Back to Fundamentals: Convex Geometry and Economic Equilibrium

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Abstract

We propose a notion of competitive equilibrium in an abstract setting called a Convex Economy using a concept of convexity borrowed from Convex Geometry. The "magic” of linear equilibrium prices is put into perspective in this abstract setting. The abstract notion of competitive equilibrium is applied to a variety of convex economies and versions of the first and second fundamental welfare theorems are proved.

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1. Introduction

In this paper, we return to the fundamental concept of competitive equilibrium. Our goal is to shed new light on the notion of competitive equilibrium by defining it in a setting where each agent has to make a choice from an abstract set.

Convexity plays a central role in the analysis of competitive equilibrium. The abstract formalization pursued here, will require a more abstract notion of convexity than in the standard setting. This notion will be built on the natural primitive phrase "an element b is between a^1 and a^2" (or more generally "b is between the elements $a^1, \ldots, a^L$"). A set of elements is understood to be convex if it contains all the elements that are between elements in the set. In an Euclidean space, the phrase "b is between a^1 and a^2" means that b is an algebraic average of a^1 and a^2. However, this phrase has a natural meaning and is even used in daily conversations in reference to elements in a set that lacks an algebraic structure. For example, it is meaningful to say that Canada is between the USA and the UK, that Game Theory is between Mathematics and Economics and that Switzerland is culturally between Germany, Italy and France.

Thus, our first step will be to borrow a formal concept of convexity from the existing literature of convex geometry. We then prove a new characterization result which identifies, for each convex geometry, a set of primitive orderings that play with respect to the abstract convexity an analogous role to that linear functions play with respect to the standard convexity. Next, we define a convex economy as a specification of: (*i*) a set of agents; (*ii*) a set of elements from which each agent chooses; (*iii*) a profile of preference relations over the set of elements; (*iv*) a feasibility constraint on choice profiles; and (*v*) a notion of convexity on the space of elements. Then, we examine a definition of competitive equilibrium that is analogous to the standard definition. The new definition is tailored to the environment that we consider, where alternatives are endowed with a convexity notion that need not be Euclidean. We then study the first and second fundamental welfare theorems for convex economies. The discussion of those theorems is used to illustrate why the fundamental welfare theorems hold for the standard exchange economy but not necessarily for other convex economies. Finally, we consider examples of convex economies: for each of them, we interpret the equilibrium concept and discuss whether the first and second welfare theorems hold.

2. **Convex Geometry**

We now introduce the reader to the basic concepts of convex geometry as given in Edelman and Jamison (1985). Let X be a set (finite unless otherwise stated) whose members we call *elements*. Convexity is defined through an operator $K : 2^X \to 2^X$ with the interpretation that $K(A)$ is the set of elements that are "between elements in A" (which includes the elements of A themselves). To fix ideas, in the standard analysis, K is the convex hull operator. A set A is *convex* if $K(A) = A$. An element $x \in X$ is said to be *extreme* in A if $K(A - x) \neq K(A)$. In other words, an extreme element of A is an element that is not between other elements of A. We denote the set of all extreme elements in A by $ext(A)$.

Note that this concept of convexity allows us to say that "c is between a and b" (by stating that $c \in K(\{a,b\})$ but does not allow us to state a phrase like "c is a quarter of the way from a to b" (as the algebraic convexity would by stating that $c = 0.75a + 0.25b$).

We follow Edelman and Jamison (1985) and define a *convex geometry* to be an operator K, which satisfies the following assumptions:

(A1) $A \subseteq K(A)$ and $K(\emptyset) = \emptyset$ (“extensitivity”).

(A2) $A \subseteq B$ implies $K(A) \subseteq K(B)$ (“monotonicity”).

(A3) $K(K(A)) = K(A)$ ("idempotence").

(A4) If A is convex, $a,b \notin A$ and $a \in K(A \cup \{b\})$, then $b \notin K(A \cup \{a\})$ (“anti-exchange”).

A1 captures the degenerate sense in which each element in a set is between the set’s elements. A2 means that any element which is between elements of the set A remains so when viewed in a larger set B. One direction of A3, $K(K(A)) \supseteq K(A)$, follows from A1 and A2. The other direction, $K(K(A)) \subseteq K(A)$, means that any element which is between elements which are themselves between elements of A is also between elements in A. A4 states that if (i) A is convex; (ii) a and b are not between elements in A; and (iii) a is between b and elements of A, then it is impossible that b will be between a and elements of A. In other words, the binary relation "a is between b and A", defined on elements not belonging to A, is asymmetric. Of course, all these properties hold for the standard case in which X is an Euclidean space and $K(A)$ is the convex hull of A.

The above definition of K implies some properties familiar from Euclidian spaces (see Koshevoy (1999); for completeness, we supply most of the proofs in footnotes):

F1: If A and B are convex, then so is $A \cap B$.[1]

F2: If $x \in K(A)$, then $K(A \cup \{x\}) = K(A)$.[2]

F3: If $x \notin ext(A)$, then $ext(A) = ext(A - x)$.[3]

F4: $K(ext(A)) = K(A)$ (and thus $ext(A) \neq \emptyset$ for all nonempty A).[4]

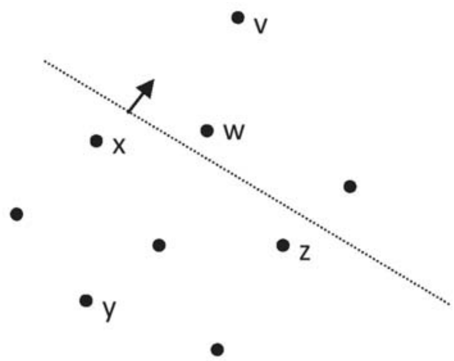
F5: $ext(A \cup B) = ext(ext(A) \cup ext(B))$.[5]

F6: Property H (heritage): If $A' \subseteq A$, then $ext(A') \supseteq ext(A) \cap A'$.[6]

F7: Property O (outcast) If $ext(A) \subseteq A' \subseteq A$, then $ext(A') = ext(A)$.[7]

We take the operator K as primitive and then derive an *ext* operator which satisfies properties H and O. The reverse is possible as well, that is, one can take an operator c which satisfies H and O and derive a convex geometry K operator such that $c(A) = ext(A)$ for all A (see Koshevoy(1999)).

We arrive now to a new representation theorem of convex geometries which will be key to our discussion of competitive equilibrium. The following claim is a generalization of a property which holds regarding the standard convex geometry in Euclidian spaces: *A point is extreme in a finite set if and only if it is the unique maximum element in the set according to at least one linear function.* For any set A, each point which is a maximum by some linear ordering is an extreme point of A. On the other hand, each extreme point in A is the maximum of some linear ordering. To illustrate, in the diagram below, w is an extreme element in $\{w, x, y, z\}$ and is a maximum of the depicted linear ordering. On the other hand, w is not an extreme element of $\{v, w, x, z\}$ since there is no linear ordering according to which w is a maximum in $\{v, w, x, z\}$.



We will show that for any convex geometry there exists a set of orderings that play a role analogous to that of the linear orderings in Euclidian setting. By an *ordering*, we mean a reflexive, complete, asymmetric and transitive binary relation.

We say that the set of orderings $\{\geq_i\}$ *generates* K and its members are *primitive orderings* for K, if for all A:

$ext(A) = \{x \in A \mid x$ is the $\geq_i$ –maximum in A for some primitive ordering $\geq_i\}$.

To illustrate, in the case that X is a finite set in an Euclidian space, a set of primitive orderings is given by the set of linear orderings which do not have any "ties" between elements of X.

Claim 1: For every convex geometry there is a set of orderings that generates it .

Proof: Let K be a convex geometry. Theorem 2.2 of Edelman and Jamison (1985) states that every maximal chain of convex sets $\emptyset \subset C_1 \subset C_2 \subset \ldots \subset X$ has length $|X|+1$. For completeness, here is a proof. As $\emptyset$ and X are convex, it suffices to show that for any two convex sets $A \subset B$ where $|B - A| > 1$ there is a convex set C such that $A \subset C \subset B$. By A1 and A2, for every $x \in B \backslash A$ we have $A = K(A) \subset K(A \cup x) \subseteq K(B) = B$ and by A3, $K(A \cup x)$ is convex. Take two elements $a, b \in B \backslash A$. If neither $K(A \cup a)$ or $K(A \cup b)$ is a proper subset of B, then $K(A \cup a) = K(A \cup b) = B$, violating A4.

For any maximal chain of convex sets $\emptyset = C_0 \subset C_1 \subset C_2 \subset \ldots \subset X$ define $c_k = C_k \backslash C_{k-1}$ and attach an ordering $c_1 < \ldots < c_{|X|-1} < c_{|X|}$. We now show that the set of the orderings attached to all maximal chains of convex sets generates K.

If $a \in ext(A)$, then $\emptyset \subseteq K(A - a) \subset X$ is a chain of convex sets and $a \notin K(A - a)$. Take any maximal chain of convex sets that extends this chain. The attached ordering will rank a above all members of $K(A - a)$ and in particular above all elements of $A - a$. Thus, a is the maximum in A according to such an ordering.

Suppose $a \in A$ is the maximum in A according to one of the attached orderings $>$ and let $J = \{x \mid x < a\}$. By the construction of $>$, the set J is convex and $a \notin J$. As $A - a \subseteq J$, by A2 and A3, $K(A - a) \subseteq K(J) = J$. Thus, $a \notin K(A - a)$ and therefore $a \in ext(A)$. ■

The next claim provides a rationale for our terminology "a set of orderings generates a convex geometry":

Claim 2: Let K be a convex geometry. If $\{\geq_k\}$ generates K then

$K(A) = \{x \mid \forall k, \exists a_k \in A \text{ s.t. } x \leq_k a_k\}$.

Proof: Take $z \in \{x | \forall k, \exists a_k \in A\ s.t.\ x \leq_k a_k\}$. If $z \in A$, then by A1, $z \in K(A)$. If $z \notin A$, then by Claim 1, $z \notin ext(A \cup z)$ which means that $K(A \cup z) = K(A)$ and thus $z \in K(A)$.

Take $z \in K(A)$. If $z \in A$, then $z \in \{x | \forall k, \exists a_k \in A\ s.t.\ x \leq_k a_k\}$. If $z \notin A$, then by F2, $K(A \cup z) = K(A)$ and thus $z \notin ext(A \cup z)$. By Claim 1, $z \in \{x | \forall k, \exists a_k \in A\ s.t.\ x \leq_k a_k\}$.
■

Claim 3: If $\{\geq_i\}$ generates K, then for each ordering $\geq_i$ and each $a \in X$, the sets $\{x|x \leq_i a\}$ and $\{x|x <_i a\}$ are convex.

Proof: Take $y \in K(\{x|x \leq_i a\})$. From Claim 2, there is a $z_i \in \{x|x \leq_i a\}$ such that $y \leq_i z_i$. Since $z_i \leq_i a$, then $y \in \{x|x \leq_i a\}$. Thus, $\{x|x \leq_i a\}$ is convex. The same argument applies to $\{x|x <_i a\}$. ■

Convex geometry and choice theory: Koshevoy (1999) pointed out an interesting connection between the literature on convex geometry and that of choice theory. The operator $ext(A)$ selects a subset of A and thus its properties can be compared to those of choice correspondences. In particular, as noted before, $ext(A)$ satisfies two familiar properties from the choice theory literature: Property H (Heritage), which is actually the α property of Sen (1970) and Property O (Outcast), which is a weaker version of the β property of Sen (1970). The connection allows us to take a familiar proposition from one literature and apply it to the other. In particular, Claim 1 can also be proved by using Aizerman and Malishevski (1981)'s result which states that a choice correspondence C satisfies H and O if and only if there is a finite number of orderings, such that an element is in $C(A)$ if and only if it is the unique maximum of at least one of the orderings over A.

Another use of a non-standard notion of gemoetry: Baldwin and Klemperer (2013) use *tropical geometry*, a new field related to algebraic geometry, to study substitutability, complementarity, and the existence of standard competitive equilibria in a Euclidean setting where the available alternatives are bundles of indivisible goods. The mathematical concept they use and their economic target are quite far from ours. Additionally, Danilov, Koshevoy, and Murota (2001), in the first of a series of three papers, show competitive equilibria existence results using the standard Euclidean convexity restricted to bundles with integer and real coordintaes as opposed to the tropical approach.

We now present some examples of convex geometries that have an economic interpretation:

Example 1: Box Convexity

The box convexity is defined by a set of orderings $\{\geq_k\}$ which can be interpreted as a set of criteria over X. Define $K(A) = \{x \mid \forall k, \exists a_k, b_k \in A \text{ s.t. } b_k \leq_k x \leq_k a_k\}$.

Thus, an element belongs to $K(A)$ if according to each criterion it is sandwiched between some pair of elements in A (which pair it is might depend upon the criterion). This geometry is generated by the set of all relations $\geq_k$ and their reversals.

Example 2: The Degenerate Convexity

The convex geometry $K(A) \equiv A$ captures the degenerate case in which no element is between any combination of other elements.

The set of all orderings on *X* generates this geometry. As is typically the case, there are many different sets of primitive orderings that generate *K*. Indeed, as every element is extreme in every set that contains it, a set of orderings generates *K* if and only if each element in *X* is top-ranked in at least one of the orderings.

Example 3: The Middle Ranked Convexity

Let *R* be a partial ordering of *X* and define the convexity as $K(A) = \{x|\exists a, b \in A \ s.t. \ aRxRb\}$. Note, it is important that the partial ordering *R* has no indifferences, otherwise *K* would violate the anti-exchange property.

For example, *X* can be the set of members of an hierarchical organization and aRb means that *a* is a superior of *b*. The expression $x \in K(A)\backslash A$ means that *x* is subordinate to one of the members of *A* and superior to another. The set of all the complete orderings that extends either *R* or the negation of *R* generates this geometry.

Example 4: **Lower Contour Set Convexity**

Let *R* be a partial ordering of a set, *X*, and define $K(A) = \{x \mid \exists a \in A$ such that $aRx\}$.

If *X* is a set of products and *aRb* means that *a* is sufficient to produce *b,* then $K(A)$ contains all the products that can be produced from single elements of *A*. By Theorem 3.2 in Edelman and Jamison (1985), every convex geometry *K* that satisfies the additional property $K(A \cup B) = K(A) \cup K(B)$ can be represented as a lower contour set convex geometry. A set of primitive orderings for this geometry is the set of all completions of the partial ordering *R*.

Example 5: **Set Union Convexity**

Let *Z* be a set of elements and *X* be the set of all menus (non-empty subsets of *Z*). Define $K(A)$ as the set of all unions of menus in *A*.

For example, if *Z* is the set of all dishes, *G* is the set of Greek dishes and *I* is the set of Italian dishes, then the menu of a restaurant that serves both Greek and Italian dishes, $G \cup I$, is between the menus of the Greek and Italian restaurants. A menu is extreme in a set of menus if it is not a union of other menus in the set.

A set of orderings that generates this geometry consists of all extensions of the partial orderings $\{\succsim_z\}_{z \in Z}$ defined by $a \succsim_z b$ if either of the following holds:

(i) $z \in a \cap b$ and $a \subset b$, or

(ii) $z \notin a \cup b$ and $a \subset b$.

To see that this set of orderings generates K, consider a menu $a \in ext(A)$. In this case, a is not a union of its strict subsets in A and thus there exists $z \in a$ such that for no $c \in A$ is it the case that $z \in c \subset a$. Therefore, a is $\succ_z$ -maximum.

On the other hand, b which is the maximum in A of one of orderings $\succsim_z$. If $b \notin ext(A)$ then b is a non-degenerate union of elements in A and includes at least two elements. We get a contradiction since:

(1) If $z \in b$, take a menu $a \in A$ such that $z \in a \subset b$. Then $b \prec_z a$ (by case (i)).

(2) If $z \notin b$, take a menu $a \in A$ such that $a \subset b$. Then $b \prec_z a$ (by case (ii)).

3. Convex economy and abstract competitive equilibrium

We turn now to define the environment in which we will define the notion of competitive equilibrium. The alternatives open to agents are taken to be the members of an abstract space X endowed with a convex geometry K. No further structure is imposed on X. We refer to members of X as *elements*. Let $N = \{1,..,n\}$ be a set of agents. Each agent has to choose an element in X. A profile assigns one element to each agent. Not all profiles are feasible. The feasibility constraint is given by a set $F \subset X^N$ and we assume that F is closed under all permutations. We have in mind environments like the following:

The Exchange Economy: $X = R_+^L$ and an element of X is interpreted as a bundle in a world with L commodities. A bundle ω is the total endowment and F is the set of all allocations of ω among the agents.

The Housing Economy: The set X contains n houses. A feasible allocation assigns to each of the n agents a distinct house.

The Set Allocation Economy: The set X is the set of all subsets of some set Z. The set F is the set of all profiles of subsets of Z such that each member of Z is allocated at most once (some members may be unallocated).

Each agent is interested only in the element he chooses and (unlike in a typical game setting) this interest is independent of the choices other agents make. Each agent i is equipped with a preference relation $\succsim^i$ on X (an upper index always stands for an index of an agent). We assume that the preference relations of all agents are *convex* (with respect to K) in the following standard sense: for every $x^* \in X$ and every agent i the upper set $\{x \mid x \succ^i x^*\}$ is convex.

We will refer to the tuple $< N, X, \{\succsim^i\}_{i \in N}, F, K >$ as a *convex economy*. Note that this notion of an economy lacks "initial endowments".

The constraints expressed by the set F will typically introduce conflicts between the agents. An equilibrium concept should provide a method to resolve the conflict in a way that expresses some form of stability. We now arrive at our central definition:

Definition: A *competitive equilibrium* is a pair $< (x^i)_{i \in N}, P >$ where $(x^i)_{i \in N}$ is a profile and P is an ordering on X, such that: (i) the profile is in F and (ii) for each i, the element x^i is $\succsim^i$ -optimal in the *opportunity set* $B(P, x^i) = \{z | x^i P z \text{ or } z = x^i\}$.

The main interpretation of this competitive equilibrium notion is that P is a social ordering of the elements which reflects worth or prestige. The term aPb means that *a is more expensive than b* or *is more prestigious than b*. For a profile to be a competitive equilibrium there must exist an ordering P such that each agent is satisfied with his assigned element given his ability to replace it only with an element which is considered less expensive or prestigious according to P. An equilibrium ordering stabilizes the equilibrium profile in the sense that each agent is happy with his assignment *given* the "worth" of his assigned element.

An alternative interpretation is that P is a socially agreed-upon or imposed motive that systematically affects the agents' preference relations. The ordering P could be thought of as an "*anti-prestige*" ordering, that is, aPb means that a is considered to be less prestigious than b. An agent can choose any alternative in X, but can't bear to suffer a loss of prestige. Thus, the ordering P systematically affects the agents' preferences: given P and an assigned element x^i, the agent's lexicographical first priority is to "not decrease the prestige of his choice" while his second priority is to maximize his original preference. An equilibrium is a profile $\{x^i\}_{i \in N}$ and a social ordering P such that no agent wishes to deviate from his assigned element given his modified preferences even if he can choose from the entire set X.

The two interpretations are dual: In the main interpretation, an agent considers only the elements further down the P ordering to be feasible, while in the alternative interpretation, all elements are feasible, but an agent finds only elements further down the P ordering to be acceptable.

Note that we require that the equilibrium price ordering to be strict. This is without loss of generality since any equilibrium profile supported by an ordering that contains indifferences will also be an equilibrium profile with arbitrary breaking of the indifferences.

To this point, any ordering can serve as an equilibrium ordering. The existence of (unrestricted) competitive equilibrium is straightforward (even without assuming that preferences are convex) as the next claim shows.

Claim 4 (Second Fundamental Welfare Theorem I): Any Pareto-efficient profile $(a^1,..,a^n)$ is a competitive equilibrium profile (supported by an unrestricted price ordering).

Proof: Define a relation R by xRy if (i) there are i and j such that $x = a^i \succ^j a^j = y$ or (ii) $y \in \{a^1,..,a^n\}$ and $x \notin \{a^1,..,a^n\}$. The first condition guarantees that if j envies the element assigned to i he is not able to replace his element with i's element. The second condition guarantees that agents cannot "afford" any unassigned element.

Since the profile is Pareto-efficient and F is closed to permutations, R does not have cycles and thus can be extended to a complete ordering P. According to this ordering, for each agent j, a^j is optimal in the $B(P,a^j)$. ■

Notice that the proof of Claim 4 applies for any profile that does not have a cycle in the envy relation. Such profiles may not necessarily be Pareto-efficient. In particular, any profile in which all agents are assigned the same element (which may not be feasible) is supported by some *unrestricted* price ordering.

4. Convex competitive equilibrium

Claim 4 relates to an equilibrium concept with an unrestricted price ordering. However, in standard economic models the price ordering is not *arbitrary*, but rather is required to be linear. Are there analogous assumptions that we can impose on the equilibrium price ordering?

A consequence of the existence of a linear price ordering P in the standard setting is that both of the relations "more expensive" and "less expensive" are convex. In other words, for any x^* *the unaffordable set* $UA(P,x^*) = \{x \mid xPx^*\}$ and its complement, the opportunity set, denoted by $B(P,x^*)$ are convex. We use the letter B to draw attention to the analogy with the familiar term "budget set".

An unaffordable set can be thought of as the set of all elements that the market maker is unwilling to exchange for x^* because he prefers them to x^*. Thus, if there is a market maker and his preferences are convex, then in turn each agent's unaffordable set is also convex.

We are not aware of a persuasive interpretation of the requirement of convex opportunity sets. Nevertheless, it is imposed in the economic literature. Aside from one example, we will focus on what we consider to be the more plausible requirement, that price orderings are convex.

We are led to the following definitions: A price ordering P is *convex* if for each element x^* the unaffordable set $UA(P,x^*)$ is convex and P is *concave* if for each x^* the opportunity set $B(P,x^*)$ is convex.

The following two examples demonstrate the non-existence of equilibrium in the case of concave prices and convex prices, respectively:

A market with no equilibrium with concave price ordering: Consider the housing economy with three agents and three houses arranged on a line $a - b - c$, with the natural convexity definition. (In other words, $K(\{a,c\}) = \{a,b,c\}$ and for all other A, $K(A) = A$. Under this geometry all sets are convex except $\{a,c\}$. The two primitive orderings $a >_1 b >_1 c$ and $c >_2 b >_2 a$ generate K.)

Assume that each of the three agents holds the convex preferences $b \succ^i c \succ^i a$. In equilibrium, one of the agents is assigned a. Since a is the worst house for all agents it must be that bPa and cPa. Since there is an agent who is assigned c and prefers b it must be that bPc. Thus, the only equilibrium price ordering is $bPcPa$, which is not concave since $B(P,c) = \{a,c\}$ is not convex.

A market with no equilibrium with convex price ordering: Consider the housing economy with four houses arranged on a line $a - b - c - d$, with the natural convexity definition. Two of the agents, 1 and 2, are "leftish" and hold the convex preferences $a \succ^i b \succ^i c \succ^i d$, while the other two, 3 and 4, are "rightish" and hold the convex preferences $a \prec^i b \prec^i c \prec^i d$. It is impossible that in equilibrium a leftish agent will be assigned a house $x \in \{c,d\}$ since then one of the rightish agents will be assigned $y \in \{a,b\}$ and it must be that both xPy and yPx. Thus, without loss of generality (a,b,c,d) is an equilibrium profile. For the ordering P to support this profile it must be that aPb and dPc. Thus, b (or c) is the P-minimal element and $UA(P,b) = \{a,c,d\}$ (or $UA(P,c) = \{a,b,d\}$) is not convex. Thus, there is no equilibrium with convex prices. Note that as proved generally in Claim 4, there exists an arbitrary price ordering ($aPdPbPc$) which supports the Pareto-efficient profile (a,b,c,d).

We now arrive at a definition of competitive equilibrium which we find most analogous to the standard definition. Recall the following two facts:

I) Linear orderings play an important role in the definition of convexity in Euclidean spaces. The set of linear orderings generates the convex geometry. Furthermore, the set has the special coupling property: for each primitive ordering the inverse ordering is primitive as well.
II) Linear orderings play an important role in the standard definition of competitive equilibrium. The price ordering is not only convex, but also linear.

Thus, the standard definition requires that the price ordering is either one of the primitive orderings or equivalently the inverse of one of the primitive orderings. Recall that primitive orderings have convex lower contour sets while convex price orderings have convex upper contour sets. Thus, our desire for the price ordering to be convex leads to the requirement that the equilibrium price ordering P should be the inverse of one of the primitive orderings. This analogy leads us to the following definition of competitive equilibrium for abstract convex economies:

Definition: Let $< N, X, \{\succsim^i\}_{i \in N}, F, K >$ be a convex economy and let $\{\geq_k\}$ be a set of primitive orderings that generates K. A *competitive equilibrium with an **inverse** primitive price ordering* is a competitive equilibrium $< (x^i)_{i=1,..,n}, P >$ where $P = -\geq_k$ for some primitive ordering.

The previous example showed that, in general, we cannot support each Pareto-efficient profile with an inverse primitive price ordering (since we could not even support it with a convex price ordering). However, the following claim shows that for a one-agent convex economy every feasible element can be supported with an inverse primitive price ordering .

Claim 5: Consider a one-agent convex economy with some set of primitive orderings. For every feasible element x^*, there exists a primitive ordering $\geq_k$ such that $(x^*, -\geq_k)$ is a competitive equilibrium.

Proof: Let x^* be a feasible element in X. By convexity of the preferences, the set $\{x \mid x \succ^1 x^*\}$ is convex. By Claim 2, since x^* is not a member of this set, there is a primitive ordering $\geq$ such that $x^* \geq x$ for every $x \succ^1 x^*$. Therefore, x^* is $\succsim^1$ -optimal in $\{x \mid x \geq x^*\}$. Let P be the inverse of $\geq$. Then, x^* is $\succsim^1$ -optimal in $\{x \mid x^* P x\} = \{x \mid x \geq x^*\}$. ■

What is special about the standard exchange economy which allows every Pareto-efficient allocation to be supported by a competitive equilibrium with a primitive price ordering? It can be attributed to the following Richness property, which is valid there:

Let $< N, X, \{\succsim^i\}_{i \in N}, F, K >$ be a convex economy with the set of primitive orderings $\{\geq_k\}$. We say that the economy satisfies the *Richness property* if the following holds: Let $\succ$ and $\succ'$ be two convex preferences over X and let (a, a') be a pair of elements in X. Let $\geq$ and $\geq'$ be two different primitive orderings such that a is $\succ$ –maximal in the set $\{x : a \geq x\}$ but not in $\{x : a \geq' x\}$ and a' is $\succ'$ –maximal in the set $\{x : a' \geq' x\}$ but not in $\{x : a' \geq x\}$. Then, there is a pair (b, b') such that (i) whenever $(a, a', a_3, .., a_n) \in F$ it is also the case that $(b, b', a_3, .., a_n) \in F$ and (ii) $b \succ a$ and $b' \succ' a'$.

Claim 6 (Second Fundamental Welfare Theorem II): Consider a convex economy with a set of primitive orderings which satisfies the Richness property. Then, any Pareto-efficient profile is supported by the inverse of a primitive ordering.

Proof: Let $\{x^i\}$ be a Pareto-efficient profile. By the convexity of preferences, for each agent i the set $U^i = \{z : z \succ^i x^i\}$ is convex and x^i is an extreme point of $V^i = U^i \cup x^i$. Therefore, x^i is maximal in V^i according to at least one primitive order. Let O^i be the (non-empty) set of primitive orderings O^i for which x^i is maximal in V^i.

The intersection $\cap_i O^i$ is not empty since otherwise there would be two agents i and j such that O^i and O^j are non-nested sets. Take $\geq_i \in O^i \backslash O^j$ and $\geq_j \in O^j \backslash O^i$. The element x^i is $\succ^i$ –maximal in the set $\{x : x^i \geq_i x\}$ but not in $\{x : x^i \geq_j x\}$, and likewise for agent j. Richness then implies that there is a pair of elements (y^i, y^j) such that the profile obtained by replacing the pair (x^i, x^j) with (y^i, y^j) in $\{x^i\}$ is feasible and Pareto-dominating.

Thus, there exists $\geq_k \in \cap_i O^i$ and the inverse of $\geq_k$ supports the profile. ■

With regard to the First Fundamental Welfare Theorem, in Example C below we construct an equilibrium with a convex price ordering that is not Pareto-efficient. The following trivial claim provides a condition under which the theorem holds. This condition is satisfied by two prominent convex economies: i) the standard exchange economy with the standard convexity concept and ii) the housing economy with any convexity concept.

Claim 7 (First Fundamental Welfare Theorem): Consider a convex economy such that for any $(a^1, \ldots, a^n) \in F$ there is no $(b^1, \ldots, b^n) \in F$ and a primitive ordering $\geq_k$ on X such that for all i either $b^i = a^i$ or $b^i >_k a^i$. Then, if $(a^1, .., a^n)$ is a competitive equilibrium profile with primitive prices, then the allocation is weak Pareto-efficient (in the sense that there is no other $(b^1, \ldots, b^n) \in F$ such that for all i either $b^i = a^i$ or $b^i \succ^i a^i$).

Proof: Assume that $((a^1, \ldots, a^n), -\geq_k)$ is a a competitive equilibrium. If $(a^1, \ldots, a^n)$ is not weak Pareto-efficient, then there is a feasible profile $(b^1, \ldots, b^n)$ such that for all i

either $b^i = a^i$ or $b^i \succ^i a^i$. Then, for all i either $a^i = b^i$ or $b^i >_k a^i$, a contradiction. ■

5. Examples

In this section we analyze the concept of competitive equilibrium in the context of several simple convex economies. In particular we examine whether the First Fundamental Welfare Theorem (FWT) and the Second (SWT) hold for each of the economies. In the first four examples, the convexity involves a nondegenerate notion of betweenness of the form "a is between $a^1, a^2 \ldots a^L$" where $L > 1$. In other words, for each of these convexities, a being a member of $K(A)$ depends on the existence of certain combinations of at least two elements in A. The convexity notion in the last two economies is less complicated as a being a member of $K(A)$ depends only on the existence of certain other single elements in A.

A. The Balanced Economy

Economy: Let $X = [-1, 1]$ with the standard convexity generated by the increasing and decreasing orderings. Assume that each agent i has convex preferences with a single peak denoted by $peak^i$. Let F be the set of all profiles of elements in X that sum up to 0. One interpretation of this economy is that an element is either a contribution to a social fund or a withdrawal from it. The feasibility constraint expresses the requirement that the social fund be balanced. Agents have different views about the tradeoff between being egalitarian and selfish. The interesting case is when $\sum peak^i \neq 0$.

One and only one of the primitive orderings is an equilibrium price ordering: Assume that $\sum peak^i < 0$. Then, there is no equilibrium in which the price ordering is the increasing ordering. If there were such an equilibrium, then all agents would be at or to the left of their peak, violating the feasibility constraint. On the other hand, any feasible profile such that all agents are assigned elements at or to the right of their peak together with the decreasing price ordering is an equilibrium. An interpretation of the decreasing price ordering is that of a social norm which allows agents only to consider giving more (or taking less) than his assigned element suggests.

FWT: By Claim 7, any equilibrium with a primitive price ordering is weak Pareto-efficient. However, there can be an equilibrium profile supported by convex price orderings that is not weakly Pareto-efficient. Consider a case with an even number of agents, half of them with $-1 < peak^i < 0$ and half of them with $0 < peak^i < 1$. Take the Pareto-inefficient profile where the "leftish" agents are assigned -1 and the "rightish" agents are assigned 1. This feasible profile is supported by the convex price ordering P, where xPy if $|x| \leq |y|$. Recall that xPy means that x is "more expensive" than y and thus this price ordering expresses a social norm according to which agents are only allowed to move from their assigned element to a "more extreme" one.

SWT: The Richness condition in Claim 5 holds for this economy and thus, every Pareto-efficient allocation is supported by either the increasing or decreasing price ordering. More explicitly, in any Pareto-efficient allocation, all agents are at or to the right of their peak (which is supported by the decreasing price ordering) or all agents are at or to the left of their peak (which is supported by the increasing price ordering).

B. An Exchange Economy with Finite Direction Convexity

Economy: Let $X = \Pi_{l=1}^{L}[0, z_l]$ be a set of bundles in an L-commodity world. Consider the convex geometry generated by M linear orderings, each represented by the function $v_l \bullet x$ where $v_1, \ldots, v_M$ are non-zero vectors in R^L. Feasibility is given by $\sum_{i=1}^{n} x^i = z$. All agents hold convex and monotonic preference relations.

FWT: The conditions of Claim 7 are satisfied and therefore any equilibrium with primitive prices is weak Pareto-efficient. It is easy to see that an equilibrium profile supported by a convex price ordering need not be weak Pareto-efficient.

SWT: The Richness property used in Claim 6 holds in this economy and thus any Pareto-efficient allocation is supported by a primitive price ordering. For example, if $M = 1$, then all agents hold the unique convex preference relation given by $-v_1x$ and the equilibrium price ordering is also $-v_1x$. As another illustration: if $v_1 = (-1, 0)$ and $v_2 = (0, -1)$ then each indifference curve is "right-angled" and each efficient allocation is such that either no agents have a surplus of good 1 or no agents have a surplus of good 2. Allocation profiles of the former type are supported by $(1, 0)$ and those of the latter type by $(0, 1)$.

C. The Set Allocation Economy

Economy: Let Z be a collection of items and X be the set of all of its subsets. The set F contains all profiles that allocate each item to at most one agent (items can be abandoned). Let K be the convexity from Example 5, namely $K(A)$ contains all unions of any collection of members of A. Agents have strict and convex preferences. Convexity of preferences is equivalent to the property that if $a, b \succ^i x$ then $a \cup b \succ^i x$, which is implied by either monotonicity ($a \succ^i x$ and $b \supset a$ implies $b \succ^i x$) or betweenness ($a \succ^i b$ implies $a \succ^i a \cup b \succ^i b$).

Counter-example to FWT: Let $Z = \{a,b,c,d\}$. There are two agents, both of whom prefer any cardinally larger set. Among the two-element sets, $ac \succ^1 ab$ and $bd \succ^2 cd$. Moreover, ab is preferred by agent 1 over any other two-element set and likewise cd for agent 2. These preference relations are convex. Now consider the equilibrium that consists of the profile (ab, cd) and a price ordering which ranks cardinally larger sets higher and among the two-element sets ab and cd are the lowest. This equilibrium profile is Pareto-dominated by (ac, bd).

SWT: Let $(x^1, \ldots, x^n)$ be a Pareto-efficient profile. Define the auxiliary binary relation D (which stands for "is desired over") by zDx^i if there exists $y \subseteq z$ such that $y \succsim^i x^i$ (notice that this relation is stronger than the envy relation). By Pareto-efficiency of the profile, D does not have a cycle and without loss of generality we can assume that if $i < j$ then i does not desire j's allocation, that is, it is not the case that x^jDx^i.

For each i, the set $U^i = \{z \mid zDx^i\}$ is *upwards-closed* (a set which includes all supersets of its members). All upwards-closed sets are convex. Additionally, the union of the upwards-closed sets $V^i = \cup_{j=i}^{n} U^j$ is upwards-closed and thus convex. Therefore, $\emptyset \subset V^n \subset \ldots \subset V^1 \subset X$ is a chain of convex sets. Each inclusion is strict because $x^j \in V^j \backslash V^{j-1}$; otherwise $x^j \in U^i$ for some $i < j$, that is, x^jDx^i, but the agents are ordered so that this is not the case.

As in the proof of Claim 1, there is a maximal chain of convex sets $\emptyset = C_0 \subset C_1 \subset \ldots \subset X$ that extends the above chain, which yields the convex ordering $y_1 > \ldots > y_{|X|}$ where $y_l = C_l \backslash C_{l-1}$. This ordering supports the profile since if $x^i > z$, then $z \notin V^i$ and in particular, $z \notin U^i$. Therefore, $x^i \succ^i z$.

D. An Economy with "Production on a Line"

This example demonstrates a possible expansion of the convex economy model to a world with production.

Economy: Let X consist of the four elements a, b, c, d arranged on a line with the natural convexity. An element in X is interpreted as an indivisible product.

There are six producers, each of whom produces one unit of a single product. The producers are divided equally across three types. Each producer in the pair $\{1,2\}$, $\{3,4\}$ or $\{5,6\}$ holds the convex production possibility set $\{a,b\}$, $\{b,c\}$ or $\{c,d\}$, respectively.

There are six consumers, each of whom consumes one unit of a single product. Consumers 1, 2 and 3 hold the preferences $a \succ b \succ c \succ d$ and consumers 4, 5 and 6 hold the preferences $a \prec b \prec c \prec d$. The feasibility constraint requires that aggregate production equals aggregate consumption for each product. A competitive equilibrium is given by a price ordering on X (possibly with indifferences) and a feasible production and consumption profile, such that all firms "maximize profits" and no agent wishes to switch from his assigned product to a (weakly) cheaper one.

This economy has a competitive equilibrium with a non-convex price ordering: Consider the feasible profile where the production vector is (a,a,b,c,d,d) and the consumption vector is (a,a,b,c,d,d). This profile is supported by the price ordering $aPdPbIc$ (bIc means that b and c are "equally priced"). Note that here we cannot apply the argument made in Section 3, according to which we can arbitrarily break ties in an equilibrium price ordering. This is because the preferences of profit-maximizing-producers depend on the price ordering. Breaking the tie between b and c in the above equilibrium will cause one of the producers, 3 or 4, to no longer be maximizing profits.

An equilibrium with a convex price ordering does not exist. If there were such an equilibrium, then either a or d would be minimal with respect to the price ordering. In such an equilibrium, the three consumers who rank this element at the top must be assigned it, while only two firms can produce it.

E. The Hierarchical Housing Economy

Economy: Consider the housing economy with the convex geometry of Example 4, that is, $K(A) = \{x : \exists a \in A \text{ s.t. } aRx\}$ where R is a partial ordering on X. The set of all extensions of R into orderings generates K. Assume that all agents have strict preferences.

Any equilibrium price ordering must be primitive: If bRa, then $a \in K(\{b\})$. By the convexity of preferences, for any i the set $\{x \mid x \succsim^i b\}$ is convex and by A2, includes $K(\{b\})$. Thus, $a \succ^i b$ for all agents, and in particular for the agent i who is assigned b. It follows that aPb. Thus, P is the inverse of a primitive order.

FWT: As commented before, the condition in Claim 7 holds. We have just seen that any equilibrium price ordering is primitive. Thus, any equilibrium profile is Pareto-efficient.

SWT: Any Pareto-efficient profile $(x^1,\ldots,x^n)$ is supported by some price ordering (see Piccione and Rubinstein (2007)) and as shown above this ordering is the inverse of a primitive ordering and therefore convex.

F. The "*Corner*" Economy

Economy: Let $X = L \cup R$ where $L = \{(a,0) : 0 < a \leq 3\}$ and $R = \{(0,b) : 0 < b \leq 3\}$. An element in X is interpreted as a consumption bundle in a world with two divisible goods where an agent can consume only one good. The feasibility constraint is $\sum_i x^i = (3,3)$ and convexity is given by $K(A) = \{x : \exists a \in A$ and $k \in \{1,2\},$ s.t. $0 < a_k \leq x_k\}$ for every A. This convexity is defined by two primitive orderings, which are decreasing on both segments L and R. One of the primitive orderings, $\geq_L$, places all the elements of L above R, and the other, $\geq_R$, places all elements of R above L. Convexity of agents' preferences is equivalent to monotonicity in both goods.

FWT: This economy satisfies the condition in Claim 7 which can be seen as follows: Let $(x^1, \ldots, x^n)$ and $(y^1, \ldots y^n)$ be two feasible profiles such that $\forall i,\ y^i \geq_L x^i$. For each agent i, $x_2^i \geq y_2^i$. Feasibility requires that $\sum x_2^i = \sum y_2^i = 1$ and therefore, for each agent, $x_2^i = y_2^i$. For any agent i who is assigned $x_2^i = y_2^i > 0$, it must be that $x^i = y^i$. For any other agent i, it must be that $x_1^i \geq y_1^i$. Feasibility again implies that for all i, $x_1^i = y_1^i$ and $x^i = y^i$. Thus, every equilibrium profile supported by an inverse primitive ordering is weakly Pareto-efficient.

SWT: Let $(x^1, \ldots, x^n)$ be a Pareto-efficient profile and thus the envy relation does not have a cycle. Without loss of generality we can assume that if $i > j$, then agent i does not envy j's assigned element. The equilibrium price ordering is defined iteratively. The bottom of the price ordering is given by the elements between 0 and x^1 in increasing order. In the i^{th} stage, the unplaced elements between 0 and x^i are added in increasing order. After the n^{th} stage, all remaining elements are added in an increasing order. This price ordering has the feature that it is increasing on L and R since the elements in each of those sets are added in an increasing order. Each agent can only afford his own bundle or bundles that are (weakly) inferior to unenvied bundles.

Comment: Note that there are Pareto-efficient allocations that can be rationalized by convex prices, but not with an ordering represented by a linear expense function. For example, suppose that there are four agents with convex preferences such that $(2,0) \succ^i (0,2) \succ^i (0,1) \succ^i (1,0)$. The profile $((2,0),(0,2),(0,1),(1,0))$ is Pareto-efficient. Suppose the equilibrium price ordering can be represented by $p_1x_1 + p_2x_2$. If $p_2 \leq p_1$, then agent 2 would deviate from $(0,2)$ to $(2,0)$. Similarly, if $p_2 > p_1$, then agent 4 would deviate from $(1,0)$ to $(0,1)$.

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Footnotes

1. By A2, $K(A \cap B) \subseteq K(A) \cap K(B)$ and thus $K(A \cap B) \subseteq A \cap B$. By A1, $A \cap B \subseteq K(A \cap B)$.

2. By A3, A1+A2, A2 accordingly $K(A) = K(K(A)) \supseteq K(A \cup \{x\})) \supseteq K(A)$.

3. If $z \in ext(A) - ext(A - \{x))$, then $K(A - z) \subset K(A)$ and $K(A - x) = K(A - x - z)$. By A2, $K(A - x - z) \subseteq K(A - z)$. Thus, $K(A - x) \subset K(A)$, contradicting $x \notin ext(A)$.
If $z \in ext(A - \{x)) - ext(A)$, then $K(A - x - z) \subset K(A - x)$ and $K(A - z) = K(A)$. By $x \notin ext(A)$, we have $K(A - x) = K(A)$ and thus, $K(A - x - z) \subset K(A)$.
To obtain a contradiction to A4, note that:
(i) By A3, $K(A - x - z)$ is convex.
(ii) $x \notin K(A - x - z)$ because otherwise by F2, $K(A - x - z) = K(A - z) = K(A)$. Similarly, $z \notin K(A - x - z)$.
(iii) $K(K(A - x - z) \cup x) \supseteq K((A - x - z) \cup x) = K(A - z) = K(A)$ (the inclusion is by A1 and A2) and thus $z \in K(K(A - x - z) \cup x)$. Similarly, $x \in K(K(A - x - z) \cup z)$.

4. Let $x \in A - ext(A)$. Then, $K(A - \{x\}) = K(A)$. It is then sufficient to notice that by F3 $ext(A - \{x)) = ext(A)$ and to iteratively remove elements of $A - ext(A)$ from A.

5. See Theorem 4 in Koshevoy (1999).

6. Let $a \in ext(A) \cap A'$. If $a \notin ext(A')$, then $a \in K(A' - a)$ and by A2 $a \in K(A - a)$ and by F2 $K(A - a) = K(A)$, contradicting $a \in ext(A)$.

7. Take $x \in A - ext(A)$. By F4, $ext(A - x) = ext(A)$. Iteratively apply to all members of $A - A'$.