# CONCAVE EXPECTED UTILITY AND EVENT SEPARABILITY

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*preliminary version; comments are welcome*

Abstract. We introduce a family of decision making models, referred to as *event-separable*, based on a non-additive probability and on a general integration scheme. To characterize such models we take a different approach to independence and present the *subjective codecomposable independence* axiom that determines when the decision maker exhibits ambiguity neutrality. The new approach allows us to: (a) introduce the *Concave Expected Utility* model of decision making, adhering to ambiguity aversion where uncertainty is captured through a non-additive probability; and (b) provide sufficient conditions, weaker than those employed by previous formulations hinging on the independence axiom, to subjective and Choquet expected utility models.

Keywords: Codecomposable independence, ambiguity aversion, concave expected utility, Choquet expected utility, Ellsberg paradox, Machina paradox.

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## 1. Introduction

In the heart of decision theory underlies Savage's [14] and Anscombe and Aumann's [1] Subjective Expected Utility (*SEU*) theory. Among other standard axioms, Savage's theory hinges on the sure thing principle while that of Anscombe–Aumann's on the independence axiom. *SEU* theory states that a decision maker entertains a subjective prior probability and uncertain alternatives (i.e., acts) are ranked according to their expected utility. Starting with Ellsberg [4], abundance of thought and lab experiments suggest that in the presence of subjective uncertainty, termed ambiguity, individuals may be unable to entertain a prior belief and hence violate the sure thing principle and the independence axiom. The main observation is that individuals are not ambiguity neutral and exhibit preferences for hedging, a phenomena that is termed *ambiguity aversion* (see Schmeidler [15] and Gilboa and Schmeidler [7]). Schmeidler [15] proposed to weaken the independence axiom and assumed that it applies only for *comonotonic* acts. He argued that comonotonic acts are structurally similar and hence there should not be a strict preference for hedging. By weakening Anscombe and Aumann's independence axiom to comonotonic-independence, Schmeidler presented Choquet Expected Utility (*CEU*) theory: ambiguity is captured through a subjective non-additive probability and alternatives are ranked according to their expected utility which is calculated by the Choquet integral.

We take a different approach to independence. The main idea is the following. Any act can be represented as a coin toss (a mixture) between betting on an event and a 'complementary' act. Typically, there are different possibilities to decompose an act this way. The independence axiom we employ requires that any act can be decomposed to a bet and a complementary act in a way that the decision maker exhibits ambiguity neutrality between them.[1] Note that according to this axiom, ambiguity neutrality is required to hold for a particular decomposition and not for all them. Also, this particular decomposition depends on the decision maker and typically differs

[1]More formally, any act can be decomposed in such a way that the decision maker exhibits ambiguity neutrality among al acts represented by the same bet and complementary act. For brevity, we use the former wording throughout the Introduction.

from one decision maker to another. We therefore refer to this axiom as *subjective codecomposable independence*.

Our approach, that hinges on codecomposable independence, enables one to characterize a large class of event-separable preferences. These preferences are represented by a subjective non-additive probability that captures uncertainty and a general integration scheme according to which expected utility is calculated. The class of event-separable preferences clearly contains *CEU* but also other economically meaningful families of preferences. We show that by further assuming the ambiguity aversion axiom, event-separable preferences are such that the concave integral (Lehrer [9]) is the integration scheme by which acts are being evaluated. These are referred to as *Concave Expected Utility* (*CavEU*) preferences.

How to motivate our codecomposable independence approach beyond the results it yields? First, codecomposable independence axiom can take different shapes, providing new ways of characterizing existing models. Thus, this approach sheds new light on known theories. For instance, given an act one could postulate that ambiguity neutrality applies to decompositions that involve bets that are comonotonic with the act. Such an axiom is sufficient to characterize the *CEU* model. Another version of codecomposable independence axiom yields the *SEU* model. This version requires that ambiguity neutrality would hold not only for a particular class of decompositions, but to any decomposition to a bet and a complementary act.

This brings us to the second motivation for subjective codecomposable independence. Comonotonic independence (Schmeidler [15]) suggests that comonotonic acts are structurally 'similar', and therefore it is reasonable to assume that a decision maker will not have a strict preference for hedging for such acts. This argument obviously applies to the comonotonic version of our codecomposable independence axiom.According to this axiom, a decision maker has no choice but to be ambiguity neutral when a bet and an act are comonotonic, since they are structurally similar. We find this assumption rather strong; a decision maker might not find this particular structural similarity sufficient to imply ambiguity neutrality.

Subjective codecomposable independence allows a decision maker the freedom to choose what similarity means and for which type of decompositions to exhibit ambiguity neutrality. Thus, ambiguity attitude towards different decompositions is completely subjective and may differ from one decision maker to another. Among the class of event-separable preferences, the ambiguity averse preferences *CavEU* are more flexible than *CEU* preferences in the sense that the acts among which ambiguity neutrality applies are subjectively determined and are not dictated by pre-specified structural similarity. This is why *CavEU* preferences are less vulnerable than *CEU* ones to 'paradoxes' such as those introduced by Machina [13].

To summarize, the contribution of the current paper is threefold. First, we introduce a subjective codecomposable independence axiom that allows us to characterize a general class of event-separable preferences. For such preferences ambiguity is captured through a non-additive probability and expected utility is determined according to a general integration scheme (not necessarily the Choquet integral). Second, this approach allows us to introduce a model of decision making that always respects ambiguity aversion where uncertainty is captured through a non-additive probability. Lastly, it provides sufficient conditions, which are weaker than the previous formulations adhering to the independence axioms, to subjective and Choquet expected utility models.

The rest of the paper is organized as follows. The next section provides an informal discussion regarding Choquet expected utility, concave expected utility and some of the differences between the two approaches. The formal framework of choice under uncertainty is presented and the basic axioms are formulated in Section 3. Subjective codecomposablity and the emergence of a capacity are presented in Section 4.1. Ambiguity aversion and *CavEU* preferences are discussed in Section 4.2 where a short literature review appears in Section 4.3. The relation of codecomposablity to *SEU* and *CEU* is present in Section 5. Lastly, Section 6 presents a recent paradox for *CEU* preferences raised by Machina [13] and shows how *CavEU* accommodates the paradox. All the proofs are in the appendix.

## 2. Choquet and Concave Expected Utility

This section provides an informal discussion and (partial) comparison between the concave and Choquet integrals. The formal study appears in the following sections. Assume that the underlying domain of alternatives is the collection of (non-negative) utility acts, or random variables[2] given a state space $S = \{s_1, ..., s_n\}$. A *capacity* $v$ over the state space is a function that assigns a number to each event in a monotonic (with respect to containment) fashion. We interpret the capacity $v(E)$ of an event $E$ as how likely $E$ is with respect to $v$. A finite collection $(a_i, E_i)_i$, where $a_i$ is a positive real number and $E_i$ is an event,[3] is a *decomposition* of an act $g$ if $\sum_i a_i \mathbb{1}_{E_i} = g$. That is, $g$ can be decomposed to the collection of simpler functions of the form $a_i \mathbb{1}_{E_i}$. Similarly to the Lebesgue integral, the value of a decomposition $(a_i, E_i)_i$ with respect to a capacity $v$ is simply $\sum_i a_i v(E_i)$.

For an act $g$, permute the state space by $\pi : \{1, ..., n\} \to \{1, ..., n\}$ such that $\pi(i) \geq \pi(j)$ if $g(s_i) \geq g(s_j)$. That is, $g(s_{\pi^{-1}(i)})$ is increasing with $i$. The Choquet integral of $g$ with respect to a capacity $v$ is

$$(1) \quad \int^C g dv = g(s_{\pi^{-1}(1)}) v(\{s_{\pi^{-1}(1)}, ..., s_{\pi^{-1}(n)}\}) + \\ + \sum_{i=2}^{n} (g(s_{\pi^{-1}(i)}) - g(s_{\pi^{-1}(i-1)}) v(\{s_{\pi^{-1}(i)}, ..., s_{\pi^{-1}(n)}\}).$$

From the right hand side of Eq. 1, the Choquet integral of $g$ with respect to $v$ is the value of a particular decomposition of $g$ of the form $\big(g(s_{\pi^{-1}(1)}), \{s_{\pi^{-1}(1)}, ..., s_{\pi^{-1}(n)}\}\big)$, $\big(g(s_{\pi^{-1}(2)}) - g(s_{\pi^{-1}(1)}), \{s_{\pi^{-1}(2)}, ..., s_{\pi^{-1}(n)}\}\big)$ , ..., $\big(g(s_{\pi^{-1}(n)}) - g(s_{\pi^{-1}(n-1)}), \{s_{\pi^{-1}(n)}\}\big)$. We refer to such a decomposition as the *Choquet decomposition*. Preferences $\succeq$ over the domain discussed are *CEU* if they can be represented by the Choquet integral with respect to a capacity. That is there exists a capacity $v$ such that $g \succeq h$ if and only if $\int^C g dv \geq \int^C h dv$.

[2]That is, we assume for the sake of simplicity that the vNM utility index was already identified.

[3]$\mathbb{1}_E$ is the indicator function of the event $E$.

Clearly, every alternative has more than one decomposition, the Choquet decomposition being one of them. The concave integral $\int^{Cav} g dv$ of a random variable $g$ is defined as the maximum value over *all* decompositions of $g$.

To illustrate how *CavEU* may be different than *CEU*, consider the following example. Let the state space be $S = \{s_1, ..., s_4\}$ and define a capacity $v$ over the state space as follows: $v(s) = \frac{1}{12}$ for every state $s$, $v(\{s_1, s_2\}) = v(\{s_1, s_3\}) = v(\{s_2, s_3\}) = v(\{s_1, s_4\}) = \frac{1}{6}$, $v(\{s_2, s_4\}) = v(\{s_3, s_4\}) = \frac{3}{12}$, $v(\{s_1, s_2, s_3\}) = v(\{s_1, s_3, s_4\}) = v(\{s_2, s_3, s_4\}) = \frac{1}{3}$, $v(\{s_1, s_2, s_4\}) = \frac{5}{6}$ and $v(S) = 1$. Note that the contribution of the state $s_2$ to any event that contains neither $s_1$ nor $s_2$ is greater than the contribution of $s_1$. Formally, for any event $E$ that does not contain the states $s_1, s_2$, $v(E \cup \{s_1\}) \leq v(E \cup \{s_2\})$. Moreover, the inequality is strict when $E = \{s_4\}$. In this sense, under the belief $v$ the state $s_2$ is more likely than $s_1$.

Now, consider the random variables $f = (0, 1, 2, 3)$ and $g = (1, 0, 2, 3)$. Note that $f$ and $g$ differ only in states $s_1$ and $s_2$. $f$ assigns the lower outcome to the less likely state and the higher outcome to the more likely one. It is the opposite case for $g$; it assigns the higher outcome to the less likely state. It is plausible then that preferences based on the capacity $v$ would rank $f$ over $g$. Nevertheless, the Choquet integral of both $f$ and $g$ is $\frac{8}{12}$: $\int^C f dv = v(\{s_2, s_3, s_4\}) + v(\{s_3, s_4\}) + v(\{s_4\}) = v(\{s_1, s_3, s_4\}) + v(\{s_3, s_4\}) + v(\{s_4\}) = \int^C g dv$. That is, *CEU* preferences represented by the capacity $v$ rank $f$ and $g$ indifferent. However, *CavEU* preferences rank $f$ strictly preferred to $g$: $\int^{Cav} f dv = v(\{s_2, s_4\}) + 2v(\{s_3, s_4\}) = \frac{9}{12} > \frac{8}{12} = v(\{s_1, s_3, s_4\}) + v(\{s_3, s_4\}) + v(\{s_4\}) = \int^{Cav} g dv$.

The capacity $v$ above can be presented as $v(E) = \min_i p_i(E)$ for every event $E$, where $p_1 = (\frac{1}{12}, \frac{1}{12}, \frac{1}{6}, \frac{2}{3}), p_2 = (\frac{1}{12}, \frac{2}{3}, \frac{1}{12}, \frac{1}{6}), p_3 = (\frac{1}{12}, \frac{2}{3}, \frac{1}{6}, \frac{1}{12})$ and $p_4 = (\frac{2}{3}, \frac{1}{12}, \frac{1}{12}, \frac{1}{6})$. That is the capacity, as a modeling tool of perception of ambiguity, displays pessimism. In this case it is natural to assume that the decision maker will exhibit ambiguity aversion. Nevertheless, the Choquet integral with respect to this capacity does not exhibit such aversion.[4] On the other hand, the concave integral does. As will be formally shown, the example above is a generic one in the sense that the capacity representing any *CavEU* preferences can always be constructed as the minimum of measures over the

[4]The capacity $v$ as defined in the example is not a convex one. According to Schmeidler [15] the Choquet integral with respect to $v$ does not adhere to ambiguity aversion.

state space. In contrast, the Choquet integral with respect to such capacities typically does not exhibit ambiguity aversion.

## 3. Environment

Consider a decision making framework in which an object of choice is an act from the state space to utility outcomes. More formally, let $S$ be a finite non–empty set of *states of nature*. An act is a function from $S$ to $\mathbb{R}_+$. The collection of acts is denoted by $\mathcal{F}$ with typical elements being $f, g, h$. Abusing notation, for an act $f \in \mathcal{F}$ and a state $s \in S$, we denote by $f(s)$ the constant act that assigns the utility $f(s)$ to every state of nature. Utils (and constant acts) will be typically denoted by $a, b, c$. Mixtures (convex combinations) of acts are performed pointwise. That is, if $f, g \in \mathcal{F}$ and $\delta \in [0, 1]$, then $\delta f + (1 - \delta)g$ is the act in $\mathcal{F}$ that yields $\delta f(s) + (1 - \delta)g(s)$ utility for every $s \in S$. Mixtures coefficients will be denoted by $\delta, \alpha$, etc.

In our framework, a decision maker is associated with a binary relation $\succeq$ over $\mathcal{F}$ representing his ranking. $\succ$ is the asymmetric part of the relation. That is $f \succ g$ if $f \succeq g$ but it is not true that $g \succeq f$. $\sim$ is the symmetric part, that is $f \sim g$ if $f \succeq g$ and $g \succeq f$.

We interpret $f(s)$ as the payoff induced by act $f \in \mathcal{F}$ in state $s \in S$ and assume it is the utility exerted by the decision maker if $f$ is chosen and $s$ is the realized state. That is, we assume that the vNM utility function of the decision maker has already been identified.[5]

A binary relation $\succeq$ is *reflexive* if $f \sim f$ for every act $f$. $\succeq$ is *complete* if for every $f, g \in \mathcal{F}$, either $f \succeq g$ or $g \succeq f$. It is *transitive* if for $f, g, h \in \mathcal{F}$, $f \succeq g$ and $g \succeq h$ imply $f \succeq h$. The following is a list of assumptions (axioms) regarding a binary relation $\succeq$ over acts. We will postulate these assumption throughout.

*Preference.* $\succeq$ is reflexive, complete and transitive.

[5]One can also consider the restatement by Fishburn [6] of the classical Anscombe-Aumann [1] set-up. In that case, standard axioms imply that the vNM utility index can be identified and that the formulation of alternatives as utility acts, as we do here, is well defined. Such results have have been established in many papers and we here rely on such results for convenience and brevity.

*Monotonicity.* For every $f, g \in \mathcal{F}$, $f(s) \geq g(s)$ for all $s \in S$ implies $f \succeq g$.

*Continuity.* For every $f \in \mathcal{F}$ the sets $\{g \in \mathcal{F} : g \succeq f\}$ and $\{g \in \mathcal{F} : g \preceq f\}$ are closed.

## 4. Decomposability and Ambiguity Aversion

4.1. **A Capacity Emerges.** A *bet* is an act that yields some utility $b \in \mathbb{R}_+$ over an event $E \subseteq S$ and the utility 0 over the complement event. Such a bet will be denoted by $b_E$. An act which is not a bet can always be represented as a convex combination, or a decomposition, of some bet and another act. That is, for $f \in \mathcal{F}$ we can find a bet $b_E$, an act $f'$ and $\delta \in [0,1]$ such that $f = \delta b_E + (1-\delta)f'$. Of course such bet $b_E$, and therefore $f'$ and $\delta$, need not be unique. The following axiom states that for at least one such decomposition to $b_E$ and $f'$, the preference relation satisfies independence over $[b_E, f'] = \{\alpha b_E + (1-\alpha)f' : \delta \in [0,1]\}$. The axiom can be restated as follows: if $f, g, h \in \mathcal{F}$ are all similar, in the sense that they can call be decomposed to a bet $b_E$ and an $f'$, then independence involving $f, g, h$ holds.

*Subjective Codecomposable Independence.* For every non-bet act $f$, there exist a bet $b_E$ and $f'$ such that $f \in [b_{E^f}, f']$ and $\succeq$ satisfies independence over $[b_{E^f}, f']$.

To explore the implication of *subjective codecomposable independence* we need to present some notations and definitions. A *capacity* $v$ over $S$ is a function $v : 2^S \to [0,1]$ satisfying: (i) $v(\phi) = 0$ and $v(S) = 1$; and (ii) $K \subseteq T \subseteq N$ implies $v(K) \leq v(T)$.

We say that a binary relation $\succeq$ over all acts $\mathcal{F}$ admits a *decomposition representation* if there exist:

1. a functional $V : \mathcal{F} \to \mathbb{R}$ that represents $\succeq$;
2. a capacity $v : 2^S \to [0,1]$, such that

$$V(f) = \sum a_E v(E) \text{ for some } \sum a_E \mathbb{1}_E = f,$$

where $a_E > 0$; and

A (finite) collection $\{(a_E, E) : a > 0, E \subseteq S\}$ is a *decomposition* of $f$ if $\sum a_E \mathbb{1}_E = f$. Given a capacity $v$ over events, the value of such a decomposition is $\sum a_E v(E)$. Thus, a binary relation admits a decomposition representation if an act is ranked according to the value, with respect to the capacity, of one of its decompositions into bets.

Before stating a result that provides a representation for *subjective codecomposable independence*, note that the axiom does not have a take on mixtures with the constant bet 0. We formulate an additional axiom that states explicitly that.

*Worst-Outcome Independence.* $\succeq$ satisfies independence over $[0, f]$ for every act $f$.

**Theorem 1.** *Let $\succeq$ be a binary relation over $\mathcal{F}$ satisfying* preferences, monotonicity, continuity, worst-outcome independence *and* subjective codecomposable independence. *Then $\succeq$ admits a decomposition representation.*

Theorem 1 states that given standard assumptions, *worse-outcome independence* and *subjective codecomposable independence*, a binary relation admits a decomposition representation. The axioms are sufficient to identify a non-additive belief and the fact that alternatives are ranked according to the value of one of their decompositions. However, *subjective codecomposable independence* is a weak assumption; it is not possible to determine exactly what is the decomposition according to which an alternative is ranked. A question is whether adding more structure to such preferences can yield interesting and natural aggregation mechanisms that are different than Choquet.

4.2. **Ambiguity Aversion and Concave Expected Utility.** Since Schmeidler [15] and Gilboa and Schmeidler [7] ambiguity aversion is one of the most studied phenomenon in the theory of decision making. Unlike Schmeidler [15] who focused on comonotonic-independence, we here wish to impose ambiguity aversion while assuming only *subjective codecomposable independence*.

*Ambiguity Aversion.* For every $f, g \in \mathcal{F}$, if $f \sim g$ then $\delta f + (1-\delta) g \succeq g$ for every $\delta \in [0, 1]$.

Lehrer [9] presented an integration scheme for capacities based on concavity: the *concave integral* of an act $f : S \to \mathbb{R}_+$ with respect to a capacity $v$ is defined by

$$\int^{Cav} f dv = \max\left\{\sum a_E v(E) : \sum a_E \mathbb{1}_E = f, a_E > 0\right\}.$$

The difference between Choquet integral and the concave integral is that, the latter considers all possible decompositions while the former takes into account only those with a chain structure.

We refer to preferences $\succeq$ over all acts $\mathcal{F}$ as *CavEU* if there exist a capacity $v : 2^S \to [0,1]$, such that for all $f, g \in \mathcal{F}$

$$f \succeq g \iff \int^{Cav} f dv \geq \int^{Cav} g dv.$$

The following result states that along with the standard assumptions, *worst-outcome independence*, *subjective codecomposable independence* and *ambiguity aversion* preferences can be represented by the concave integral.

**Theorem 2.** *Let $\succeq$ be a binary relation over $\mathcal{F}$. Then the following are equivalent:*
*1. $\succeq$ are* preferences *that satisfy* monotonicity, continuity, worst-outcome independence, subjective codecomposable independence *and* ambiguity aversion*; and*
*2. $\succeq$ is CavEU.*

For a capacity $v$, define $\hat{v}(E) = \int^{Cav} \mathbb{1}_E dv$. $\hat{v}$ is termed the *totally balanced cover* of $v$. If $\hat{v} = v$ we say that $v$ is *totally balanced*. A capacity $v$ is said to be *exhibiting pessimism* if it can be written as a minimum of measures. That is, there exist a finite collection of measures $\{\mu_i\}_i$ such that $v = \min_i \mu_i$. The following is our uniqueness result regarding the representation of *CavEU* preferences.

**Proposition 1.** *Let $\succeq$ be* CavEU*. Then: 1. there exists a unique totally balanced capacity $v$ representing $\succeq$; 2. there exists a unique capacity $v'$ exhibiting pessimism that represents $\succeq$. Furthermore, $v = v'$.*

4.3. **How Does It Fit in the Lit?** There are numerous models of choice under uncertainty. The most related ones are *confidence preferences* presented by Chateauneuf and Faro [3], maxmin expected utility (*MEU*) that were axiomatized by Gilboa and Schmeidler [7] and, of course, *CEU* preferences.

*CavEU* are clearly a particular case of confidence preferences, but require more structure since not every confidence preferences satisfy the decomposability property. To see this, consider *MEU* preferences, which are a particular case of confidence preferences.

Not every *MEU* preference relation can be represented as a (concave) integral; *MEU* satisfy translation covariance (due to the c-independence axiom) while it is clear from *subjective codecomposable independence* that it does not have to be satisfied by *CavEU*. The subclass of *CavEU* preferences that do admit an *MEU* representation are those that can be represented with a capacity having a *large core* (see, Lehrer [9]).[6] This brings us to *CEU* preferences. Schmeidler [15] that the Choquet integral is a concave one if and only if the capacity is *convex*. Hence we have that when the capacity is not convex *CavEU* and *CEU* differ. In addition, due to Lehrer [9] and Teper and Lehrer [11], *CEU* and *CavEU* coincide if and only if the capacity representing the preferences is *convex* (and in this case it is also *MEU*).

The latter point emphasizes that given ambiguity aversion, the class of *CavEU* preferences is more general than that of *CEU*. In Section 6 we illustrate this point by discussing an example by Machina [13] and showing that *CavEU* can explain behavior that may lead to a "paradox" for *CEU*.

## 5. Codecomposable Independence and Expected Utility Models

It is interesting to see the relation between the codecomposable independence approach to existing models. Clearly, both *SEU* and *CEU* are particular classes of preferences admitting a decomposition representation. In what follows, we provide stronger versions of our independence axiom that will yield exactly *SEU* and *CEU*.

Fix an act $f$. Recall that *subjective codecomposable independence* states that independence holds over at least one interval $[b_E, f']$ that contains $f$. The following axiom postulates that for every such decomposition to $b_E$ and $f'$, the preference relation satisfies independence over $[b_E, f']$.

*Codecomposable Independence.* For every bet $b_E$ and act $f'$, $\succeq$ satisfies independence over $[b_E, f']$.

Assuming *codecomposable independence* along with the axioms specified above allows us to formulate the following result.

[6] The definition of large core is due to Sharkey [16].

**Proposition 2.** *The following two statements are equivalent:*

*1.* $\succeq$ *satisfies* preference, continuity, monotonicity *and* codecomposable independence*;*

*2.* $\succeq$ *admits an* SEU *representation.*

Proposition 2 states that given the standard axioms, *codecomposable independence* allows us to identify a subjective probability with respect to which the decision maker calculates the expected utility of the different alternatives and ranks them accordingly. Note that *worst-out independence* is no longer needed as it is implied by *codecomposable independence*.

For an act $f$ and a utility level $a \in \mathbb{R}_+$, let $E_a^f = \{s \in S : f(s) \geq a\}$ be the event in which $f$ performs better that $a$. We refer to such an event as a *cumulative* event for $f$. When considering a cumulative event for an act $f$, we may ignore the utility level at times and write $E^f$. A weaker codecomposable independence axiom can be formulated taking into account only decomposition of acts to bets over (respectively) cumulative events.

*Cumulative Codecomposable Independence.* For every act $f$, bet $b_{E^f}$ and $f'$ such that $f \in [b_{E^f}, f']$, $\succeq$ satisfies independence over $[b_{E^f}, f']$.

The axiom postulates that if $f, g, h \in \mathcal{F}$ can all be decomposed to a bet $b_{E^f}$ and an $f'$, then independence involving $f, g, h$ holds. Note that $f, g$ and $h$ are comonotonic. Resulting from such weakening of codecomposable independence is the following proposition. Again, *worst-out independence* is implied by *cumulative codecomposable independence*.

**Proposition 3.** *The following two statements are equivalent:*

*1.* $\succeq$ *satisfies* preference, continuity, monotonicity *and* cumulative codecomposable independence*;*

*2.* $\succeq$ *admits a* CEU *representation.*

## 6. On an Example by Machina

Machina [13] in a recent paper "exploits" the structural event-separability (in particular, tail separability, as he refers to it) exhibited by *CEU* prefernences and constructs several examples, in the spirit of Ellsberg, in which such preferences can not accommodate choices that may be considered natural. This has been reinforced by L'Hardion and Placido [12] who showed that a large number of subjects exhibit such choices. As discussed in the Introduction, *CavEU* are more flexible than *CEU* preferences in the sense that event-separability is subjective and is not pre-specified structurally. This is why *CavEU* preferences are less vulnerable than *CEU* ones to such 'paradoxes'.

One of Machina's examples is the following. Consider an urn containing 100 balls, each marked with a number from 1 through 4. All you know is that there are 50 balls that are marked either 1 or 2, and 50 balls that are marked either 3 or 4. You are being offered a pair of bets $f$ and $g$, as described in Table 1, that depend on a draw of one ball from the urn.[7] In addition you are being offered another pair of bets $h$ and $k$ that depend as well on a draw of one ball from the urn.

TABLE 1. The Reflection Example

| Bet | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|
| $f$ | 0 | 200 | 100 | 100 |
| $g$ | 0 | 100 | 200 | 100 |
| $h$ | 100 | 200 | 100 | 0 |
| $k$ | 100 | 100 | 200 | 0 |

Machina notices that by tail separability, as he refers to it, *CEU* maximizer prefers $f$ to $g$ if and only if she prefers $h$ to $k$. From the tables above it is clear that acts $h$ and $k$ are obtained from $f$ and $g$ by a pair of common-outcome tail shifts; *CEU* preferences cannot explain a "reversal" such as the preference of $f$ over $g$ and at the same time the preference of $k$ over $h$.

*CavEU* preferences can accommodate the reversal of preferences indicated by Machina.[8] If $v(s_2) > v(s_3) = 0$, $v(s_2, s_3, s_4) > v(s_2, s_3) + v(s_3, s_4)$ and $v(s_2, s_4) = v(s_2)$ then

[7]Even though the analysis would go through if entries are monetary, we consider utils for brevity and simplicity.

[8]In an unpublished manuscript, Lehrer [10] shows that the concave integral can accommodate the other reversal, that is, the preference of $g$ over $f$ and that of $h$ over $k$. L'Hardion and Placido [12] find that the more common reversal is the one discussed in the discussion above. Baillon, L'Haridon,

$\int^{Cav} f dv = v(s_2, s_3, s_4) + v(s_2)$, $\int^{Cav} g dv = v(s_2, s_3, s_4) + v(s_3)$ and $\int^{Cav} f dv > \int^{Cav} g dv$. On the other hand, if in addition $v(s_1, s_2, s_3) - v(s_2, s_3) > v(s_1, s_2) - v(s_2)$ and $v(s_1, s_2, s_3) < v(s_1, s_3) + v(s_2, s_3)$ then $\int^{Cav} h dv = v(s_1, s_2, s_3) + v(s_2)$, $\int^{Cav} k dv = v(s_1, s_3) + v(s_2, s_3)$ and $\int^{Cav} k dv > \int^{Cav} h dv$.

## References


[1] F. J. Anscombe, and R. J. Aumann, A definition of subjective probability, *Annals of Mathematical Statistics*, **34** (1963), 199-295.

[2] A. Baillon, O. L'Haridon and L. Placido, Ambiguity Models and the Machina Paradoxes, *American Economic Review,* **101** (2011), 1546-1560.

[3] A. Chateauneuf and J. H. Faro, Ambiguity Through Confidence Functions, *Journal of Mathemtical Economics,* **45** (2009), 535-558.

[4] D. Ellsberg, Risk, Ambiguity, and the Savage Axioms, *Quarterly Journal of Economics*, **75** (1961), 643-669.

[5] Y. Even and E. Lehrer, Decomposition-Integral: Unifying Choquet and the Concave Integrals, Mimeo, Tel Aviv University.

[6] C. P. Fishburn, *Utility Theory for Decision Making*, Wiley, New York (1970).

[7] I. Gilboa and D. Schmeidler, Maxmin Expected Utility with a Non-Unique Prior, *Journal of Mathematical Economics*, **18** (1989), 141-153.

[8] E. Kalai and E. Zemel, Totally Balanced Games and Games of Flow, *Mathematics of Operations Research*, **7** (1982), 476-478.

[9] E. Lehrer, A New Integral for Capacities, *Economic Theory*, **39** (2009), 157-176.

[10] E. Lehrer, A comment on an Example by Machina, Mimeo, Tel Aviv University.

[11] E. Lehrer and R. Teper, The Concave Integral Over Large Spaces, *Fuzzy Sets and Systems*, **16** (2008), 2130-2144.

[12] O. L'Haridon and L. Placido, Betting on Machinas Reflection Example : An Experiment on Ambiguity, *Theory and Decision,* **69** (2009), 375-393.

[13] M. J. Machina, Risk, Ambiguity and the Rank-Dependent Axion, *The American Economic Review*, **99** (2009), 385-392.

[14] L.J. Savage, *The Foundations of Statistics*, Wiley, New York (1954).

[15] D. Schmeidler, Subjective Probabilities and Expected Utility Without Additivity, *Econometrica*, **57** (1989), 571-587.


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and Placido [2] rely on the particular example in Lehrer's note and claim that *CavEU* preferences cannot accommodate the common reversal. The example above shows that this is in fact inaccurate.

[16] W. W. Sharkey, Cooperative Games with Large Cores, *International Journal of Game Theory,* **11** (1982), 175-182.

## Appendix: Proofs

*Proof of Theorem 1. preferences* and *continuity* imply that $\succeq$ admits a (continuous) representation $V : \mathbb{R}_+^S \to \mathbb{R}$. That is, for every $f, g \in \mathcal{F}$, $f \succeq g$ if and only if $V(f) \geq V(g)$.

Now, pick a non-bet act $f \in \mathcal{F}$. *Subjective codecomposable independence* implies that there exist an event $E_1 \subseteq S$ and an $f_1$ such that $f = \delta b_{E_1} + (1-\delta)f_1$, $V$ is affine on the interval spanned by $b_{E_1}$ and $f_1$, and in particular $V(f) = \delta V(b_{E_1}) + (1-\delta)V(f_1)$. Let $\delta_1^*$ and $b^{*1}$ such that their product is maximized across all pairs of $\delta$ and $b$ that satisfy the latter equalities. Such a maximum exists due to *continuity*. Let $f_1^*$ be the act in the decomposition of $f$ corresponding to such $\delta_1^*$ and $b^{*1}$. If $f_1^*$ is not a bet, the process above can be repeated and $f_1^*$ can be represented as well by $f_1^* = \delta_2^* b_{E_2}^{*2} + (1-\delta_2^*)f_2^*$, where $V$ is affine on the interval spanned by $b_{E_2}^{*2}$ and $f_2^*$, and in particular $V(f_1^*) = \delta_2^* V(b_{E_2}^{*2}) + (1-\delta_2^*)V(f_2^*)$. This process can be repeated and in the $n$th step, if $f_{n-1}^*$ is not a bet, we get that there exist an $E_n, \delta_n^*, f_n^*$ such that $f_{n-1}^* = \delta_n^* b_{E_n}^{*n} + (1-\delta_n^*)f_n^*$ where $V$ is affine over the interval spanned by $b_{E_n}^{*n}$ and $f_n^*$. Now, due to maximality of the $\delta_j^* b^{*j}$'s and the fact that $V$ is affine along the path of decompositions, it cannot be the case that there are $k \neq j$ such that $E_k = E_j$. Hence, since the state space is finite, the procedure above must be of finite $m$ iterations.

We obtained that $f = \delta_1^* b_{E_1}^{*1} + (1-\delta_1^*)\delta_{E_2}^* b_{E_2}^{*2} + \cdots + (1-\delta_1^*)\cdots(1-\delta_{m-1}^*)\delta_m^* b_{E_m}^{*m} + (1-\delta_1^*)\cdots(1-\delta_m^*)b^*m + 1_{E_{m+1}}$ and $V(f) = \delta_1^* V(b_{E_1}^{*1}) + (1-\delta_1^*)\delta_{E_2}^* V(b_{E_2}^{*2}) + \cdots + (1-\delta_1^*)\cdots(1-\delta_{m-1}^*)\delta_m^* V(b_{E_m}^{*m}) + (1-\delta_1^*)\cdots(1-\delta_m^*)V(b_{E_{m+1}}^{*m+1}) = \delta_1^* b^{*1} V(\mathbb{1}_{E_1}) + (1-\delta_1^*)\delta_{E_2}^* b^{*2} V(\mathbb{1}_{E_2}) + \cdots + (1-\delta_1^*)\cdots(1-\delta_{m-1}^*)\delta_m^* b^{*m} V(\mathbb{1}_{E_m}) + (1-\delta_1^*)\cdots(1-\delta_m^*)b^{*m+1}V(\mathbb{1}_{E_{m+1}})$, where the last equality is due to homogeneity of $V$ which is a result of *worst-outcome independence*. Defining a set function $v : 2^S \to [0,1]$ by $v(E) = V(\mathbb{1}_E)$, we have that $V(f) = \sum a_E v(E)$ for some $\sum a_E \mathbb{1}_E = f$. Now, $v$ is a capacity. Indeed, due to homogeneity of $V$ we have that $v(\emptyset) = 0$, $v(S)$ can be normalized to 1 without loss of generality, and $v$ is monotonic since $V$ is monotone.

*Proof of Proposition 3.* It is clear that the axioms are satisfied by the *CEU* preferences. As for the other implication, all that is needed to show is that given *cumulative codecomposable independence* the decomposition of any act obtained in the proof of Theorem 1 is the Choquet one.

To see that pick an act $f \in \mathcal{F}$ and, let $a_1 = \max\{f(s) : s \in S\}$ and $E_1 = \{s \in S : f(s) = a_1\}$. Also denote $a_2 = \max\{f(s) : s \in E_1^c\}$. Let $f'$ be the act defined by $f'(s) = f(s)$ whenever $s \in E_1^c$ and $a_2$ otherwise (that is, $f'$ coincides with $f$ over the complement of $E_1$, and over $E_1$ it is defined as the second highest value $f$ attains). Now, $f = f' + (a_1 - a_2)1\!\!1_{E_1} = \frac{a_2}{a_1}(\frac{a_1}{a_2}f') + \frac{a_1-a_2}{a_1}(a_{1E_1})$. Note that $E_1$ is cumulative to $f$, hence by *cumulative codecomposable independence* we have that $V(f) = \frac{a_2}{a_1}V\left(\frac{a_1}{a_2}f'\right) + \frac{a_1-a_2}{a_1}V\left(a_{1E_1}\right) = V(f') + (a_1 - a_2)V(1\!\!1_{E_1}) = V(f') + (a_1 - a_2)v(E_1)$. Repeating the same procedure to $f'$ we get that the desired result.

*Proof of Proposition 2.* It is clear that the axioms are satisfied by the EU preferences. As for the other implication, all that is needed to show is that given *codecomposable independence* the capacity obtained in the proof of Theorem 1 is additive, hence a probability.

Pick any event $E \subset S$ and state $s \in S \setminus E$ and consider an act of the form $f = 21\!\!1_{\{s\}} + 1\!\!1_E$. On one hand, from the proof of Proposition 3 we know that $V(f) = v(E \cup \{s\}) + v(\{s\})$. On the other hand, we can write $f = \frac{1}{2}(41\!\!1_{\{s\}}) + \frac{1}{2}(21\!\!1_E)$ and due to *codecomposable independence* we have that $V(f) = \frac{1}{2}(41\!\!1_{\{s\}}) + \frac{1}{2}(21\!\!1_E) = 2v(\{s\}) + v(E)$. Thus, $v(\{s\} + v(E \cup \{s\}) = 2v(\{s\}) + v(E)$, implying that $v(\{s\}) + v(E) = v(E \cup \{s\})$. Since $E$ is an arbitrary event, we get that $v(F) = \sum_{s \in F} v(s)$ for any event $F \subset S$, implying that $v$ is a probability over $S$.

*Proof of Theorem 2.* The concave integral satisfies *subjective codecomposable independence* due to Proposition 5 in Even and Lehrer [5] (and it is immediate that the rest of the axioms are implied by integral).

Following Proposition 1, *worst-outcome independence* and *continuity* we have that $\succeq$ is represented by a homogeneous and continuous $V$ such that $V(1\!\!1_E) \geq v(E)$. *Ambiguity aversion* implies that $V$ is a concave functional. By Lemma 1 in Lehrer [9] we have that $V(\cdot) \geq \int^{Cav}(\cdot)dv$. However, for every $f \in \mathcal{F}$ concavity of $V$ implies that $V(u(f)) \leq \sum \alpha_E V(\mathbb{1}_E)$ for all decompositions of $f$, implying that $V(u(f)) \leq \int^{Cav} u(f)dv$. Therefore $V(\cdot) = \int^{Cav} (\cdot)dv$.

*Proof of Proposition 1.* Due to Lemma 1 in Lehrer and Teper [11], without loss of generality we can assume that $v$ is totally balanced. Now, let $\tilde{v}$ be a different totally balanced capacity and assume that $\tilde{v}$ represents $\succeq$. Since $v$ and $\tilde{v}$ are different there exist $E \subseteq S$ such that without loss of generality $v(E) < \tilde{v}(E)$. Let $b \in \mathbb{R}_+$ such that $v(E) < b < v'(E)$. Then we have that $\int^{Cav} \mathbb{1}_E dv < \int^{Cav} bdv$ and $\int^{Cav} \mathbb{1}_E d\tilde{v} > \int^{Cav} bd\tilde{v}$, which contradicts the assumption that $\tilde{v}$ represents $\succeq$. Thus, 1 is proven. 2 is due to 1 and Theorem 1 in Kalai and Zemel [8].