 2q_1^{12} + 3q_2^{12} - s^{12} & 1/4 + \alpha \\ 2q_1^{23} + 3q_2^{23} - s^{23} & \ge & 2q_1^{22} + 3q_2^{22} - s^{22} & 2\alpha \end{array} \tag{4}$$

(the first inequality is IR at $(1,1)$, and the others are various IC constraints). Multiplying these inequalities by the multipliers on the right (which are all

[22] This explains the reason for including the outcome $x_1 + x_2 - 4$ in the mechanism.

nonnegative when $0 \leq \alpha \leq 1/12$) and then adding them up yields:

$$-\left(3/4-3\alpha\right)q_2^{11}-2\alpha q_1^{12}+\left(1/4-3\alpha\right)q_2^{12}+2\alpha q_1^{22}+q_1^{23}+(3/2)q_2^{23}$$
$$\geq (1/4)s^{11}+\left(1/4-\alpha\right)s^{12}+\alpha s^{22}+(1/2)s^{23}.$$

The right-hand side is precisely the expected revenue at $\mathcal{F}_\alpha$, and the left-hand side is at most $0+0+(1/4-3\alpha)+2\alpha+1+3/2 = 11/4-\alpha$ (since $q_2^{11}, q_1^{12} \geq 0$ and $q_2^{12}, q_1^{22}, q_1^{23}, q_2^{23} \leq 1$). Therefore the revenue cannot exceed $11/4-\alpha$, and so the revenue of $11/4-\alpha$ achieved by b of (3) is indeed maximal. ■

To get some intuition: the mechanism of (3) is:

Valuation	**Outcome**	
$x = (x_1, x_2)$	$q(x) = (q_1(x), q_2(x))$	$s(x)$
$(1,1)$	$(0,0)$	0
$(2,2)$	$(1,0)$	1
$(1,2)$	$(0,1)$	2
$(2,3)$	$(1,1)$	4

(5)

When the value of good 1 goes up (e.g., from $x = (1,2)$ to $x' = (2,2)$), the probability of getting good 1 also goes up (i.e., $q_1(x') = 1 > 0 = q_1(x)$); this is always so, as it is a consequence of the convexity of the buyer's payoff function b). However, at the same time the probability of getting good 2 may go down (e.g., $q_2(x') = 0 < 1 = q_2(x)$); moreover, it can do so in such a way that the allocation is worth less to the buyer, and so his payment to the seller goes down (i.e., $s(x') = 1 < 2 = s(x)$).

Remarks. *(a)* The mechanism (5) is the *unique* optimal mechanism at each $\mathcal{F}_\alpha$ with $0 \leq \alpha < 1/12$; indeed, in order to get the revenue of $11/4-\alpha$ one needs all relevant inequalities to become equalities (thus $q_2^{11} = q_1^{12} = 0$ and $q_2^{12} = q_1^{23} = q_1^{23} = q_2^{23} = 1$, which together with (4) as equalities can be easily shown to yield $q_1^{11} = 1, q_2^{22} = 0, s^{11} = 1, s^{12} = 2, s^{22} = 1, s^{23} = 4$— which is precisely (5)).

(b) Any small enough perturbation of the example—such as having full support on a square like $[0,3]^2$, or increasing all valuations as α increases—will not affect the nonmonotonicity, since the inequality $\textsc{Rev}(\mathcal{F}_0) > \textsc{Rev}(\mathcal{F}_{1/12})$ is strict.

3.3 Nonmonotonicity for independent and identically distributed goods

We now provide an example of nonmonotonicity where the goods are independent and identically distributed.

Example E2. Let F_1 and F_2 be the following one-dimensional distributions:

$$F_1 = \begin{cases} 10, & \text{with probability } \frac{4}{15}, \\ 46, & \text{with probability } \frac{1}{90}, \\ 47, & \text{with probability } \frac{1}{3}, \\ 80, & \text{with probability } \frac{7}{30}, \\ 100, & \text{with probability } \frac{7}{45}. \end{cases} \qquad F_2 = \begin{cases} 10, & \text{with probability } \frac{2399}{9000}, \\ 13, & \text{with probability } \frac{1}{9000}, \\ 46, & \text{with probability } \frac{1}{90}, \\ 47, & \text{with probability } \frac{1}{3}, \\ 80, & \text{with probability } \frac{7}{30}, \\ 100, & \text{with probability } \frac{7}{45}. \end{cases}$$

Clearly F_2 first-order stochastically dominates F_1 (since F_2 is obtained from F_1 by moving a probability mass of 1/9000 from 10 to 13), which of course implies that $F_2 \times F_2$ first-order stochastically dominates $F_1 \times F_1$. However, the optimal revenue from $F_1 \times F_1$ turns out to be *higher* than the optimal revenue from $F_2 \times F_2$.

Proposition 7 *In Example E2:*

$$\textsc{Rev}(F_1 \times F_1) \approx 69.47145 > \textsc{Rev}(F_2 \times F_2) \approx 69.47126.$$

Proof. Maximizing revenue for a distribution with finite support is a linear programming problem (the unknowns are the $q_i(x)$ and $s(x)$ for all x in the support, the constraints are the IR and IC inequalities, and the objective function is the expected revenue). Using Maple yields the following.

The unique[23] optimal mechanism for $F_2 \times F_2$ consists of 11 outcomes (ordered in the table below according to increasing payment to the seller s):

Valuations	**Outcome**	
x	$q(x)$	$s(x)$
$(10,10),(10,13),(13,10),(13,13),$ $(10,46),(46,10),(13,46),(46,13),(46,46),$ $(13,47),(47,13),(10,47),(47,10)$	$(0,0)$	0
$(46,47)$	$(\frac{32}{1187},\frac{384}{13057})$	$\frac{34240}{13057}\approx 2.6$
$(47,46)$	$(\frac{384}{13057},\frac{32}{1187})$	$\frac{34240}{13057}\approx 2.6$
$(47,47)$	$(\frac{35}{1187},\frac{35}{1187})$	$\frac{3258}{1187}\approx 2.7$
$(13,80)$	$(\frac{32}{1187},\frac{5647}{5935})$	$\frac{90672}{1187}\approx 76.4$
$(80,13)$	$(\frac{5647}{5935},\frac{32}{1187})$	$\frac{90672}{1187}\approx 76.4$
$(46,80)$	$(\frac{35}{1187},\frac{5647}{5935})$	$\frac{90810}{1187}\approx 76.5$
$(80,46)$	$(\frac{5647}{5935},\frac{35}{1187})$	$\frac{90810}{1187}\approx 76.5$
$(10,80),(10,100),(13,100)$	$(0,1)$	80
$(80,10),(100,10),(100,13)$	$(1,0)$	80
$(46,100),(100,46),$ $(47,80),(80,47),(47,100),(100,47),$ $(80,80),(80,100),(100,80),(100,100)$	$(1,1)$	126

For $F_1 \times F_1$ the same mechanism is optimal; however, the 5^{th} and 6^{th} outcomes are not used (the value 13 has probability 0) and may be dropped. This yields:

$$\begin{aligned}\textsc{Rev}(F_1 \times F_1) &= \frac{408189937}{5875650} = 69.47145...\ ,\\ \textsc{Rev}(F_2 \times F_2) &= \frac{30614162731}{440673750} = 69.47126...\ .\end{aligned}$$

[23] Uniqueness is proved using the dual linear progamming problem, as in the previous sections.

The nonmonotonicity of the payments is seen at $s(10, 80) > s(13, 80), s(46, 80)$ and $s(80, 10) > s(80, 13), s(80, 46)$. ■

4 Lotteries and Revenue

In the case of a single good (i.e., $k = 1$), in order to maximize revenue it suffices to consider deterministic mechanisms (specifically, "posted-price" mechanisms; see Myerson 1981). That is *not* so in the multi-good case. Examples where the optimal mechanism requires randomization (i.e., in some of the outcomes the probability of getting a good is strictly between 0 and 1) have been provided by Thanassoulis (2004) (in the slightly different context where the buyer's demand is limited to one good), Pycia (2006), Manelli and Vincent (2006, 2007),[24] and Pavlov (2011, Example 3(ii)). However, most of these examples are relatively complicated and require non-trivial computations, and it is not clear why and how randomization helps only when there are multiple goods.

We will provide two examples that are simple and transparent enough that the need for randomization becomes clear. In the first, the values of the two goods are correlated; in the second, the values are independent and identically distributed.

4.1 Lotteries for multiple goods

Consider the following example with two goods and three possible valuations[25] (the values of the two goods are correlated).

[24] Manelli and Vincent (2007) provide an example (Example 1) of an "undominated mechanism" that requires lotteries. While it is clear that an undominated mechanism is optimal for some distribution $\mathcal{F}$, it is claimed there (Theorem 9) that any undominated mechanism is optimal for some distribution with *independent* goods (i.e., a product distribution). However, there is an error in the proof of Theorem 9, as the set of product distributions (specifically, the set G in their proof) is not convex.

[25] Pycia (2006) solves the seller's problem when there are exactly two valuations and shows that randomizations may be needed. For instance, when the valuations are $(2, 3)$ and $(6, 1)$ with equal probabilities, the *unique* optimal mechanism gives buyer $(2, 3)$, for the total price of 4, good 2 and a 1/2 chance of getting good 1; and gives buyer $(6, 1)$ both goods for the total price of 7. However, we have found that Example E3, with

Example E3. Let $\mathcal{F}$ be the following two-dimensional probability distribution:

$$\mathcal{F} = \begin{cases} (1,0), & \text{with probability } 1/3, \\ (0,2), & \text{with probability } 1/3, \\ (3,3), & \text{with probability } 1/3. \end{cases}$$

Proposition 8 *The mechanism* (q, s) *defined by*

Valuation	**Outcome**	
x	$q(x)$	$s(x)$
$(1,0)$	$(\frac{1}{2}, 0)$	$\frac{1}{2}$
$(0,2)$	$(0,1)$	2
$(3,3)$	$(1,1)$	5

(6)

with

$$b(x_1, x_2) = \max\left\{0, \frac{1}{2}x_1 - \frac{1}{2},\ x_2 - 2,\ x_1 + x_2 - 5\right\} \tag{7}$$

is the unique *revenue-maximizing IC and IR mechanism for* $\mathcal{F}$ *of Example E3.*

Thus, the buyer can get both goods for price 5, or get good 2 for price 2, or *get good* 1 *with probability* 1/2 for price 1/2; the optimal revenue is $5/2 = 2.5$. It can be shown[26] that if the seller were restricted to deterministic mechanisms (where each q_i is either 0 or 1), then the optimal revenue would decrease to $7/3 = 2.33...$ (which is attained for instance by selling separately, at the optimal-single-good prices of 3 for good 1 and 2 for good 2). A detailed explanation of the role of randomization, and why it is needed only when there are multiple goods, follows the proof below.

Proof. Let $\langle(\alpha_1, \beta_1); \sigma_1\rangle, \langle(\alpha_2, \beta_2); \sigma_2\rangle$, and $\langle(\alpha_3, \beta_3); \sigma_3\rangle$ be the outcome $\langle(q_1(x), q_2(x)); s(x)\rangle$ at $x = (1,0), (0,2)$, and $(3,3)$, respectively (thus $\alpha_i, \beta_i \in$

three possible valuations, provides slightly more transparent insights (as there is a clearer separation between the IC and IR constraints).

[26]See footnote 27.

$[0,1]$). The objective function is $S := \sigma_1 + \sigma_2 + \sigma_3$ (this is 3 times the revenue). Consider the relaxed problem of maximizing S subject only to the individual rationality constraints at $(1,0)$ and $(0,2)$, and to the two incentive compatibility constraints at $(3,3)$, i.e.,

$$\begin{aligned} \alpha_1 - \sigma_1 &\geq 0, \\ 2\beta_2 - \sigma_2 &\geq 0, \\ 3\alpha_3 + 3\beta_3 - \sigma_3 &\geq 3\alpha_1 + 3\beta_1 - \sigma_1, \\ 3\alpha_3 + 3\beta_3 - \sigma_3 &\geq 3\alpha_2 + 3\beta_2 - \sigma_2. \end{aligned}$$

These inequalities can be rewritten as:

$$\begin{aligned} \sigma_3 + 3\alpha_1 + 3\beta_1 - 3\alpha_3 - 3\beta_3 \;&\leq\; \sigma_1 \;\leq\; \alpha_1, \\ \sigma_3 + 3\alpha_2 + 3\beta_2 - 3\alpha_3 - 3\beta_3 \;&\leq\; \sigma_2 \;\leq\; 2\beta_2. \end{aligned}$$

Therefore, in order to maximize $S = \sigma_1 + \sigma_2 + \sigma_3$ we must take $\sigma_1 = \alpha_1$ and $\sigma_2 = 2\beta_2$, which gives:

$$\begin{aligned} \sigma_3 &\leq 3\alpha_3 + 3\beta_3 - 2\alpha_1 - 3\beta_1, \\ \sigma_3 &\leq 3\alpha_3 + 3\beta_3 - 3\alpha_2 - \beta_2. \end{aligned}$$

Thus we must take $\alpha_3 = \beta_3 = 1$, $\beta_1 = \alpha_2 = 0$, and then $\sigma_3 = \min\{6 - 2\alpha_1, 6 - \beta_2\}$, and so $S = \alpha_1 + 2\beta_2 + \min\{6 - 2\alpha_1, 6 - \beta_2\} = \min\{2\beta_2 - \alpha_1, \beta_2 + \alpha_1\} + 6$. Since S is increasing in β_2 we must take $\beta_2 = 1$, and then $S = \min\{2 - \alpha_1, 1 + \alpha_1\} + 6$ is maximized at[27] $\alpha_1 = 1/2$. This is pecisely the mechanism (6), which is easily seen to satisfy also all the other IR and IC constraints. ■

To understand the use of randomization, consider the outcome $\langle(1/2, 0); 1/2\rangle$ at $x = (1,0)$ in (6): it is a lottery ticket that costs 1/2 and gives

[27] For deterministic mechanisms (i.e., $\alpha_i, \beta_i \in \{0,1\}$), everything is the same up to this point, but now S is maximized at both $\alpha_1 = 0$ and $\alpha_1 = 1$; the optimal revenue for deterministic mechanisms is thus $S/3 = 7/3$.

a $1/2$ probability of getting good 1; alternatively,[28] it is a $1/2 - 1/2$ lottery between getting good 1 for the price 1 (i.e., $\langle(1,0);1\rangle$), and getting nothing and paying nothing (i.e., $\langle(0,0);0\rangle$). It is thus the average of these two deterministic outcomes, and we now consider what happens when we replace the lottery by either one of them (see Table 1 below). It turns out that in *both* cases the *revenue strictly decreases*. In the first case, replacing $\langle(1/2,0);1/2\rangle$ by $\langle(1,0);1\rangle$ forces the price of the bundle to decrease to 4 (otherwise the $(3,3)$-buyer would switch from paying 5 for the bundle to paying 1 for good 1); therefore the net change in the revenue is $1/3 \cdot (1 - 1/2) + 1/3 \cdot (4 - 5)$, which is negative.[29] In the second case, replacing $\langle(1/2,0);1/2\rangle$ by $\langle(0,0);0\rangle$ results in the loss of the revenue from the $(1,0)$-buyer, without, however, increasing the revenue from the $(3,3)$-buyer: indeed, if we were to increase the bundle price, then $(3,3)$ would switch to $\langle(0,1);2\rangle$, i.e., would get good 2 for price 2 (and, if we were to drop this outcome $\langle(0,1);2\rangle$ altogether in order to increase the bundle price to 6, the total revenue would again decrease).[30]

x	$q(x)$	$s(x)$	$q^{(1)}(x)$	$s^{(1)}(x)$	$q^{(2)}(x)$	$s^{(2)}(x)$
$(1,0)$	$(\frac{1}{2},0)$	$\frac{1}{2}$	$(1,0)$	1	$(0,0)$	0
$(0,2)$	$(0,1)$	2	$(0,1)$	2	$(0,1)$	2
$(3,3)$	$(1,1)$	$\underline{\mathbf{5}}$	$(1,1)$	4	$(1,1)$	$\underline{\mathbf{5}}$

Table 1: Replacing a lottery outcome when there are *two* goods

It is instructive to compare this with a similar example but with a single good. Assume the values are $x = 1, 0, 3$, with equal probabilities of $1/3$ each (just like good 1 in Example E3). Take the mechanism with outcomes $\langle 1/2; 1/2\rangle, \langle 0;0\rangle, \langle 1;2\rangle$ (see Table 2 below); it is easy to see that it is IC and IR, and its revenue is $5/6$. The lottery outcome $\langle 1/2;1/2\rangle$—get the good with probability $1/2$ for price $1/2$—is the average of $\langle 0;0\rangle$ and $\langle 1;1\rangle$. Replacing the lottery $\langle 1/2;1/2\rangle$ by $\langle 1;1\rangle$ lowers the revenue to $2/3$: the 3-buyer switches

[28]Because of risk-neutrality.

[29]The buyer's payoff function in this mechanism is $b^{(1)}(x) = \max\{x_1 - 1, x_2 - 2, x_1 + x_2 - 4\}$.

[30]The buyer's payoff function in this mechanism is $b^{(2)}(x) = \max\{0, x_2 - 2, x_1 + x_2 - 5\}$.

to $\langle 1;1\rangle$. Replacing the lottery $\langle 1/2;1/2\rangle$ by $\langle 0;0\rangle$ *increases* the revenue to 1: the 3-buyer is now offered, and chooses, $\langle 1;3\rangle$. The revenue of 5/6 of the original mechanism with the lottery outcome is precisely the average of the revenues from these two resulting mechanisms, 2/3 and 1 (this averaging property holds at each valuation x).

x	q	s	$q^{(1)}$	$s^{(1)}$	$q^{(2)}$	$s^{(2)}$
1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	0	0
0	0	0	0	0	0	0
3	1	$\underline{\mathbf{2}}$	1	1	1	$\underline{\mathbf{3}}$

Table 2: Replacing a lottery outcome when there is *one* good

This is a general phenomenon when there is only one good: the revenue from a mechanism that includes an outcome that is a probabilistic mixture of two outcomes (a "lottery outcome") is the *average* of the revenues obtained by replacing the lottery with each one of these two outcomes and then adapting the remaining outcomes.[31] Formally, this is the counterpart of expressing the corresponding $b \in \mathcal{B}^1$ as an average of two functions in $\mathcal{B}^1$; in the example above, $b(x) = \max\{0, x/2 - 1/2, x - 2\}$ is the $1/2 - 1/2$ average of $b^{(1)}(x) = \max\{0, x - 1\}$ and $b^{(2)}(x) = \max\{0, x - 3\}$. Thus lotteries are indeed not needed when there is only one good.

Example E3 illustrates why this is *not* the case for multiple goods: replacing the lottery outcome with $\langle (0,0); 0\rangle$ yields the mechanism $(q^{(2)}, s^{(2)})$, whose revenue is *lower* than that of (q, s) (whereas replacing $\langle 1/2;1/2\rangle$ with $\langle 0;0\rangle$ yields a *higher* revenue). In fact, the function b of (7) is an extreme point in $\mathcal{B}^2$ (in particular, it is *not* the average of $b^{(1)}$ and $b^{(2)}$).

This is exactly where having more than one good matters. In the case of *one* good there is only *one* binding constraint per value x, namely, the outcome chosen by the next lower value. Consequently, dropping an outcome (such as a lottery outcome) chosen by x enables the seller to increase the rev-

[31] This statement, which is easily proved in general, provides another proof of Myerson's result that in the one-good case it suffices to consider deterministic mechanisms.

enue obtained from all higher-valuation buyers, as they can no longer switch to the outcome that has been removed and they strictly prefer their own outcome to any of the outcomes chosen by values below x. In contrast, when there are *multiple goods*, such an increase in revenue may not be possible because there may be *multiple* binding constraints per each valuation x (in our example, buyer $(3,3)$ is indifferent between reporting truthfully and reporting either $(1,0)$ or $(0,2)$). These buyer types may switch to other outcomes that involve other goods, and so the total revenue may well decrease.

Next, how does a lottery outcome increase revenue? The seller would like to earn positive revenue from selling good 1 to the $(1,0)$ buyer, but without jeopardizing the higher revenue obtained from selling the bundle of both goods to the $(3,3)$ buyer (and, as we have seen, he cannot increase the price of the bundle because of the "good 2 for price 2," alternative, i.e., $\langle(0,1);2\rangle$). If the price of good 1 is above 1 then $(1,0)$ will not buy it; if it is below 1, then $(3,3)$ will switch from buying the bundle to buying good 1 (since his payoff will increase from 1 to 2 or more).[32] Thus selling good 1 does not help. What does help is selling only a *fractional part* of good 1, which has the effect of making this option less attractive to the high-valuation buyer $(3,3)$ (since his possible gain is smaller: it is only *that fraction* of the difference in values). Thus, the two conflicting desiderata—getting some revenue from a low-valuation buyer, and not jeopardizing the higher revenue from a higher-valuation buyer—are reconciled by offering for sale fractions of the goods, i.e., lotteries. In the present example, that optimal fraction turns out to be $1/2$; it comes from balancing the incentives between the two goods (specifically, it is the ratio of two value differences, $3-2$ for good 2 and $3-1$ for good 1; see the Proof of Proposition 8 above).[33]

Finally, we note that mechanism design is a sequential game, with the seller moving first. In such games, the use of randomization may in general be strictly advantageous to the first mover (take for instance the sequential

[32] As we saw above, lowering the price of the bundle to 4 (while keeping the price of good 1 at 1) will not help either, because the total revenue decreases.

[33] Thus one can easily get other probabilities by changing the values. Moreover, the example is highly robust: it has a large neighborhood where the optimal mechanisms always require lotteries.

"matching pennies" game). Thus, the surprising fact here is not that randomizations can increase revenue (when there are multiple goods), but that they *cannot* do so when there is only one good.[34,35]

4.2 Lotteries for independent and identically distributed goods

We now provide a simple example where lotteries are necessary to achieve the maximal revenue for two goods that are independent and identically distributed.

Example E4. Let F be the following one-dimensional probability distribution:

$$F = \begin{cases} 1, & \text{with probability } 1/6, \\ 2, & \text{with probability } 1/2, \\ 4, & \text{with probability } 1/3, \end{cases}$$

and take two independent F-distributed goods, i.e., $\mathcal{F} = F \times F$.

Proposition 9 *The mechanism (q, s) defined by*

Valuations	**Outcome**	
x	$q(x)$	$s(x)$
$(1,1)$	$(0,0)$	0
$(2,1)$	$(\frac{1}{2},0)$	1
$(1,2)$	$(0,\frac{1}{2})$	1
$(1,4),(4,1),(2,2),(2,4),(4,2),(4,4)$	$(1,1)$	4

(8)

with

$$b(x_1, x_2) = \max\left\{0,\ \frac{1}{2}x_1 - 1,\ \frac{1}{2}x_2 - 1,\ x_1 + x_2 - 4\right\} \tag{9}$$

is the unique *optimal mechanism for $\mathcal{F} = F \times F$ of Example E4.*

[34]We thank Bob Aumann for this comment.

[35]Pycia (2006) shows how in the multiple-goods case non-deterministic mechanisms are generically needed to maximize revenue.

Proof. First, the revenue from the mechanism (8) is easily computed: it equals 61/18.

Second, take the following inequalities, which are various individual rationality and incentive compatibility constraints[36]:

$$\begin{array}{rcl|l}
q_1^{11}+q_2^{11}-s^{11} & \geq & 0 & 3\\
q_1^{12}+2q_2^{12}-s^{12} & \geq & 0 & 8\\
2q_1^{21}+q_2^{21}-s^{21} & \geq & 0 & 8\\
2q_1^{22}+2q_2^{22}-s^{22} & \geq & 0 & 17\\
q_1^{12}+2q_2^{12}-s^{12} & \geq & q_1^{11}+2q_2^{11}-s^{11} & 1\\
2q_1^{21}+q_2^{21}-s^{21} & \geq & 2q_1^{11}+q_2^{11}-s^{11} & 1\\
2q_1^{22}+2q_2^{22}-s^{22} & \geq & 2q_1^{12}+2q_2^{12}-s^{12} & 3\\
2q_1^{22}+2q_2^{22}-s^{22} & \geq & 2q_1^{21}+2q_2^{21}-s^{21} & 3\\
q_1^{14}+4q_2^{14}-s^{14} & \geq & q_1^{12}+4q_2^{12}-s^{12} & 3\\
4q_1^{41}+q_2^{41}-s^{41} & \geq & 4q_1^{21}+q_2^{21}-s^{21} & 3\\
2q_1^{22}+2q_2^{22}-s^{22} & \geq & 2q_1^{14}+2q_2^{14}-s^{14} & 1\\
2q_1^{22}+2q_2^{22}-s^{22} & \geq & 2q_1^{41}+2q_2^{41}-s^{41} & 1\\
2q_1^{24}+4q_2^{24}-s^{24} & \geq & 2q_1^{22}+4q_2^{22}-s^{22} & 8\\
4q_1^{42}+2q_2^{42}-s^{42} & \geq & 4q_1^{22}+2q_2^{22}-s^{22} & 8\\
4q_1^{44}+4q_2^{44}-s^{44} & \geq & 4q_1^{24}+4q_2^{24}-s^{24} & 2\\
4q_1^{44}+4q_2^{44}-s^{44} & \geq & 4q_1^{42}+4q_2^{42}-s^{42} & 2
\end{array} \tag{10}$$

Multiplying each inequality by the weight on the right and adding up yields:

$$\begin{aligned}
& s^{11}+3s^{12}+3s^{21}+9s^{22}+2s^{14}+2s^{41}+6s^{24}+6s^{42}+4s^{44}\\
& \quad \leq\ 2q_1^{22}+q_1^{14}+10q_1^{41}+8q_1^{24}+24q_1^{42}+16q_1^{44}\\
& \qquad +2q_2^{22}+10q_2^{14}+q_2^{41}+24q_2^{24}+8q_2^{42}+16q_2^{44}.
\end{aligned} \tag{11}$$

The left-hand side turns out to be precisely 36 times the expected revenue of the seller for the distribution $\mathcal{F}=F\times F$, i.e., $36\mathbb{E}_{\mathcal{F}}\left[s(x)\right]$, and the right-hand side is bounded from above by 122 (replace all q_1 and q_2 there by their upper bound of 1). Therefore $\mathbb{E}_{\mathcal{F}}\left[s(x)\right]\leq 122/36=61/18$. Recalling that 61/18 is

[36] These specific inequalities and their corresponding multipliers below were obtained by solving the dual of the linear programming problem of maximizing the revenue.

precisely the revenue of the mechanism (8) shows that (8) is optimal.

Finally, to see that (8) is the only optimal mechanism: by the proof above, for the maximal revenue of 61/18 to be achieved, all the inequalities must become equalities. First, all the q_1 and q_2 appearing on the right-hand side of (11) must equal 1:

$$\begin{aligned} 1 &= q_1^{22} = q_1^{14} = q_1^{41} = q_1^{24} = q_1^{42} = q_1^{44} \\ &= q_2^{22} = q_2^{14} = q_2^{41} = q_2^{24} = q_2^{42} = q_2^{44}. \end{aligned} \tag{12}$$

Second, the inequalities in (10), which are now equalities, yield after substituting (12):

$$\begin{aligned} &s^{44} = s^{24} = s^{42} = s^{22} = s^{14} = s^{41} = 4, \quad s^{12} = s^{21} = 1, \quad s^{11} = 0, \\ &q_1^{11} = q_2^{11} = q_1^{12} = q_2^{21} = 0, \;\; q_1^{21} = q_2^{12} = \frac{1}{2}. \end{aligned}$$

Together with (12) this yields precisely the mechanism (8). ■

It can be checked that the maximal revenue achievable by a deterministic mechanism is 10/3 (obtained by the mechanism with price 2 for each good).

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