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INFORMATION, INTERDEPENDENCE, AND INTERACTION: WHERE DOES THE VOLATILITY COME FROM?

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Information, Interdependence, and Interaction: Where Does the Volatility Come From?*

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Abstract

We analyze a class of games with interdependent values and linear best responses. The payoff uncertainty is described by a multivariate normal distribution that includes the pure common and pure private value environment as special cases. We characterize the set of joint distributions over actions and states that can arise as Bayes Nash equilibrium distributions under any multivariate normally distributed signals about the payoff states. We characterize maximum aggregate volatility for a given distribution of the payoff states. We show that the maximal aggregate volatility is attained in a noise-free equilibrium in which the agents confound idiosyncratic and common components of the payoff state, and display excess response to the common component. We use a general approach to identify the critical information structures for the Bayes Nash equilibrium via the notion of Bayes correlated equilibrium, as introduced by Bergemann and Morris (2013b).
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1 Introduction

In an economy with widely dispersed private information about the fundamentals, actions taken by agents in response to the private information generate individual as well as aggregate volatility in the economy. How does the nature of the volatility in the fundamentals translates into volatility of the outcomes, the actions take by the agents. To what extent does heterogeneity in the fundamentals dampen or accentuates volatility in the individual or aggregate actions? At one extreme, the fundamental state, the payoff relevant state of the economy could consist of a common value that affects the utility or the productivity of all the agents in the same way. At the other extreme, the payoff relevant states could consist of purely idiosyncratic, distinct and independent, states across the agents.

We analyze these questions in a general class of linear quadratic economies with a continuum of agents while allowing for heterogeneity in the fundamentals, ranging from pure common to interdependent to pure private values. We restrict our attention to environments where the fundamentals are distributed according to a multivariate normal distribution. We describe the payoff relevant state of each agent by the sum of a common component and an idiosyncratic component. As we vary the variance of the common and the idiosyncratic component, we change the nature of the heterogeneity in the fundamentals and can go from pure common to interdependent to pure private value environments.

The present objective is to analyze how the volatility in the fundamentals translates into volatility of the outcomes. The nature of the private information, the information structure of the agents, matters a great deal for the transmission of the fundamental volatility. We therefore want to know which information structure leads to the largest (or the smallest) volatility of the outcomes for a given structure in the fundamental volatility. Thus, we are lead to investigate the behavior of the economy for a given distribution of the fundamentals across all possible information or signal structures that the agents might have. By contrast, the established literature is mostly concerned with the characterization of the behavior, the Bayes Nash equilibrium for a *given* information structure.

In earlier work, two of us suggested an approach to analyze the equilibrium behavior of the agents for a given description of the fundamentals for *all* possible information structures. In Bergemann and Morris (2013a) we define a notion of correlated equilibrium for games with incomplete information, which we call *Bayes correlated equilibrium*. We establish that this solution concept characterizes the set of outcomes that could arise in any Bayes Nash equilibria of an incomplete information game where agents may or may not have access to more information beyond the given common prior over the fundamentals. In Bergemann and Morris (2013b) we pursue this argument in detail and characterize the set of Bayes correlated equilibria in the class of games with quadratic payoffs and normally distributed uncertainty, but there we restricted our attention to the special environment with *pure common values* and *aggregate interaction*. In the present contribution, we substantially generalize the environment to interdependent values (and more general interaction structures) and analyze the interaction between the heterogeneity in the fundamentals and the information structure of the agents. We continue to restrict attention to multivariate normal distribution for the fundamentals and for the signals, but allowing for arbitrarily high-dimensional signals. With the resulting restriction to multivariate normal distributions of the joint equilibrium distribution over actions and fundamentals, we can describe the set of equilibria across all (normal) information structures completely in terms of restrictions on the first and second moments of the equilibrium joint distribution.

We begin with an exact characterization of the set of all possible Bayes correlated equilibrium distributions. We show in Proposition 1 that the joint distribution of any BCE is completely characterized by three correlation coefficients. In particular, the mean of the individual and the aggregate action is uniquely determined by the fundamentals of the economy. But, the second moments, the variance and covariances can differ significantly across equilibria. The set of BCE is explicitly described by a triple of correlation coefficients: (i) the correlation coefficient of any pair of individual actions (or equivalently the individual action and the average action), (ii) the correlation coefficient of an individual action and the associated individual payoff state, and (iii) the correlation coefficient of the aggregate action and an individual state. The description of the Bayes correlated equilibrium set arises from two distinct set of conditions. The restrictions on the coefficients themselves are purely statistical in nature, and come from the requirement that the variance-covariance matrix is positive definite, and thus these conditions do not depend at all on the nature of the interaction and hence the game itself. It is only the determination of the moments themselves, the mean and the variance that reflect the best response conditions, and hence the interaction structure of the game. This striking decomposition of the equilibrium conditions into statistical *and* incentive conditions arises as we allow for all possible information structures consistent with the common prior. Subsequently, in Section 5, we restrict attention to the subset of information structures in which each agent knows his own payoff state with certainty. This restriction on the informational environment is commonly imposed in the literature, and indeed we analyze the resulting implications for the equilibrium behavior. But with respect to the above separation result, we find that the decomposition fails whenever we impose any additional restrictions, such as knowing one's own payoff state.

We then characterize the upper boundary of the Bayes correlated equilibrium in terms of the correlation coefficients. We show in Proposition 2 that the upper boundary consists of equilibria that share a property that we refer to as *noise-free*.[1] The noise-free equilibria have the property that the conditional variance of the individual and aggregate action is zero, conditional on the idiosyncratic and common component of the state. That is, given the realization of the components of each agent's payoff state, the action of each agent is deterministic in the sense that it is a function of the common and the idiosyncratic components only. In fact, this description of the noise-free equilibria directly links the Bayes correlated equilibria to the Bayes Nash equilibria. In Proposition 3 we show that we can represent every noise-free Bayes correlated equilibrium in terms of the action chosen by the agent as a convex combination of the idiosyncratic and the common component of the state. Thus, a second surprising result of our analysis is that the boundary of the equilibrium set is formed by action state distributions that do no contain any additional noise. However as each agent implicitly responds to a convex sum of the components, each agent is likely to confound the idiosyncratic and common components with weights that differ from the equal weights by which they enter the payoff state, and hence lead to under or overreaction relative to the complete information Nash equilibrium.

We show for every Bayes correlated equilibrium, there exists an information structure under which the equilibrium distribution can be (informationally) decentralized as a Bayes Nash equilibrium, and conversely. The logic of Proposition 4 follows an argument presented in Bergemann and Morris (2013a) for *canonical* finite games and information structures. The additional insight here is that in the present environment any Bayes correlated equilibrium outcome can be achieved by a minimal information structure, in which each agent only receives a one-dimensional signal. The exact construction of the information structure is suggested by the very structure of the BCE. With the one-dimensional signal, each agent receives a signal that is linear combination of the common and idiosyncratic part of their payoff state and an additive noise term, possibly with a component common to all agents. Under this signal structure the action of each agent is a constant times the signal, and thus the agent only needs to choose this constant optimally. This construction allows us to understand how the composition of the signal affects the reaction of the agent to the signal.

With the compact description of the noise-free Bayes correlated equilibrium, we can then ask which information structure generates the largest aggregate (or individual) volatility, and which information structure might support the largest dispersion in the actions taken by the agents. In Proposition 5, we provide very mild sufficient conditions, essentially very weak monotonicity conditions, on the objective functions such that we can restrict attention to the noise-free equi-

[1] We would like to thank Marios Angeletos, our discussant, for suggesting this term.

libria. Any of these second moments is maximized by a noise-free equilibrium, but importantly not all of these quantities are maximized by the same equilibrium. The compact representation of the noise-free equilibria, described above, is particularly useful as we can find noise-free equilibria by maximizing over a single variable without constraints, rather than having to solve a general constrained maximization program. In Proposition 8 we use this insight to establish the comparative statics for maximal individual or aggregate volatility that can arise in any Bayes correlated equilibrium as we change the nature of the strategic interaction.

With a detailed characterization of the feasible equilibrium outcomes we ask what does interdependence add or to change relative to equilibrium behavior in a model with pure common values. With pure common values, any residual uncertainty about the common payoff state lowers the responsiveness of the agent to the information, it attenuates their response to the true payoff state. Thus, the maximal responsiveness is attained in the complete information equilibrium in which consequently the aggregate volatility is maximized across all possible information structures, see Proposition 6. At the other end of the spectrum, with pure private values, the complete information Nash equilibrium has zero aggregate volatility. But if the noisy information has an error term common across the agents, then aggregate volatility can arise with incomplete information, even though there is no aggregate payoff uncertainty. Now, the presence of the error term still leads every agent to attenuate his response to the signal compared to his response if he were to observe the true payoff state, as a Bayesian correction to the signal. Nonetheless, we show that as the variance of the idiosyncratic component increases, the maximal aggregate volatility increases as well, in fact linearly in the variance of the idiosyncratic component, see Proposition 9. Now, strikingly, with interdependent values, that is in between pure private and pure common values, the aggregate volatility is still maximal in the presence of residual uncertainty, but the response of the agent is not *attenuated* anymore, quite to contrary, it *exceeds* the complete information responsiveness in some dimensions. In the noise-free equilibrium, whether implicit in the BCE, or explicit in the BNE, the information about the idiosyncratic component is bundled with the information about the common component. In fact, the signal in the noise-free BNE represents the idiosyncratic and the common component as a convex combination. Thus, even though the signal does not contain any payoff irrelevant information, hence noise-free, the agent still faces uncertainty about his payoff state as the convex weights typically differ from the equal weights the components receives in the payoff state. As a consequence, any given convex combination apart from the equal weigh gives each agent a reason to react stronger to some component of the fundamental state than he would if he had complete information about either component, thus making it an excess response to at least some component. Now, if the excess response occurs with respect to the common component, then we can observe an increase in the aggregate volatility far beyond the one suggested by the complete information analysis. Moreover, as Proposition 9 shows there is a strong positive interaction effect with respect to the maximal aggregate volatility between the variance of the idiosyncratic and the common component. This emphasizes the fact that the increased responsiveness to the common component cannot be understood by merely combining the separate intuition gained from the pure private and the pure common value case with which we start above.

We proceed to extend the analysis in two directions. First, we constrain the set of possible information structures and impose the common restriction that every agent knows his own value but may still be uncertain about the payoff state of the other agents. We show that this common informational restriction can easily be accommodated by our analysis. But once we impose any such restriction on the class of permissible information structures, then the separability between the correlation structure and the interaction structure, that we observe without any restriction, does not hold anymore. Second, we extend the analysis to accommodate more general interaction structures. In particular, we consider the case when the best response of every agent is to a convex combination of the average action and the action of a specific, matched, agent. This allows us in particular to analyze the case of pairwise interaction, and we consider both random matching as well as positive or negative assortative matching. The following analysis shows that the current framework is sufficiently flexible to eventually accommodate an even more ambitious analysis of general information and network (interaction) structures.

The remainder of the paper is organized as follows. Section 2 introduces the model and the equilibrium concept. Section 3 offers the analysis of the model with aggregate interaction. Section 4 establishes the link between information structure and volatility. Section 5 considers the case of restricted information structures, in particular it assumes that every agent knows his own payoff state. Section 6 considers the more general interaction structures, and in particular analyzes the case of pairwise interaction. Section 7 discusses the relationship of our results to recent contribution in macroeconomics on the source and scope of volatility and concludes. Section 8 constitutes the appendix and contains most of the proofs.

2 Model

2.1 Payoffs

We consider a continuum of agents, with mass normalized to 1. Agent $i \in [0,1]$ chooses an action $a_i \in \mathbb{R}$ and is assumed to have a quadratic payoff function which is function of his action a_i, the mean action taken by agents, A, and the individual payoff state, $\theta_i \in \mathbb{R}$:

$$u_i : \mathbb{R}^3 \to \mathbb{R}.$$

In consequence, agent i has a linear best response function:

$$a_i = r\mathbb{E}[A|\mathcal{I}_i] + \mathbb{E}[\theta_i|\mathcal{I}_i], \tag{1}$$

where $\mathbb{E}[\cdot|\mathcal{I}_i]$ is the expectation conditional on the information agent i has prior to taking an action a_i. The parameter $r \in \mathbb{R}$ of the best response function represents the strategic interaction among the agents. If $r < 0$, then we have a game of *strategic substitutes*, if $r > 0$, then we have a game of *strategic complements*. We shall assume that the interaction parameter r is bounded above, or $r \in (-\infty, 1)$.

We assume that the individual payoff state θ_i is given by the linear combination of a *common component* $\bar{\theta}$ and an *idiosyncratic component* $\Delta\theta_i$:

$$\theta_i = \bar{\theta} + \Delta\theta_i. \tag{2}$$

Each component is assumed to be normally distributed. While $\bar{\theta}$ is common to all agents, the idiosyncratic component $\Delta\theta_i$ is identically distributed across agents, independent of the common component. Formally, we describe the payoff uncertainty in terms of the pair of random variables $(\overline{\theta}, \Delta\theta_i)$:

$$\begin{pmatrix} \overline{\theta} \\ \Delta\theta_i \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu_{\overline{\theta}} \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2_{\overline{\theta}} & 0 \\ 0 & \sigma^2_{\theta_i} \end{pmatrix}\right). \tag{3}$$

It follows that the population average satisfies $\mathbb{E}_i[\Delta\theta_i] = 0$, and we denote the average taken across the entire population, that is across all i, as $\mathbb{E}_i[\cdot]$. The common component can be interpreted as the mean or average payoff state as $\bar{\theta} = \mathbb{E}_i[\theta_i]$.

Given the independence and the symmetry of the idiosyncratic component $\Delta\theta_i$ across agents, the above joint distribution can be expressed in terms of the variance $\sigma^2_\theta = \sigma^2_{\overline{\theta}} + \sigma^2_{\theta_i}$ of the individual state, and the correlation (coefficient) $\rho_{\theta\theta}$ between any two states of any two agents i and j, θ_i and θ_j. After all, by construction the covariance of θ_i and θ_j is equal to the covariance between θ_i and $\overline{\theta}$, and in turn also represents the variance of the common component, or $\sigma^2_{\overline{\theta}} = \rho_{\theta\theta}\sigma^2_\theta$:

$$\begin{pmatrix} \theta_i \\ \overline{\theta} \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu_\theta \\ \mu_\theta \end{pmatrix}, \begin{pmatrix} \sigma^2_\theta & \rho_{\theta\theta}\sigma^2_\theta \\ \rho_{\theta\theta}\sigma^2_\theta & \rho_{\theta\theta}\sigma^2_\theta \end{pmatrix}\right). \tag{4}$$

We assume that the above joint normal distribution, given by (4), is the commonly known *common prior*. We refer to the special cases of $\rho_{\theta\theta} = 0$ and $\rho_{\theta\theta} = 1$ as the case of *pure private* and *pure common* values. We shall almost exclusively use the later representation (4) of payoff uncertainty, and only occasionally the former representation (3).

2.2 Bayes Correlated Equilibrium

We now define the main solution concept we will use throughout the paper.

Definition 1 (Bayes Correlated Equilibrium)
The variables $(\theta_i, \bar{\theta}, a_i, A)$ form a symmetric and normally distributed Bayes correlated equilibrium (BCE) if their joint distribution is given by a multivariate normal distribution and for all i and a_i :

$$a_i = r\mathbb{E}[A|a_i] + \mathbb{E}[\theta_i|a_i]. \tag{5}$$

We denote the variance-covariance matrix of the joint distribution of $(\theta_i, \bar{\theta}, a_i, A)$ by $\mathbb{V}$.

3 Bayes Correlated Equilibrium

We now find conditions such that the random variables $(\theta_i, \bar{\theta}, a_i, A)$. These conditions can be separated into two distinct sets of requirements: the first set consists of conditions such that the variance-covariance matrix $\mathbb{V}$ of the joint multivariate distribution constitutes a valid variance-covariance matrix, namely that it is positive-semidefinite; and a second set of conditions that guarantee that the best response conditions (7) holds. The first set of conditions are purely statistical requirements. The second set of conditions are necessary for any BCE, and these later conditions merely rely on the linearity of the best response. Importantly, both set of conditions are necessary *independent* of the assumption of normal distributed payoff uncertainty. The normality assumption will simply ensure that the equilibrium distributions are completely determined by the first and second moment. Thus, the normality assumptions allows us to describe the set of BCE in terms of restrictions that are necessary *and* sufficient.

3.1 Characterization of Bayes Correlated Equilibrium

We begin the analysis of the Bayes correlated equilibrium by reducing the dimensionality of the variance-covariance matrix. We appeal to the symmetry condition to express the aggregate variance in terms of the individual variance and the correlation between individual terms. Just as we described above the variance $\sigma^2_{\overline{\theta}}$ of the common component $\overline{\theta}$ in terms of the covariance between any two individual payoff states in (4), or $\sigma^2_{\overline{\theta}} = \rho_{\theta\theta}\sigma^2_{\theta}$, we can describe the variance of aggregate action σ^2_A in terms of the covariance of any two individual actions, or $\sigma^2_A = \rho_{aa}\sigma^2_a$. We write $\rho_{a\theta}$ for the correlation coefficient between action a_i and payoff state θ_i of player i:

$$\operatorname{cov}(a_i, \theta_i) \triangleq \rho_{a\theta}\sigma_a\sigma_\theta.$$

We denote by $\rho_{a\phi}$ the correlation coefficient between the action a_i of agent i and the payoff state θ_j of an arbitrary other agent j:

$$\operatorname{cov}(a_i, \theta_j) \triangleq \rho_{a\phi}\sigma_a\sigma_\theta.$$

These three correlation coefficients, $(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$, parameterize the whole variance-covariance matrix. To see why, observe that the covariance between a purely idiosyncratic random variable and a common random variable is always 0. This implies that both the covariance between the aggregate action A and the payoff state θ_j of player j and the covariance between the agent i's action, a_i, and the common component of the payoff state, $\overline{\theta}$, are the same as the covariance between the action of player i and the payoff state θ_j of player j, or $\rho_{a\phi}\sigma_a\sigma_\theta$. Thus we can reduce the number of variance terms, and in particular the number of correlation coefficients needed to describe the variance covariance matrix $\mathbb{V}$ without loss of generality.

Lemma 1 (Symmetric Bayes Correlated Equilibrium)
The variables $(\theta_i, \bar{\theta}, a_i, A)$ form a symmetric and normally distributed Bayes correlated equilibrium (BCE) if and only if there exist parameters of the first and second moments $(\mu_a, \sigma_a, \rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$ such that the joint distribution is given by:

$$\begin{pmatrix} \theta_i \\ \bar{\theta} \\ a_i \\ A \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu_\theta \\ \mu_\theta \\ \mu_a \\ \mu_a \end{pmatrix}, \begin{pmatrix} \sigma^2_\theta & \rho_{\theta\theta}\sigma^2_\theta & \rho_{a\theta}\sigma_a\sigma_\theta & \rho_{a\phi}\sigma_a\sigma_\theta \\ \rho_{\theta\theta}\sigma^2_\theta & \rho_{\theta\theta}\sigma^2_\theta & \rho_{a\phi}\sigma_a\sigma_\theta & \rho_{a\phi}\sigma_a\sigma_\theta \\ \rho_{a\theta}\sigma_a\sigma_\theta & \rho_{a\phi}\sigma_a\sigma_\theta & \sigma^2_a & \rho_{aa}\sigma^2_a \\ \rho_{a\phi}\sigma_a\sigma_\theta & \rho_{a\phi}\sigma_a\sigma_\theta & \rho_{aa}\sigma^2_a & \rho_{aa}\sigma^2_a \end{pmatrix} \right), \tag{6}$$

and for all i and a_i :

$$a_i = r\mathbb{E}[A|a_i] + \mathbb{E}[\theta_i|a_i]. \tag{7}$$

Henceforth, we shall therefore refer to the variance of the individual action as the *individual volatility:*

$$\sigma_a^2; \tag{8}$$

to the variance of the average action A as the *aggregate volatility*:

$$\rho_{aa}\sigma_a^2; \tag{9}$$

and to the variance of $\Delta a_i = a_i - A$ as the *dispersion:*

$$(1-\rho_{aa})\sigma_a^2. \tag{10}$$

As for the best response condition (7), with a multivariate normal distribution the conditional expectations have the familiar linear form:

$$\mathbb{E}\left[\begin{array}{c|c} \theta_i & \\ \overline{\theta} & a_i \\ A & \end{array}\right] = \begin{pmatrix} \mu_\theta \\ \mu_\theta \\ \mu_a \end{pmatrix} + \sigma_a^{-2} \begin{pmatrix} \rho_{a\theta}\sigma_a\sigma_\theta \\ \rho_{a\phi}\sigma_\theta^2 \\ \rho_{aa}\sigma_a^2 \end{pmatrix} (a_i - \mu_a),$$

and we can write the best response condition (7):

$$a_i = r(\mu_a + \rho_{aa}(a_i - \mu_a)) + (\mu_\theta + \rho_{a\theta}\frac{\sigma_\theta}{\sigma_a}(a_i - \mu_a)). \tag{11a}$$

By the law of iterated expectation we obtain:

$$\mu_a = r\mu_a + \mu_\theta \Rightarrow \mu_a = \frac{\mu_\theta}{1-r}. \tag{12}$$

Taking the derivative of (11a) with respect to a_i we get:

$$1 = r\rho_{aa} + \rho_{a\theta}\frac{\sigma_\theta}{\sigma_a} \Leftrightarrow \sigma_a = \frac{\rho_{a\theta}\sigma_\theta}{1-r\rho_{aa}}. \tag{13}$$

We thus have a complete determination of the first and second moment. Alternatively, we could obtain these conditions by directly using the best response condition, and applying the law of iterated expectation (or total expectation and similarly the law of total variance) to obtain the moment conditions.

Now, for $\mathbb{V}$ to be a valid variance-covariance matrix, it has to be positive semi-definite. Now, we can provide a sharp characterization of the set of parameters that constitute a BCE.

Proposition 1 (Characterization of BCE)

A multivariate normal distribution of $(\theta_i, \bar{\theta}, a_i, A)$ *is a symmetric Bayes correlated equilibrium if and only if :*

1. *the mean of the individual action is:*
$$\mu_a = \frac{\mu_\theta}{1-r};$$
2. *the standard deviation of the individual action is:*
$$\sigma_a = \frac{\rho_{a\theta}\sigma_\theta}{1 - r\rho_{aa}} \geq 0;$$
3. *the correlation coefficients* $\rho_{aa}, \rho_{a\theta}, \rho_{a\phi}$ *satisfy the nonnegativity conditions* $\rho_{aa}, \rho_{a\theta} \geq 0$ *and the inequalities:*
$$(i) \quad (\rho_{a\phi})^2 \leq \rho_{\theta\theta}\rho_{aa}, \quad (ii) \quad (1-\rho_{aa})(1-\rho_{\theta\theta}) \geq (\rho_{a\theta} - \rho_{a\phi})^2. \tag{14}$$

Proof. The moment equalities (1) and (2) were established in (12) and (13). Thus we proceed to verify that the inequality constraints (3) are necessary and sufficient to guarantee that the matrix $\mathbb{V}$ is positive semi-definite.

Here we express the equilibrium conditions, by a change of variables, in terms of different variables, which facilitates the calculation. Let:

$$M \triangleq \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus, we have that:

$$\begin{pmatrix} \Delta\theta_i \\ \bar{\theta} \\ \Delta a_i \\ A \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ \mu_\theta \\ 0 \\ \mu_a \end{pmatrix}, M\mathbb{V}M' \right),$$

where

$$\mathbb{V}_\perp \triangleq M\mathbb{V}M' = \begin{pmatrix} (1-\rho_{\theta\theta})\,\sigma_\theta^2 & 0 & (\rho_{a\theta} - \rho_{a\phi})\,\sigma_a\sigma_\theta & 0 \\ 0 & \rho_{\theta\theta}\sigma_\theta^2 & 0 & \rho_{a\phi}\sigma_a\sigma_\theta \\ (\rho_{a\theta} - \rho_{a\phi})\,\sigma_a\sigma_\theta & 0 & (1-\rho_{aa})\,\sigma_a^2 & 0 \\ 0 & \rho_{a\phi}\sigma_a\sigma_\theta & 0 & \rho_{aa}\sigma_a^2 \end{pmatrix}. \tag{15}$$

We use $\mathbb{V}_{\perp}$ to denote the variance/covariance matrix expressed in terms of $(\Delta\theta_i, \bar{\theta}, \Delta a_i, A)$. It is easy to verify that $\mathbb{V}_{\perp}$ is positive semi-definite if and only if the inequality conditions (3) are satisfied. To check this it is sufficient to note that the leading principal minors are positive if and only if these conditions are satisfied, and thus $\mathbb{V}_{\perp}$ is positive semi-definite if and only if these conditions are satisfied. ■

The orthogonal representation of the random variables, in terms of the idiosyncratic and the common component, namely $\Delta\theta_i$ and $\bar{\theta}$, or Δa_i and A, as represented by the variance/covariance matrix $\mathbb{V}_{\perp}$ in (15) is useful in many ways. In particular, it establishes why aggregate variance is the product of the individual variance σ_a^2 and the correlation coefficient ρ_{aa}.

There are two aspects of a BCE that we would like to highlight. First, note that condition (1) of Proposition 1 completely pins down the expected value of the action in terms of the fundamentals, which implies that any difference across Bayes correlated equilibria will manifest itself in the second moments only. Thus, during the rest of the paper we will adopt the normalization that $\mu_\theta = 0$, as any change in μ_θ will affect only the expected action of each agent and then by a constant value across all BCE.

Second, the restrictions on the equilibrium correlation coefficients do not at all depend on the interaction parameter r. It is easy to see from the proof of Proposition 1 that the restrictions on the set of equilibrium correlations are purely statistical. They come exclusively from the condition that the correlation matrix $\mathbb{V}$ is a valid variance/covariance matrix, namely that $\mathbb{V}$ is positive semi-definite matrix. As we will show later, this disentanglement of the set of feasible correlations and the interaction parameter is only possible when we allow for all possible information structures, i.e. when we do impose any restrictions on the private information that agents may have. By contrast, the mean μ_a and the variance σ_a^2 of the individual actions does depend on the interaction parameter r, as it is determined by the best response condition (7).

We note that in the special case of pure private (or pure common) values the set of outcomes in terms of the correlation coefficients is only two dimensional. The reduction in dimensionality arises as the correlation between the average action and the individual state is either zero (as in the pure private value case) or is in a constant linear relationship to correlation between the individual action and individual state (as in the pure common value case), and thus redundant in either case.

3.2 Geometric Representation of BCE

Before we continue with the analysis, it is useful to visualize the set of feasible Bayes correlated equilibria, and see how the mechanics of the different constraints change the set of feasible equilibria. To visualize the constraints of (14) of Proposition 1 we plot the feasible regions in the $(\rho_{aa}, \rho_{a\theta})$ space, which are the most relevant for volatility, and thus the most relevant for us to plot. Since this depends on the values of $\rho_{a\phi}$ and $\rho_{\theta\theta}$ we do this in several plots.

In Figure 1 we vary $\rho_{\theta\theta}$ and find the full set of feasible equilibria in the $(\rho_{a\theta}, \rho_{aa})$ space. As we will show later, for any given ρ_{aa}, the upper bound on the values of $\rho_{a\theta}$ that can be achieved is found by imposing that the constraints (14) of Proposition 1 bind. We get the following:

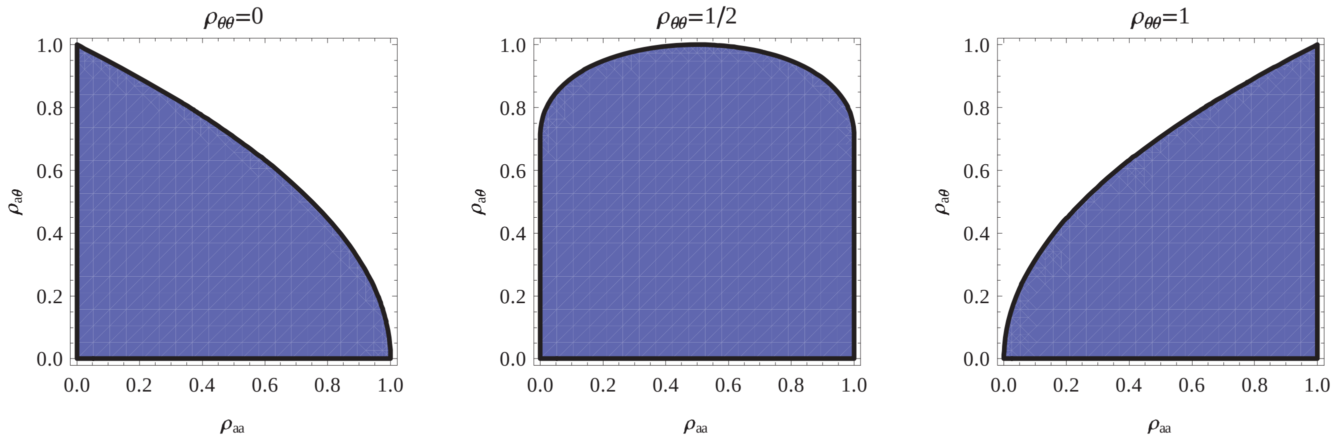

Figure 1: The set of BCE in the $(\rho_{aa}, \rho_{a\theta})$ space: varying $\rho_{\theta\theta}$

In Figure 2 we fix $\rho_{\theta\theta} = 0.5$, and impose a linear relationship between the correlation coefficient $\rho_{a\phi}$ and $\rho_{a\theta}$:

$$\rho_{a\phi} = \alpha \rho_{a\theta},$$

and plot the equilibrium set for different values of α. As we will show later, the case $\alpha = 0$ and $\alpha = 1$ will be the cases in which agent only responds to variations in the idiosyncratic and the common component of his payoff state respectively (that is, $\Delta\theta_i$ and $\bar{\theta}$ separately), while the case of $\alpha = 1/2$ is the case in which each agent responds to $\Delta\theta_i$ and $\bar{\theta}$ with equal intensity, and thus only responds to the sum given by θ_i (shown in the last plot). More generally, increasing α will correspond to increasing the information agents have about $\bar{\theta}$ and decreasing the information they have about $\Delta\theta_i$.

We can see that the case of $\alpha = 0$ and $\alpha = 1$ is a particular case of analyzing private and common values, as agent only have information on $\Delta\theta_i$ and $\bar{\theta}$ respectively. Thus the similarity in both of the plots for the cases $\alpha = 0$ and $\alpha = 1$ with $\rho_{\theta\theta} = 0$ and $\rho_{\theta\theta} = 1$ respectively. Also, note that the envelope of the kinks in Figure 2 correspond to the frontier of $\rho_{\theta\theta} = 0.5$ in the Figure 1. As we will show next, this will also be a result of the fact that this frontier is given by

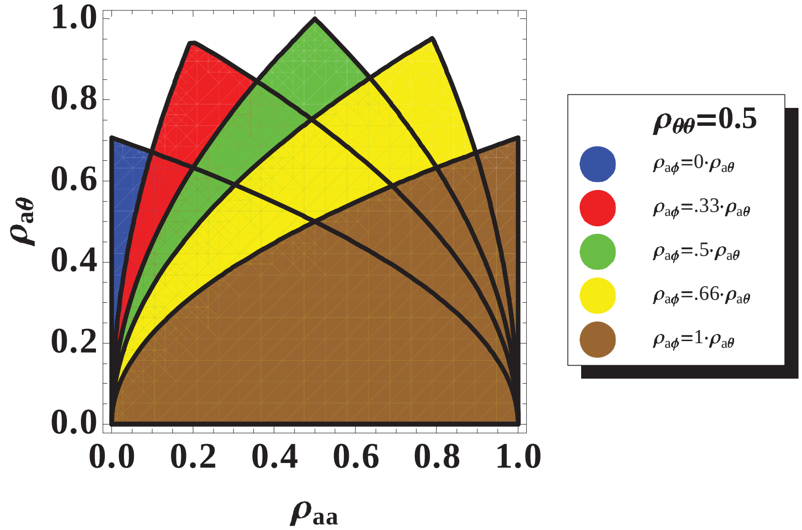

Figure 2: The set of BCE in the $(\rho_{aa}, \rho_{a\theta})$ space: varying $\rho_{a\phi}$

"noise-free" equilibria.

3.3 Noise-Free Equilibria

We can decompose the action of each agent in terms of his responsiveness to the components of his payoff state θ_i, namely the idiosyncratic component $\Delta\theta_i$ and the common component $\bar{\theta}$, and any residual responsiveness has to be attributed to noise. Given the multivariate normal distribution, the responsiveness of the agent to the components of his payoff state is directly expressed by the covariance:

$$\text{cov}(a_i, \Delta\theta_i) = \frac{\partial \mathbb{E}[a_i|\Delta\theta_i]}{\partial \Delta\theta_i}, \quad \text{cov}(a_i, \overline{\theta}) = \frac{\partial \mathbb{E}[a_i|\overline{\theta}]}{\partial \overline{\theta}}. \tag{16}$$

The action a_i itself also has an idiosyncratic and a common component as $a_i = A + \Delta a_i$. The conditional variance of these components of a_i can be expressed in terms of the correlation coefficients $(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi}, \rho_{\theta\theta})$, which are subject to the restrictions of Proposition 1.3. By using the familiar property of the multivariate normal distribution for the *conditional variance*, we obtain a diagonal matrix:

$$\text{var}\left[\begin{array}{c|c} \Delta a_i & \Delta\theta_i \\ A & \bar{\theta} \end{array}\right] = \sigma_a^2 \begin{pmatrix} (1-\rho_{aa}) - \frac{(\rho_{a\theta}-\rho_{a\phi})^2}{1-\rho_{\theta\theta}} & 0 \\ 0 & \rho_{aa} - \frac{\rho_{a\phi}^2}{\rho_{\theta\theta}} \end{pmatrix}. \tag{17}$$

If the components A and Δa_i of the agent's action are completely explained by the components of the payoff state, $\bar{\theta}$ and $\Delta\theta_i$, then the conditional variance of the action components, and a fortiori of the action itself, is equal to zero. Now, by comparing the conditional variances above with the conditions of Proposition 1.3, it is easy to see that the conditional variances are equal to zero if and only if the conditions of Proposition 1.3 are satisfied as equalities. Moreover, by the conditions of Proposition 1.3, in any BCE, the conditional variance of action a_i can be equal to zero if and only if the conditional variances of the components, A and Δa_i are each equal to zero. This suggests the following definition.

Definition 2 (Noise-Free BCE)
A BCE is noise-free *if a_i has zero variance, conditional on $\bar{\theta}$ and $\Delta\theta_i$.*

We observe that the above matrix of conditional variances is only well-defined for interdependent values, that is for $\rho_{\theta\theta} \in (0, 1)$. For the case of pure private or pure common values, $\rho_{\theta\theta} = 0$ or $\rho_{\theta\theta} = 1$, only one of the off diagonal terms is meaningful, as the other conditioning terms, $\bar{\theta}$ or $\Delta\theta_i$, have zero variance by definition. Using the equilibrium conditions of Proposition 1.3, we therefore obtain an explicit characterization of the noise-free Bayes correlated equilibria.

Proposition 2 (Characterization of Noise-Free BCE)
For all $\rho_{\theta\theta} \in (0, 1)$, the set of noise-free BCE (in the space of correlation coefficients $(\rho_{aa}, \rho_{a\theta})$) is given by:

$$\{(\rho_{aa}, \rho_{a\theta}) \in [0, 1]^2 : \rho_{a\theta} = |\sqrt{\rho_{aa}\rho_{\theta\theta}} \pm \sqrt{(1 - \rho_{\theta\theta})(1 - \rho_{aa})}|\}. \quad (18)$$

To visualize the noise-free equilibria in the $(\rho_{aa}, \rho_{a\theta})$ space we plot them in Figure 3. The set

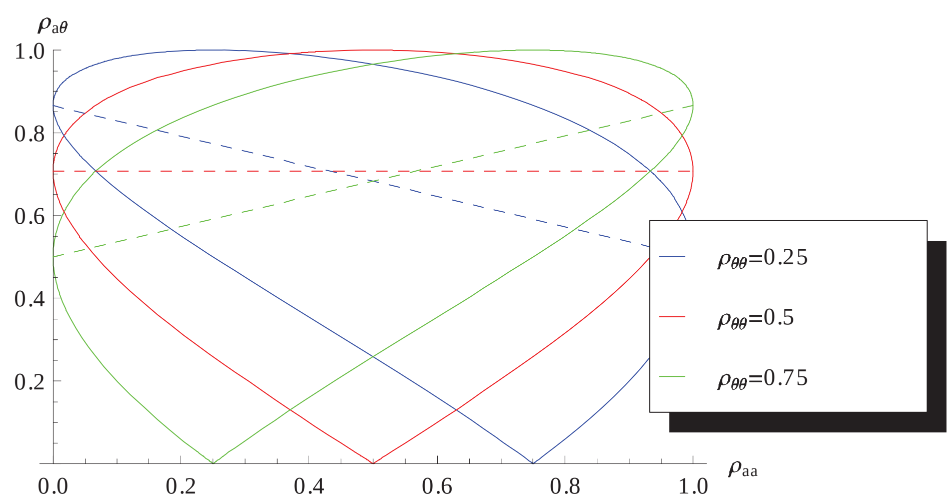

Figure 3: The set of noise-free BCE

of noise-free equilibria is described by the two roots of a quadratic equation, visually separated by the dashed line in the above illustration. The positive root corresponds to the upper part of the ellipse, the negative root to the lower part. The kink in the lower segment arises as the sign of the term inside the absolute value of (19) changes. For our purposes, it is exclusively the positive root that is of interest. Namely, in the set of *all* Bayes correlated equilibria it describes for any given correlation coefficient ρ_{aa} the highest attainable correlation coefficients of $\rho_{a\theta}$ *and* $\rho_{a\phi}$ simultaneously. For this reason, we shall refer from now on to the positive root only when we speak of the noise-free equilibria, that is:

$$\rho_{a\theta} = \sqrt{\rho_{aa}\rho_{\theta\theta}} + \sqrt{(1-\rho_{\theta\theta})(1-\rho_{aa})}, \quad \rho_{a\phi} = \rho_{aa}\rho_{\theta\theta}, \quad \text{for all } \rho_{aa} \in [0,1]\,. \tag{19}$$

Now, by construction the variance of the individual actions conditional on the components $\Delta\theta_i$ and $\bar{\theta}$ of the payoff state θ_i is zero in all noise-free BCE. In other words, the action of each action has to be a deterministic function of the components $\Delta\theta_i$ and $\bar{\theta}$ only. Given the linearity of the best response function, and of the conditional expectation in the normal environment, this suggests that we can express the action a_i of each agent as a linear combination of the components $\Delta\theta_i$ and $\bar{\theta}$ of the state θ_i. In fact, we can provide a canonical construction of the action a_i chosen in any noisy free BCE by means of a (convex) combination of the idiosyncratic component $\Delta\theta_i$ and the common component $\bar{\theta}$ of the individual state θ_i :

$$s_i(\lambda) \triangleq \lambda\Delta\theta_i + (1-\lambda)\bar{\theta} \text{ with } \lambda \in [0,1]. \tag{20}$$

We refer to the convex combination of $\Delta\theta_i$ and $\bar{\theta}$ as a "signal" $s_i(\lambda)$, as the present construction will shortly also lead us to a canonical information structure to decentralize the Bayes correlated equilibrium as a Bayes Nash equilibrium. In any case, given the convex combination represented by λ, we can then ask how strongly the agent responds to the "signal", and we denote the responsiveness by $\nu(\lambda)$. The next result establishes that the entire set of noise-free Bayes correlated equilibria can be described by a family of functions indexed by λ and linear in $\Delta\theta_i$ and $\bar{\theta}$. Importantly given λ, the action a_i of each agent is determined by the realizations of $\Delta\theta_i$ and $\bar{\theta}$ as follows:

$$a_i = \nu(\lambda)s_i(\lambda) = \nu(\lambda)\left(\lambda\Delta\theta_i + (1-\lambda)\bar{\theta}\right) \tag{21}$$

In addition, the next proposition establishes that for every weight $\lambda \in [0,1]$, we can uniquely and explicitly determine the responsiveness of the agent to the signal $s_i(\lambda)$:

$$\nu(\lambda) = \frac{(1-\lambda)\rho_{\theta\theta} + \lambda(1-\rho_{\theta\theta})}{(1-r)(1-\lambda)^2\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta})}. \tag{22}$$

Given a weight λ, we can then compute the joint distribution of the variables.

Proposition 3 (Deterministic and Convex Representation of Noise-free BCE)
Every noise-free BCE is uniquely characterized by $\lambda \in [0,1]$ and the associated correlation coefficients are:

$$\rho_{aa} = \frac{(1-\lambda)^2\rho_{\theta\theta}}{(1-\lambda)^2\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta})},\ \rho_{a\theta} = \frac{(1-\lambda)\rho_{\theta\theta} + \lambda(1-\rho_{\theta\theta})}{\sqrt{(1-\lambda)^2\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta})}},\ \rho_{a\phi} = \frac{(1-\lambda)\rho_{\theta\theta}}{\sqrt{(1-\lambda)^2\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta})}}. \tag{23}$$

In the case of pure private or pure common values, $\rho_{\theta\theta} = 0$ and $\rho_{\theta\theta} = 1$, respectively, the payoff uncertainty is described completely by either $\Delta\theta_i$ or $\bar{\theta}$, and in this sense is one dimensional. In both cases, there are only two possible noise-free equilibria, players either respond perfectly to the state of the world (complete information) or players do not respond at all (zero information). By contrast, in the general model with interdependent values, $\rho_{\theta\theta} \in (0,1)$, the payoff uncertainty of each player is described by the pair $\left(\Delta\theta_i, \bar{\theta}\right)$ and hence each player faces a two dimensional uncertainty. Unlike the case of pure private or pure common values, there is now a continuum of noise-free equilibria with interdependent values. The continuum of equilibria arises as each player may respond to a linear combination of $\Delta\theta_i$ and $\bar{\theta}$, but the weights may not be equal to 1 as they would be in a complete information world, where θ_i is known to be the sum of $\Delta\theta_i$ and $\bar{\theta}$, namely $\theta_i = \Delta\theta_i + \bar{\theta}$.

It is then evident that there is a discontinuity at $\rho_{\theta\theta} \in \{0,1\}$ in what we describe as the set of noise-free Bayes correlated equilibria. The reason is simple and comes from the fact that as $\rho_{\theta\theta}$ approaches zero or one, one of the dimensions of the uncertainty about fundamentals vanishes. Yet we should emphasize, that even as the payoff types approach the case of pure common or pure private values, the part of the fundamental that becomes small can be arbitrarily amplified by the weight λ. For example, as $\rho_{\theta\theta} \to 1$, the environment becomes arbitrarily close to pure common values, yet the shock $\Delta\theta_i$ still can be amplified by letting $\lambda \to 1$ in the construction of signal (20) above. Thus, the component $\Delta\theta_i$ acts similarly to a purely idiosyncratic noise in an environment with pure common values. After all, the component $\Delta\theta_i$ only affects the payoffs in a negligible way, but with a large enough weight, it has a non-negligible effect on the actions that the players take. Thus, for the case in which the the correlation of types approach the case of pure common or pure independent values, there is no longer a sharp distinction between what is noise and what is fundamentals. Thus, although there is a discontinuity in the set of noise-free BCE, the upper boundary of the Bayes correlated equilibria described by (19) remains valid even for $\rho_{\theta\theta} \in \{0,1\}$. But at the end points $\rho_{\theta\theta} \in \{0,1\}$, the set of equilibria will contain noise relative the fundamental components $\Delta\theta_i$ and $\bar{\theta}$, as either the idiosyncratic or the common component of the fundamental state cease to have variance, and hence cannot support the requisite variance in the actions anymore.

The above representation of the noise-free Bayes correlated equilibria in terms of a convex combination of the idiosyncratic and the common component of payoff state naturally suggests a signal structure that can implement the noise-free Bayes correlated equilibria as the corresponding Bayes Nash equilibria. We therefore turn our attention now to the relationship between the Bayes correlated and Bayes Nash equilibria, and will find that the link suggested above extends more generally beyond the noise-free equilibria.

3.4 Bayes Nash Equilibrium

So far, our solution concept has been the Bayes correlated equilibrium, and we now provide the connection between BCE and Bayes Nash Equilibrium (BNE). First, consider the Nash equilibrium under complete information. Here the linear best response game has a unique Nash equilibrium in which the action of agent i is given by:

$$a_i = \Delta\theta_i + \frac{\bar{\theta}}{1-r},$$

and the mean action is given by:

$$\mu_a = \frac{\mu_\theta}{1-r}.$$

Of course, the Nash equilibrium under complete information is indeed a noise-free Bayes correlated equilibrium, with values given by:

$$\lambda^* = \frac{1-r}{2-r} \quad \text{and} \quad v(\lambda^*) = \frac{2-r}{1-r}. \tag{24}$$

The value of $\lambda^* = (1-r)/(2-r)$ will be an important benchmark as we analyze how the information structure changes the responsiveness of the agents relative to the complete information equilibrium. Moreover, if we insert the expression λ^* in Proposition 3, then we obtain the correlation coefficients that form the joint distribution of the complete information Nash equilibrium, and we identify the complete information Nash equilibrium outcomes notationally by *.

Corollary 1 (Nash Equilibrium Distribution)
The normal distribution of the complete information Nash equilibrium is:

1. *the mean of the individual action:*

$$\mu_a^* = \frac{\mu_\theta}{1-r};$$

2. the standard deviation of the individual action:

$$\sigma_a^* = \sigma_\theta \sqrt{(1-\rho_{\theta\theta}) + \frac{\rho_{\theta\theta}}{(1-r)^2}};$$

3. and the correlation coefficients:

$$\rho_{aa}^* = \frac{\rho_{\theta\theta}}{(1-\rho_{\theta\theta})(1-r)^2 + \rho_{\theta\theta}},\ \rho_{a\theta}^* = \frac{\rho_{\theta\theta} r + (1-r)}{\sqrt{(1-\rho_{\theta\theta})(1-r)^2 + \rho_{\theta\theta}}},\ \rho_{a\phi}^* = \frac{\rho_{\theta\theta}}{\sqrt{(1-\rho_{\theta\theta})(1-r)^2 + \rho_{\theta\theta}}}.$$

We observe that the complete information Nash equilibrium could also be achieved by less than complete information. In particular, if each agent would only receive a one-dimensional signal s_i of the form given by (20) evaluated at λ^*, and respond with the corresponding intensity $\nu(\lambda^*)$, then we would also achieve the complete information Nash equilibrium. But the signal s_i would of course constitute a noisy signal in the sense that it would induce a posterior belief over the payoff state θ_i and the components $\overline{\theta}$ and $\Delta\theta_i$, but such that the posterior belief would be a sufficient statistic with respect to the equilibrium action.

To give a complete description of the Bayesian Nash equilibrium we need to specify a type space or an information structure. While we can allow for any finite-dimensional normally distributed information structure, for the present purpose, it will suffice to consider the following one-dimensional class of signals:

$$s_i = \lambda \Delta\theta_i + (1-\lambda)\overline{\theta} + \varepsilon_i, \tag{25}$$

where ε_i is normally distributed with mean zero and variance σ_ε^2. Similar to the definition of the payoff relevant fundamentals, the individual error term ε_i can have a common component:

$$\overline{\varepsilon} \triangleq \mathbb{E}_i[\varepsilon_i],$$

and an idiosyncratic component:

$$\Delta\varepsilon_i \triangleq \varepsilon_i - \overline{\varepsilon},$$

while being independent of the fundamental component, so that the joint distribution of the states and signals is given by:

$$\begin{pmatrix} \Delta\theta_i \\ \overline{\theta} \\ \Delta\varepsilon_i \\ \overline{\varepsilon} \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ \mu_\theta \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} (1-\rho_{\theta\theta})\sigma_\theta^2 & 0 & 0 & 0 \\ 0 & \rho_{\theta\theta}\sigma_\theta^2 & 0 & 0 \\ 0 & 0 & (1-\rho_{\varepsilon\varepsilon})\sigma_\varepsilon^2 & 0 \\ 0 & 0 & 0 & \rho_{\varepsilon\varepsilon}\sigma_\varepsilon^2 \end{pmatrix} \right), \tag{26}$$

and $\sigma_\varepsilon > 0$ and $\rho_{\varepsilon\varepsilon} \in [0,1]$ are the parameters of the fully specified information structure $\mathcal{I} = \{\mathcal{I}_i\}_{i\in[0,1]}$.

Definition 3 (Bayes Nash Equilibrium)

The random variables $\{a_i\}_{i\in[0,1]}$ form a normally distributed Bayes Nash Equilibrium under information structure $\mathcal{I}$ if and only if the random variables $\{a_i\}_{i\in[0,1]}$ are normally distributed and

$$a_i = \mathbb{E}[\theta_i + rA|s_i].$$

In an environment with multi-dimensional signals s_{ij}, with $j = 1, ..., J$, the definitions would extend and we would have to specify the variance and correlation of the error of each signal (ε_{ij}), the composition of each signal (λ_j), and also the correlation between the errors of the different signals:

$$s_{ij} = \lambda_j \Delta\theta_i + (1-\lambda_j)\bar{\theta} + \varepsilon_{ij}.$$

Yet, as the next result establishes these more general information structures are not needed in the sense that any distribution over action and payoff state they would induce can already be achieved with the above one-dimensional class of information structures, see (25) and (26). Moreover, the entire set of Bayes correlated equilibria can already by decentralized as Bayes Nash equilibria of these one-dimensional information structures.

Proposition 4 (Equivalence Between BCE and BNE)

The variables $(\theta_i, \bar{\theta}, a_i, A)$ form a (normal) Bayes correlated equilibrium if and only if there exists some information structure $\mathcal{I}$ under which the variables $(\theta_i, \bar{\theta}, a_i, A)$ form a Bayes Nash equilibrium.

One of the important insights from the analysis of BCE is that the set of outcomes that can be achieved as a BNE for some information structure, can also be described as a BCE. Thus, the solution concept of BCE allow us to study the set of outcomes that can be achieved as a BNE, without the need to specify information structures.

The equivalence between Bayes correlated equilibrium and Bayes Nash equilibrium is established for canonical finite games and arbitrary information structures in Bergemann and Morris (2013a) as Theorem 1. The proposition specializes the proof to the environment with linear best responses and multivariate normal distributed outcomes. The additional result here is that the entire set of multivariate Bayes correlated equilibria can be decentralized as Bayes Nash equilibria with the class of one-dimensional information structures given by (25) and (26). In the case of pure common values Bergemann and Morris (2013b) show that any BCE can be decentralized by considering a pair of noisy signals, a private and a public signal of the payoff state. By contrast, in the present general environment this class of binary information structures cannot decentralize the entire set of BCE, as it is not possible to express all linear combinations of the fundamental components by means of the two dimensional class of signals in which only the payoff state θ_i itself receives a private and public error term.

4 Aggregate Volatility and Information

In this Section, we use the characterization of the Bayes correlated equilibria, and in particular, the characterization of the noise-free equilibria to analyze what drives the volatility of the outcomes, in particular the aggregate outcome in this class of linear quadratic environments. We begin with the special case of pure common values in Section 4.1 and then analyze the general environment in Section 4.2. We focus mainly on aggregate volatility, $\rho_{aa}\sigma_a^2$, as it is often the most interesting empirical quantity and this lens keeps our discussion focused, but we also state results for the individual volatility σ_a^2, and the dispersion $(1-\rho_{aa})\sigma_a^2$.

Before we proceed, we need an auxiliary result which establishes that we can restrict attention without loss of generality to the noise-free BCE as defined earlier. Given our characterization of the set of Bayes correlated equilibria, we know that the mean and variance is determined by the interaction structure and the correlation coefficients, where the later are restricted by the inequalities derived in Proposition 1. Thus, if we are interested in identifying the equilibrium with the maximal aggregate or individual volatility, or essentially any other function of the first and the second moments of the equilibrium distribution, we essentially look at some function $\psi : [-1,1]^3 \to \mathbb{R}$, where the domain is given by the triple of correlation coefficients: $(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$.

Proposition 5 (Maximal Equilibria)
Let $\psi(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$ be an arbitrary continuous function, strictly increasing in $\rho_{a\theta}$ and weakly increasing in $\rho_{a\phi}$. Then, the BCE that maximizes ψ is an noise-free BCE.

An immediate consequence of the above result is the following

Corollary 2 (Volatility and Dispersion)
The individual volatility σ_a^2, the aggregate volatility $\rho_{aa}\sigma_a^2$ and the dispersion $(1-\rho_{aa})\,\sigma_a^2$ are all continuous functions of $(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$, strictly increasing in $\rho_{a\theta}$ and weakly increasing in $\rho_{a\phi}$.

Thus we can already conclude that equilibria that maximize either aggregate volatility or individual volatility or the dispersion are all noise-free BCE. This means that we can always represent the individual action as a linear combination of the two sources of fundamental uncertainty, $\Delta\theta_i$ and $\overline{\theta}$. In the present contribution, we are mostly interested in the narrow interpretation of Proposition 5 as given by Corollary 2. But clearly, the content of Proposition 5 would be relevant if we were to conduct a more comprehensive welfare analysis as the associated objective functions, in particular if it were linear quadratic as well, would satisfy the mild monotonicity conditions of Proposition 5. In particular, we note that the conditions of Proposition 5 are silent about the correlation coefficient ρ_{aa}, and thus we can accommodate arbitrary behavior, in particular we can accommodate environments (and payoffs and associated objective functions) with strategic substitutes or complements.

4.1 Volatility: The Case of Pure Common Values

As a benchmark, it is instructive to recall how the aggregate volatility behaves in a model with pure common values, $\rho_{\theta\theta} = 1$, as analyzed in Bergemann and Morris (2013b).

Proposition 6 (Aggregate Volatility with Common Values)
With common values, $\rho_{\theta\theta} = 1$, the maximum aggregate volatility across all BCE is given by:

$$\sigma_A^2 = \rho_{aa}\sigma_a^2 = \frac{\sigma_\theta^2}{(1-r)^2}.$$

The maximal aggregate volatility is increasing in r and is attained by the complete information Nash equilibrium.

Thus, with pure common values the maximum aggregate volatility is bounded by the aggregate volatility in the complete information equilibrium and the responsiveness of the agents to the common state $\overline{\theta}$ is always bounded above by the responsiveness achieved in the complete information equilibrium. Interestingly, this property is not going to hold anymore in environments with interdependent values. That is, information structures with less than complete information will lead to larger aggregate volatility than could be observed with complete information.

4.2 Volatility: Information and Fundamentals

We now establish that in an environment with interdependent values equilibria that maximize volatility or dispersion are not complete information equilibria, and hence will involve residual uncertainty from the point of view of each individual agent. With pure common values, any residual uncertainty about the payoff state inevitably reduces the responsiveness of the individual agent to the common state, and hence ultimately reduces the aggregate responsiveness. By contrast, with pure private values, the residual uncertainty might be correlated across agents, and hence allow for aggregate volatility to arise in the absence of aggregate uncertainty. Still, for each individual agent the residual uncertainty attenuates the responsiveness to his payoff state

θ_i. Now, with interdependent values, the interaction between the idiosyncratic and the common component in the payoff state can correlate the responsiveness of the agents without attenuating the individual response. It thus allows for aggregate behavior to emerge that cannot arise under either pure common or pure private values. More precisely, the residual uncertainty about idiosyncratic component can be present because it is bundled with, but not distinguishable from information about the common component. But this bundling of information can be achieved precisely by the convex combinations of the components suggested earlier in the characterization of the noise-free equilibria, see (20). And importantly, it occurs without the introduction of additional noise.

Now, if we are concerned with volatility, then by Corollary 2, it is indeed sufficient to consider precisely these noise-free equilibria in which each agent responds to a linear combination of the two fundamental sources of uncertainty. As the signal confounds the sources, the agents cannot disentangle them, and in turn lead the agent to overreact to one component and underreact to the other. In the noise-free equilibria, the responsiveness of the agent to the fundamentals can be expressed directly in terms of the weight λ that the idiosyncratic component receives in the convex combination:

$$\lambda\Delta\theta_i + (1-\lambda)\bar{\theta} \quad \text{with} \quad \lambda \in [0,1].$$

We note that we can use (and interpret) λ in two different, but closely related ways. It is either a parametrization of the noise-free Bayes correlated equilibria as suggested by Proposition 3 or it is a specific information structure of the form given by (25) that leads to Bayes Nash equilibria which informationally decentralize the noise-free Bayes correlated equilibria as suggested by Proposition 4. In either case, the weight λ is measure of how confounded the idiosyncratic and the common component are.

Proposition 7 (Responsiveness to Fundamentals)
In any noise-free BCE,

1. $\lambda \in (\lambda^*, 1) \Longleftrightarrow \operatorname{cov}(a_i, \Delta\theta_i) > 1$;
2. $\lambda \in (0, \lambda^*) \Longleftrightarrow \operatorname{cov}(a_i, \bar{\theta}) > \frac{1}{1-r}$.

Thus the responsiveness of the action to one of the components of the payoff state is typically stronger than in the complete information environment. In Figure 4 we plot the responsiveness to the two components of the fundamental state for the case of $\rho_{\theta\theta} = 0.5$ and different interaction parameters r. The threshold values λ_r^* simply correspond to the critical value λ^* for each of the considered interaction parameters $r \in \left\{-\frac{3}{4}, 0, +\frac{3}{4}\right\}$. The horizontal black lines represent the responsiveness to common component $\bar{\theta}$ in the complete information equilibrium which is equal to $1/(1-r)$, and the responsiveness to the idiosyncratic part, which is always equal to 1. By contrast, the red curves represent the responsiveness to the common component along the noise-free equilibrium, and the blue curves represent the responsiveness to the idiosyncratic component. Thus if $\lambda < \lambda^*$, then the responsiveness to the common component $\bar{\theta}$ is larger

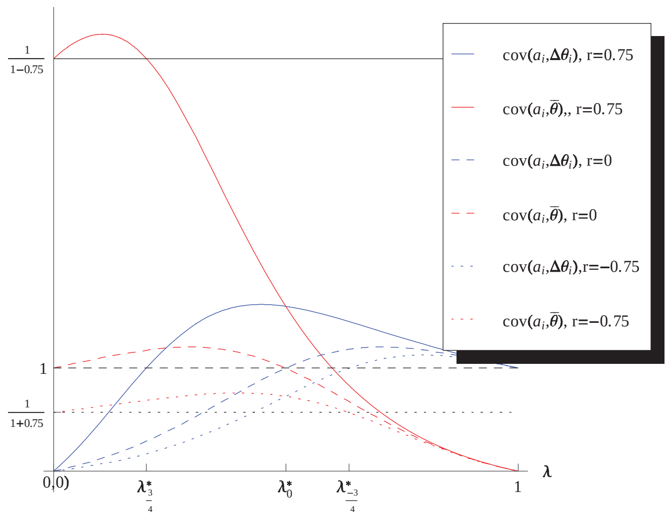

Figure 4: Responsiveness to Fundamentals for $\rho_{\theta\theta} = 1/2$

than in the complete information equilibrium, and conversely for $\Delta\theta_i$. Moreover, we observe that the maximum responsiveness to the common component is never attained in the complete information equilibrium or at the boundary value of λ, that is at 0 or 1. This immediately implies that the responsiveness is not monotonic in the informational content. We now provide some general comparative static results.

Proposition 8 (Comparative Statics)
For all $\rho_{\theta\theta} \in (0,1)$*:*

1. *the maximal individual volatility* $\max_\lambda\ \sigma_a^2$*, the maximal aggregate volatility* $\max_\lambda\ \rho_{aa}\sigma_a^2$ *and the maximal dispersion* $\max_\lambda (1-\rho_{aa})\sigma_a^2$ *are strictly increasing in* r;

2. *the corresponding weights on the idiosyncratic component,* $\operatorname{argmax}_\lambda \sigma_a^2$, $\operatorname{argmax}_\lambda\ \rho_{aa}\sigma_a^2$*,* $\operatorname{argmax}_\lambda (1-\rho_{aa})\sigma_a^2$ *are strictly decreasing in* r*;*

3. *the corresponding weights on the idiosyncratic component satisfy:*
$\operatorname{argmax}_\lambda\ (1-\rho_{aa})\sigma_a^2 > \operatorname{argmax}_\lambda\ \sigma_a^2 > \operatorname{argmax}_\lambda\ \rho_{aa}\sigma_a^2.$

Thus, the maximal volatility, both individual and aggregate, is increasing in the level of complementarity r. Even the maximal dispersion is increasing in r. In the maximally dispersive equilibrium, the agents confound the idiosyncratic and aggregate component of the payoff type and overreact to the idiosyncratic part, this effect increases with r. This implies that the responsiveness to the common component $\overline{\theta}$ increases, and hence the overreaction to the idiosyncratic component $\Delta\theta_i$ increases as well. Moreover, the optimal weight on the aggregate component increases in r for all of the second moments.

We can contrast the behavior of aggregate volatility with pure common values and with interdependent values even more dramatically. Up to now, we stated our results with a representation of the payoff uncertainty in terms of the total variance σ_θ^2 of the payoff state and the correlation coefficient $\rho_{\theta\theta}$, see (4). The alternative parametrization, see (3), represented the uncertainty by the variance $\sigma_{\overline{\theta}}^2$ of the common component $\overline{\theta}$, and the variance $\sigma_{\theta_i}^2$ of the idiosyncratic component $\Delta\theta_i$ separately (and then the correlation coefficient across agents is determined by the two variances terms.) For the next result, we shall adopt this second parametrization, as we want to ask what happens to the aggregate volatility as we keep the variance $\sigma_{\overline{\theta}}^2$ of the common component $\overline{\theta}$ constant, and increase the variance $\sigma_{\theta_i}^2$ of the idiosyncratic component $\Delta\theta_i$, or conversely.

Proposition 9 (Aggregate Volatility)
For all $r \in (-\infty, 1)$, the maximal aggregate volatility is equal to

$$\max_{\lambda} \left\{\rho_{aa}\sigma_a^2\right\} = \frac{\sigma_{\theta_i}^4}{4\left(\sqrt{\sigma_{\overline{\theta}}^2 + (1-r)\sigma_{\theta_i}^2} - \sigma_{\overline{\theta}}\right)^2} \tag{27}$$

and is strictly increasing without bound in $\sigma_{\theta_i}^2$. As $\sigma_{\theta_i}^2 \to 0$, this converges to the

$$\max_{\lambda} \left\{\rho_{aa}\sigma_a^2\right\} = \sigma_{\overline{\theta}}^2/(1-r)^2,$$

which is also the aggregate volatility in the complete information equilibrium. As $\sigma_{\overline{\theta}}^2 \to 0$, this converges to

$$\max_{\lambda}\{\rho_{aa}\sigma_a^2\} = \sigma_{\theta_i}^2/(4(1-r)).$$

In other words, as we move away from the model of pure common values, that is $\sigma_{\theta_i}^2 = 0$, the aggregate volatility larger with some amount of incomplete information. In consequence, the maximum aggregate volatility is not bounded by the aggregate volatility under the complete information equilibrium as it is the case with common values. In fact, the aggregate volatility is increasing without bounds in the variance of the idiosyncratic component even in the absence of variance of the common component $\overline{\theta}$. The latter result in stark contrast to the complete information equilibrium in which the aggregate volatility is unaffected by the variance of the idiosyncratic component. This illustrates in a simple way that in this model the aggregate volatility may result from uncertainty about the aggregate fundamental or the idiosyncratic fundamental.

Earlier, we suggested that the impact of the confounding information on the equilibrium behavior is distinct in the interdependent value environment relative to either the pure private and pure common value environment. We can make this now precise by evaluating the impact the introduction of a public component has in a world of pure idiosyncratic uncertainty. By evaluating the aggregate volatility and ask how much can it be increased by adding a common payoff shock with arbitrarily small variance, we find from (27) that:

$$\frac{\partial \max_{\lambda} \{\rho_{aa}\sigma_a^2\}}{\partial \sigma_{\bar{\theta}}}|_{\sigma_{\bar{\theta}}=0} = \frac{\sigma_{\theta_i}^2}{2(1-r)^{3/2}}.$$

More generally, there is positive interaction between the variance of the idiosyncratic and the common term with respect to the aggregate volatility than can prevail in equilibrium as the cross-derivative inform us:

$$\frac{\partial \max_{\lambda} \{\rho_{aa}\sigma_a^2\}}{\partial \sigma_{\theta_i}\partial \sigma_{\bar{\theta}}} = \frac{\sigma_{\theta_i}^3}{2\left(\sigma_{\bar{\theta}}^2 + (1-r)\sigma_{\theta_i}^2\right)^{3/2}} > 0.$$

Interestingly, for a given variance of the common component, the positive interaction effect as measured by the cross derivatives occurs at finite values of the variance of the idiosyncratic component.

We complete our discussion by returning to the special cases of pure common and pure private values. In Proposition 6 we showed that with common values the information structure that maximizes aggregate volatility is always the complete information equilibrium. By contrast, for pure private values the information structure that maximizes aggregate volatility has to involve a common error term, as the aggregate volatility would otherwise be zero. By a similar argument, the information structure that maximizes dispersion with common values has to involve idiosyncratic noise, as otherwise the equilibrium would trivially have no dispersion. When we identify the information structures that maximize individual volatility under pure common or pure private values, the results are more subtle and depend on the nature of the interaction.

Proposition 10 (Comparative Statics for Common and Independent Values)

1. *If* $\rho_{\theta\theta} = 1$ *and* $r \geq -1$*, then the maximal individual volatility is achieved in the complete information equilibrium; and if* $r < -1$ *then the maximal individual volatility is achieved by a signal with idiosyncratic noise:* $s_i = \bar{\theta} + \varepsilon_i$, $\sigma^2_{\varepsilon_i} = -r - 1$.

2. *If* $\rho_{\theta\theta} = 0$ *and* $r \leq 1/2$*, then the maximal individual volatility is achieved in the complete information equilibrium; and if* $r > 1/2$ *then the maximal individual volatility is achieved by a signal with common noise* $s_i = \Delta\theta_i + \varepsilon$, $\sigma^2_{\varepsilon} = (2r - 1) / (1 - r)$.

The result of Proposition 10.1 follows from the analysis in Bergemann and Morris (2013b). There we show in Proposition 2 that in the model with pure common values, if $r < -1$ then the maximal individual volatility requires less than perfect correlation across actions, or $\rho_{aa} = \rho^2_{a\theta} < 1$. In fact, in Bergemann and Morris (2013b), we consider an application to large Cournot markets, and find that the socially optimal information is in fact provided by the noisy information structure identified in the above Proposition 10.1. More broadly, this indicates that the present results on individual and aggregate volatility would have direct implications for a more extensive welfare analysis, one that we do not pursue here.

The above result illustrates that for pure private and pure common value environments, it is important to consider noisy signals to find the equilibria that maximize aggregate or individual volatility. Put differently, in the exceptional circumstances of pure private and pure common values, the noise has to pick up what otherwise, that is in interdependent value environments, would be picked up by payoff relevant information, as in purification arguments. With interdependent values, the fundamental components $\overline{\theta}$ and $\Delta\theta_i$ provide enough richness to increase or decrease the equilibrium correlation of the actions, and thus it is not necessary to recur to noise.

5 A Priori Restrictions on Information Structures

In the analysis of games of incomplete information, the context and the specific application might suggest natural restrictions on the information structure. For example in oligopoly games, each firm may know its own marginal cost, but be uncertain about the cost of the competing firms. In a trading environment, as in Angeletos and La'O (2013), the trader knows his own technology, but is uncertain about the technology of the other agents. In this section we investigate how the set of feasible BCE changes given certain a priori restrictions on the information structures. A particularly relevant and natural restriction is the case in which each agent i knows his own payoff state θ_i, which we shall analyze in this section. Of course, all of the information structures in which each agent knows his payoff state are contained in the set of unrestricted information structures, and thus we shall characterize a subset of BCE that are equivalent in terms of outcomes to the set of Bayes Nash equilibria in which agents know their own payoff state.

We say that the variables $(\theta_i, \bar{\theta}, a_i, A)$ form a (normal) Bayes correlated equilibrium in which each agent knows his own payoff state if the joint distribution is given by (6), and in addition, the best response is conditioning on the action a_i and the state θ_i :

$$a_i = r\mathbb{E}[A|a_i, \theta_i] + \mathbb{E}[\theta_i|a_i, \theta_i]. \tag{28}$$

The set of feasible BCE when each agent knows his payoff state can be described by an argument similar to the unrestricted case. We simply have to add an additional constraint that reflects the fact that agent i knows, and hence conditions on, his payoff state θ_i. From a qualitative point of view, the most important aspect of restricting the set of possible information structures is the result that the set of feasible correlations is now *no longer* independent of the interaction structure.

Proposition 11 (Characterization BCE Agents Know Own Payoff)
The variables $(\theta_i, \bar{\theta}, a_i, A)$ form a (normal) BCE if and only if the conditions of Proposition 1 hold and in addition:

$$\rho_{a\theta}\sigma_a = \sigma_\theta + r\rho_{a\phi}\sigma_\theta. \tag{29}$$

Thus, imposing the restriction that each agent knows his own payoff state adds a restriction to the characterization of the feasible BCE. But the nature of the restriction is such that the set of feasible correlations is no longer independent of the interaction structure, represented by r.

We continue to describe the set of feasible correlations in the space of correlation coefficients $(\rho_{aa}, \rho_{a\theta})$, and can contrast it directly with the case of unrestricted information structures. The characterization consists of two parts. First, we describe the set of feasible action correlations ρ_{aa} when agents know their own payoff state. If each agent only knows his own payoff state, then the correlation across actions ρ_{aa} is equal to $\rho_{\theta\theta}$, $\rho_{aa} = \rho_{\theta\theta}$. Since the only information of agent i is his payoff state θ_i, the actions of two agents can only be correlated to the extent that their payoff states are correlated. By contrast, if the agents have complete information, the correlation is given, as established earlier, by $\rho_{aa} = \rho^*_{aa}$. We find that the set of feasible action correlations is always between these two quantities, providing the lower and upper bounds. If $r > 0$, then the complete information bound is the upper bound, if $r < 0$, it is the lower bound, and they coincide for $r = 0$. Second, we describe the set of feasible correlations between action and state, $\rho_{a\theta}$, for any feasible ρ_{aa}. More precisely, the set of feasible $\rho_{a\theta}$ is determined by two functions of

ρ_{aa}, which provide the lower and upper bound for the feasible $\rho_{a\theta}$. We denote these functions by $\rho^c_{a\theta}$ and $\rho^i_{a\theta}$ as these bounds are achieved by information structures in which the agents either get a noisy signal of the form given by (25) with either a *common* error term or with an *idiosyncratic* error term. The equilibria with $\rho^c_{a\theta}$ and $\rho^i_{a\theta}$ correspond to Bayes Nash equilibria in which each agent gets an additional signal of the form defined by (25) earlier:

$$s_i = \lambda \Delta\theta_i + (1-\lambda)\bar{\theta} + \varepsilon_i,$$

with either a common or an idiosyncratic error term. The value λ of the convex weight itself is irrelevant as long as $\lambda \neq 1/2$, as the agent can always filter the additional information relative to his knowledge of his own payoff state θ_i. For a given ρ_{aa}, $\rho^c_{a\theta}$ and $\rho^i_{a\theta}$ represent the solutions of the following equations:

$$\frac{1}{r}(-(1-r)(\rho^c_{a\theta})^2 + 1 - r\rho_{aa}) - \rho^c_{a\theta}\sqrt{(1-\rho_{aa})(1-\rho_{\theta\theta})} = 0, \tag{30}$$

$$\frac{1}{r}((\rho^i_{a\theta})^2 - (1 - r\rho_{aa})) - \rho^i_{a\theta}\sqrt{\rho_{aa}\rho_{\theta\theta}} = 0. \tag{31}$$

Proposition 12 (Bounds on BCE Correlation Coefficients)

1. *The action correlation* ρ_{aa} *is BCE feasible if and only if* $\rho_{aa} \in [\min\{\rho^*_{aa}, \rho_{\theta\theta}\}, \max\{\rho^*_{aa}, \rho_{\theta\theta}\}]$;

2. *For all* $\rho_{aa} \in [\min\{\rho^*_{aa}, \rho_{\theta\theta}\}, \max\{\rho^*_{aa}, \rho_{\theta\theta}\}]$, *the correlation pair* $(\rho_{aa}, \rho_{a\theta})$ *is BCE feasible if and only if*

$$\rho_{a\theta} \in [\min\{\rho^c_{a\theta}, \rho^i_{a\theta}\}, \max\{\rho^c_{a\theta}, \rho^i_{a\theta}\}].$$

In Figure 5, we illustrate the Bayes correlated equilibrium set for different values of the interaction parameter r with a given interdependency $\rho_{\theta\theta} = 1/2$. Each interaction value r is represented by a differently colored pair of lower and upper bounds. For each value of r, the entire set of BCE is given by the area enclosed by the lower and upper bound. Notably, the bounds $\rho^c_{a\theta}(\rho_{aa})$ and $\rho^i_{a\theta}(\rho_{aa})$ intersect in two points, corresponding to each agent knowing his payoff state θ_i only (at $\rho_{aa} = \rho_{\theta\theta} = 1/2$) and to complete information, at the low or high end of ρ_{aa} depending on the nature of the interaction, respectively. In fact these, and only these, two points, are also noise-free equilibria of the unrestricted set of BCE. When $r \geq 0$, the upper bound is given by signals with idiosyncratic error terms, while the lower bound is given by signals with common error terms, and conversely for $r \leq 0$. With $r > 0$, as the additional signal contains an idiosyncratic error, it forces the agent to stay closely to his known payoff state, as this is where the desired correlation with the other agents arises, and only slowly incorporate the information about the idiosyncratic state, thus overall tracking as close as possible his own payoff state θ_i, and achieving the upper bound. The argument for the lower bound follows a similar logic.

Thus the present restriction to information structures in which each agent knows at least his own payoff state dramatically reduces the set of possible BCE. With the exception of these two points, all elements of the smaller set are in the interior of unrestricted set of BCE. Moreover, the nature of the interaction has a profound impact on the shape of the correlation that can arise in equilibrium, both in terms of the size as well as the location of the set in the unit square of $(\rho_{aa}, \rho_{a\theta})$.

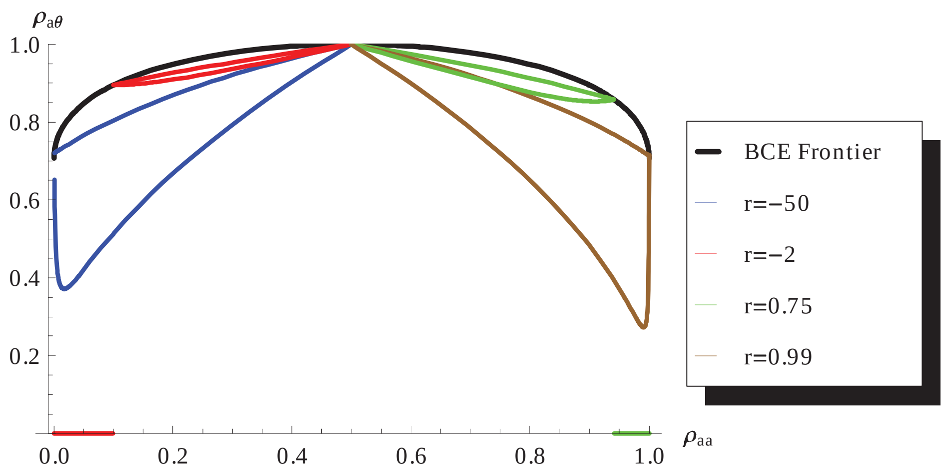

Figure 5: BCE with Prior Information $\rho_{\theta\theta} = 1/2$

Finally, we observe that for the case of pure common or pure private values, the set of Bayes correlated equilibria when agents know their own payoff state is degenerate. Under either pure common or pure private values if agents know their own payoff state then there is no uncertainty left, and thus the only possible outcome corresponds to the complete information outcome. In consequence, there is then either no aggregate volatility if the payoff states are pure private values or there is no dispersion in actions if payoff states are pure common values, and correspondingly $\rho_{a\theta} = 1$, but ρ_{aa} is either 0 or 1 depending on private or common values.

6 Pairwise Interaction

We now extend our model to allow for heterogenous interaction among agents. We shall not present the most general possible framework, but rather propose a symmetric model of interaction that includes individual as well as aggregate interaction. In particular, each agent's best response is now given by a linear function with weights on the aggregate (or average) action and an individual action of a specifically matched agent. The interaction with an individual agent is the result of match between agent i and j, and the matched pair can arise as the result of a random matching or an assortative matching.

6.1 Statistical Model of Pairwise Interaction

Agent $i \in [0,1]$ chooses an action $a_i \in \mathbb{R}$ and is assumed to have a quadratic payoff function which is function of his action a_i, the action taken by his match (a_j), the average action A taken by the entire population and his individual payoff state θ_i:

$$u_i : \mathbb{R}^4 \to \mathbb{R}.$$

In consequence, we write the best response function of agent i as follows,

$$a_i = r_a \mathbb{E}[a_j|\mathcal{I}_i] + r_A \mathbb{E}[A|\mathcal{I}_i] + \mathbb{E}[\theta_i|\mathcal{I}_i]. \tag{32}$$

If $r_a = 0$ then we have a model of aggregate interactions as previously studied. If $r_A = 0$ then there is only pairwise interaction between agents. Of course, we maintain that the payoff states in the entire population have a given distribution to which the payoff states of particular pairs must conform to when aggregated. To keep the best responses bounded we assume that $|r_A| + |r_a| < 1$.

We will allow for matched agents to be assorted according to their payoff states, thus we assume that if agents (i,j) are matched, then

$$\text{cov}(\theta_i, \theta_j) = \rho_{\varphi\varphi}\sigma_\theta^2$$

where the new correlation coefficient $\rho_{\varphi\varphi}$ has to satisfy $\rho_{\varphi\varphi} \in [2\rho_{\theta\theta} - 1, 1]$. Under random matching, we have that $\rho_{\varphi\varphi} = \rho_{\theta\theta}$, yet this will not be the general case and thus $\rho_{\varphi\varphi}$ will be the parameter of assortative matching. We say there is *positive assortative matching* if $\rho_{\varphi\varphi} \geq \rho_{\theta\theta}$ and we will say there is *negative assortative matching* if $\rho_{\varphi\varphi} \leq \rho_{\theta\theta}$.

Thus, given a matched pair of agents (i,j), we can describe the basic uncertainty by:

$$\begin{pmatrix} \theta_i \\ \theta_j \\ \mathbb{E}_i[\theta_i] \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu_\theta \\ \mu_\theta \\ \mu_\theta \end{pmatrix}, \begin{pmatrix} \sigma_\theta^2 & \rho_{\varphi\varphi}\sigma_\theta^2 & \rho_{\theta\theta}\sigma_\theta^2 \\ \rho_{\varphi\varphi}\sigma_\theta^2 & \sigma_\theta^2 & \rho_{\theta\theta}\sigma_\theta^2 \\ \rho_{\theta\theta}\sigma_\theta^2 & \rho_{\theta\theta}\sigma_\theta^2 & \rho_{\theta\theta}\sigma_\theta^2 \end{pmatrix} \right), \tag{33}$$

where we recall that $\mathbb{E}_i[\cdot]$ is used to denote the average taken across the entire population, and thus $\mathbb{E}_i[\theta_i]$ is the common component of the payoff state of every agent. We assume that the joint normal distribution given by (33), is the *common prior*.

We require some additional notation to describe the random variables associated with the matching of agents. Given a match (i, j) we define:

$$\theta_i^+ \triangleq \frac{\theta_i + \theta_j}{2} \quad \text{and} \quad \theta_i^- \triangleq \frac{\theta_i - \theta_j}{2},$$

and subsequently, we use the $+$ and $-$ superscript to denote the average and weighted difference of any pair of variables that is generated by a pair of matched agents (i, j). Thus, an equivalent way to describe the basic uncertainty is given by, where we denote by extension $\Delta\theta_i^+ = \theta_i^+ - \bar{\theta}$:

$$\begin{pmatrix} \theta_i^- \\ \Delta\theta_i^+ \\ \bar{\theta} \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \\ \mu_\theta \end{pmatrix}, \begin{pmatrix} (1-\rho_{\varphi\varphi})\frac{\sigma_\theta^2}{2} & 0 & 0 \\ 0 & (1+\rho_{\varphi\varphi}-2\rho_{\theta\theta})\frac{\sigma_\theta^2}{2} & 0 \\ 0 & 0 & \rho_{\theta\theta}\sigma_\theta^2 \end{pmatrix}\right). \tag{34}$$

Note that the formulations (33) and (34) are equivalent, yet, by writing the model in terms of orthogonal random variables we can simplify the analysis and interpret the results in a easier way. Finally, as shorthand notation, we define,

$$\rho_{\varphi\varphi}^+ \triangleq \frac{1}{2}(1+\rho_{\varphi\varphi}-2\rho_{\theta\theta}), \quad \rho_{\varphi\varphi}^- \triangleq \frac{1}{2}(1-\rho_{\varphi\varphi}),$$

and we observe that under random matching $\rho_{\varphi\varphi}^+ = \rho_{\varphi\varphi}^-$.

6.2 BCE with Pairwise Matching

As before, we will define a BCE with pairwise matching and characterize it.

Definition 4 (Bayes Correlated Equilibrium)
The variables $(\theta_i, \theta_j, \bar{\theta}, a_i, a_j, A)$ form a symmetric and normally distributed Bayes Correlated Equilibrium with pairwise matching if their joint distribution is given by a multivariate normal distribution and for every matched agent i (to some j), and for all a_i :

$$a_i = r_a\mathbb{E}[a_j|a_i,\theta_i] + r_A\mathbb{E}[A|a_i,\theta_i] + \mathbb{E}[\theta_i|a_i,\theta_i]. \tag{35}$$

With minor abuse of notation, we now refer to $\mathbb{V}$ as the larger variance-covariance matrix of the joint distribution of $(\theta_i, \theta_j, \bar{\theta}, a_i, a_j, A)$. The matrix $\mathbb{V}$ can be reduced in its dimensions by the same use of symmetry and orthogonal random variables as earlier in Lemma 1. Relative to the benchmark model with aggregate interaction we introduce one new exogenous variable, the correlation coefficient $\rho_{\varphi\varphi}$ of payoff state between matched agents. We then require two extra endogenous correlation coefficients to fully specify an equilibrium: (i) the correlation of actions between matched agents, denoted by ρ_{aa}, as opposed to the correlation between the action of an individual and the aggregate action, now denoted by ρ_{aA}, and (ii) the correlation between an individual agent's action and the payoff state of his match, denoted by $\rho_{a\varphi}$. As before, the set of BCE is equivalent to the set of BNE for some information structure by the same general argument as in Bergemann and Morris (2013a). Analogous to the benchmark model we now characterize the set of BCE.

Proposition 13

The variables $(\theta_i, \theta_j, \bar{\theta}, a_i, a_j, A)$ form a Bayes correlated equilibrium if and only if:

1. *the individual variance is* $\sigma_a^2 = \left(\frac{\rho_{a\theta}\sigma_\theta}{1-r_a\rho_{aa}-r_A\rho_{aA}}\right)^2$;
2. *the aggregate variance is* $\sigma_A^2 = \rho_{aA} \cdot \left(\frac{\rho_{a\theta}\sigma_\theta}{1-r_a\rho_{aa}-r_A\rho_{aA}}\right)^2$;
3. *the correlation coefficients satisfy:*

$$\rho_{a\theta} \geq 0, \;\; \rho_{aA}\rho_{\theta\theta} \geq \rho_{a\phi}^2, \tag{36}$$

and

$$1 - \rho_{aa} \geq \frac{(\rho_{a\theta} - \rho_{a\varphi})^2}{1 - \rho_{\varphi\varphi}}, \quad 1 + \rho_{aa} - 2\rho_{aA} \geq \frac{(\rho_{a\theta} - 2\rho_{a\phi} + \rho_{a\varphi})^2}{1 - 2\rho_{\theta\theta} + \rho_{\varphi\varphi}}. \tag{37}$$

Note that the characterization proceeds in the same way as in the benchmark model. As before, the constraints on the feasible correlations are purely statistical, and thus do not depend on the nature of the interaction structure. Thus it is easy to see that all restrictions on correlations for the benchmark model must also hold with random matching, as these are purely statistical constraints. We should add that the separation between statistical constraints and incentives constraints remains to hold in general linear interaction networks, and thus we could extend the present analysis to network models such as Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Salehi (2012), but now allowing for incomplete information.

Similar as in the benchmark model with aggregate interaction, we can define a BCE with matching and under the restriction that each agent i knows his payoff state θ_i. As before, the joint distribution of variables is given by (6), and the best response conditions are given by:

$$a_i = r_a\mathbb{E}[a_j|a_i,\theta_i] + r_A\mathbb{E}[A|a_i,\theta_i] + \mathbb{E}[\theta_i|a_i,\theta_i], \quad \forall i, \forall a_i, \forall \theta_i. \tag{38}$$

In this case the characterization of the BCE with matching in which agents know their own payoff state is given by:

Proposition 14 (Characterization of BCE when Agents Know Own Payoff State) *The variables $(\theta_i, \theta_j, \bar{\theta}, a_i, a_j, A)$ form a (normal) Bayes correlated equilibrium if and only if the conditions of Proposition 13 hold and*

$$\rho_{a\theta}\sigma_a = \sigma_\theta + r_a\rho_{a\varphi}\sigma_a + r_A\rho_{a\phi}\sigma_a. \tag{39}$$

6.3 BCE with Pairwise Matching: Comparative Statics

The pairwise matching adds some features to our benchmark model which are worth highlighting. Henceforth we will assume random matching, that is $\rho_{\varphi\varphi} = \rho_{\theta\theta}$, unless otherwise noted. First, the comparative statics with respect to the pairwise interaction parameter r_a are distinct from those with the aggregate interaction parameter r_A. For the model with aggregate interaction, we showed that the volatility (individual, aggregate and dispersion) were monotone increasing in r_A, see Proposition 8. But this is no longer true with respect to the pairwise interaction r_a. We illustrate this in the absence of aggregate interaction, that is we set $r_A = 0$. In this case, the maximum individual and aggregate volatility has an interior minimum, and maxima at the positive *and* the negative boundary of the interaction, namely at $|r_a| = 1$, as illustrated by Figure 6 and 7. With individual interaction, we have positive correlation among the actions with strategic complements, and negative correlation with strategic substitutes. Now, in the aggregate interaction model, the individual actions cannot *all* be negatively correlated, but this restriction is not present anymore at the level of pairwise interaction. Here, as the strategic substitute property becomes stronger, that is as r_a decreases and approaches -1, the individual agent seeks to respond strongly *and* negatively to his expectation about the matched agent's action. But now, a common error in the information for both agents will introduce positive volatility among the pair, and in fact across all the pairs, hence allowing for aggregate volatility even in the presence of strong negative interaction.

In Figure 6 we display the maximum individual volatility for the benchmark model, $r_a = 0$, and the pairwise interaction model, $r_A = 0$, with *random matching* for different parameters $\rho_{\theta\theta}$ of interdependence, without any restrictions on the information structure. With pairwise interaction, the maximum individual volatility is in minimal at an interior level of $r_a = 0$, while with aggregate interaction the maximum individual volatility is always increasing in r_A. We observe in Figure 7 that the aggregate volatility requires a larger absolute level of substitutability to display increasing aggregate volatility, but eventually as $r_a \to -1$, it behaves as the individual volatility.

Second, as the interaction arises in matched pairs, there is an additional element of uncertainty for each agent, namely the payoff state of the matched agent. This induces particularly striking

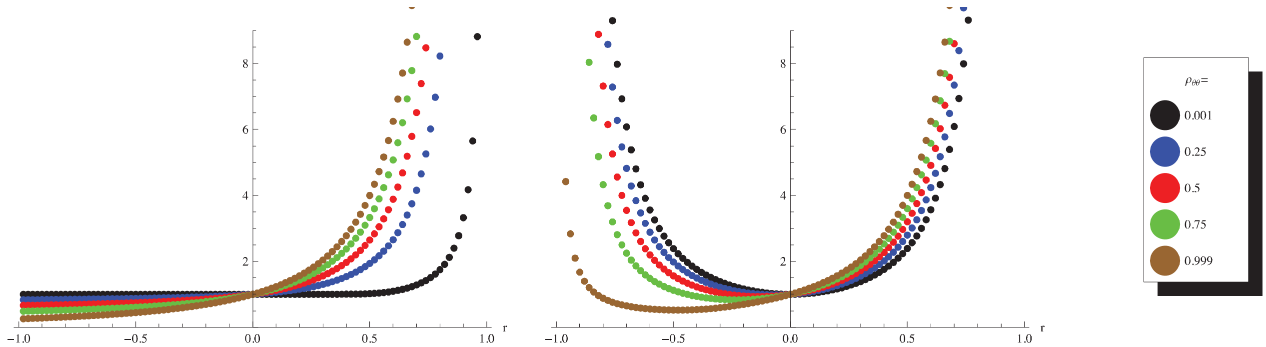

Figure 6: Maximal individual volatility: aggregate vs. pairwise interaction.

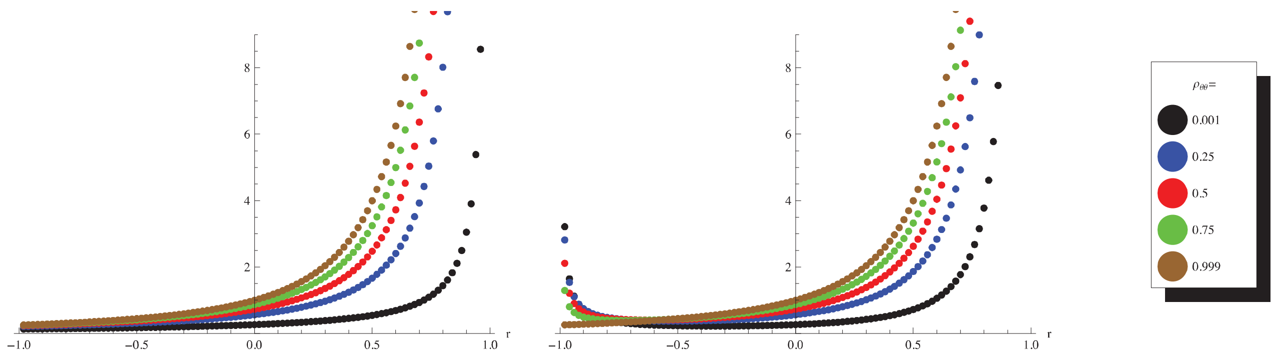

Figure 7: Maximal aggregate volatility: aggregate vs. pairwise interaction.

results in the pure private value model where the payoff states are independently distributed, $\rho_{\theta\theta} = 0$, and the agents know their own payoff state. Under these assumptions, the model with aggregate interaction has a unique equilibrium, the complete information Nash equilibrium, in which there is zero aggregate volatility. By contrast, if we impose random matching and pairwise interactions, $r_A = 0, r_a \neq 0$, then this result does no longer hold true. The agents do not have complete information anymore when they know their payoff state, as there is still uncertainty about the payoff state of their matched partner. The residual uncertainty about the specific payoff state of the matched partner means that it is possible to support equilibria in which the aggregate volatility of actions is not identically 0, even though there is no aggregate uncertainty about the payoff states.

In Figure 8 we plot the maximum aggregate volatility for the model with aggregate interaction: $r_a = 0$, and the model with pairwise interaction: $r_A = 0$, for different parameters of payoff interdependency, while imposing the informational restriction that each agent knows his own

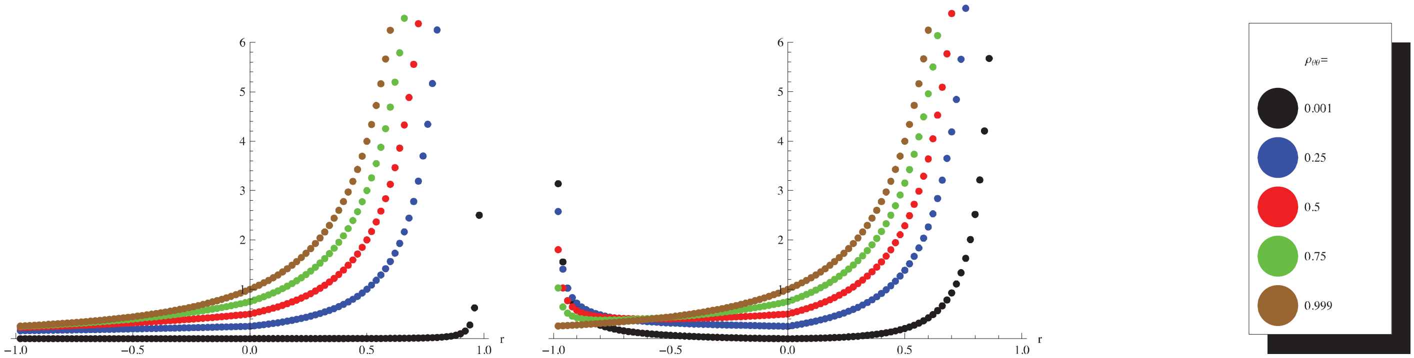

Figure 8: Maximal aggregate volatility with known payoff type: aggregate vs. pairwise interaction.

payoff state. Once again, the comparative statics and the behavior of the equilibrium changes with respect to the interaction parameter r and the payoff interdependency $\rho_{\theta\theta}$. As $r_a \to -1$, each agent reacts strongly to the idiosyncratic shock of his partner as he seeks negative correlation. In consequence, the highest aggregate variance is achieved when each agent confounds the idiosyncratic shock of his partner with the aggregate shock, and thus react very strongly to the aggregate shock. This also explains the shape of the maximum aggregate volatility with respect to the pairwise interaction parameter in Figure 8. For an increase in the aggregate volatility to occur, it is necessary for agents to be positively correlated with the population average, as this means that for a fixed individual volatility the amount of aggregate volatility is bigger. But, when $r_a < 0$, then it is optimal for a given agent to be negatively correlated with his partner to increase the responsiveness of each agent to his signal. Yet, there is a statistical constraint on how much agents can negatively correlate with their partner and yet positively correlate with the population average. This restriction imposes a balance on how much the agents can be positively correlated with the average while still be negatively correlated with their individual partners, leading to lower aggregate volatility with strategic substitutes than with strategic complements.

Third, the behavior in the pairwise interaction model is in even starker contrast to the aggregate interaction model if we were to display positive (or negative) assortative matching. In Figure 9 we remain with the pairwise matching model and pure private values $\rho_{\theta\theta} = 1$. We display the maximum aggregate volatility with restricted information structures and unrestricted information structures, left to right respectively. By comparison with Figure 8, which considers random matching at a range of Interdepedent values, we find in Figure 9 that we can recover a very similar pattern of aggregate volatility with assortative matching, ranging from random matching with $\rho_{\varphi\varphi} = 0$ to perfect positive assortative matching $\rho_{\varphi\varphi} = 1$ alone, without relying at all on correlation in the common component.

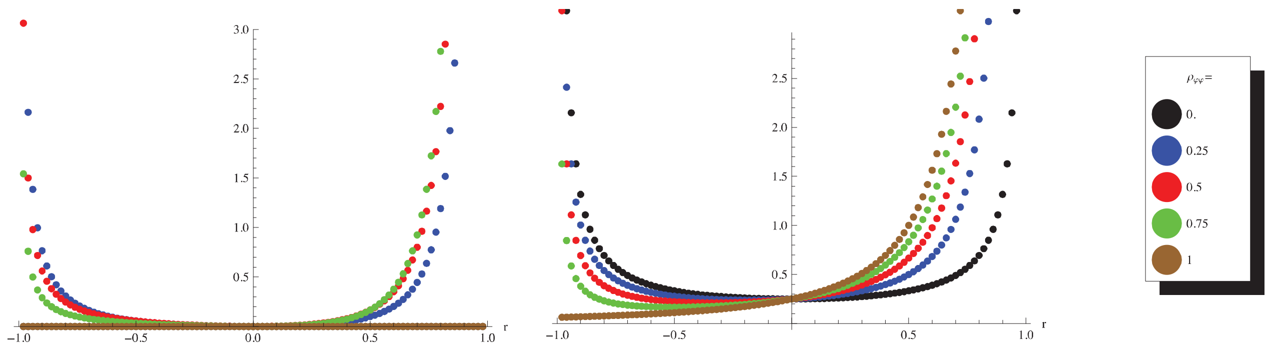

Figure 9: Maximal aggregate volatility for assortative matching with pure private values: restricted vs. unrestricted information structures.

7 Discussion

In this section we review the main results of the paper and at the same time provide a very broad connection to other papers found in the literature. We will explain how our results connect to the results found in different papers, and explain how one can understand them in a common framework.

The Drivers of Responsiveness to Fundamentals A literature in which the responsiveness to fundamentals has been widely studied is the study of nominal rigidities driven by informational frictions. Lucas (1972) established informational frictions can cause an economy to respond to monetary shocks that would not have real effects under complete information. In the model agents are subject to real and monetary shocks, but only observe prices. Since the unique signal they observe does not allow them to disentangle the two shocks, the demand of agents (in real terms) responds to monetary shocks. Thus, even when under complete information there would be no "real" effects of monetary shocks, as the agents confound the shocks which allows monetary shocks to have real effects.

In more recent contributions, Hellwig and Venkateswaran (2009) and Mackowiak and Wiederholt (2009) both present dynamic models in which the intertemporal dynamics are purely informational and in which the maximization problem in each period is strategically equivalent to the one in the present model with aggregate interactions only ($r_a = 0$).

The informational assumptions in Hellwig and Venkateswaran (2009) are derived from a micro-founded model. Each agent observes a signal that is a result of his past transaction, thus a result of past actions and past shocks. The persistence of shocks means that such a signal serves as a predictor of future shocks. The main feature of the signal is the property that it is composed of idiosyncratic and common shocks, and hence the informational content of the signal is similar to the one-dimensional signal analyzed in Section 3.4. They suggest that in empirically relevant information structure the idiosyncratic shocks might be much larger than the common shocks, and thus agents have very little information about the realization of the common shocks. Thus, even when agents have little information about the common shocks, they respond strongly to the idiosyncratic shock in order to be able to respond to the common shock at all.

By contrast, the informational assumptions in Mackowiak and Wiederholt (2009) are derived from a model of rational inattention. Here the agents receive separate signals on the idiosyncratic and common part of their shocks. In consequence, the response to each signal can be computed independently and the responsiveness to each signal is proportional to the quality of information. However, the amount of noise on each signal is chosen optimally by the agents, subject to a global informational constraint motivated by limited attention. Again, motivated by empirical observations, it is assumed that the idiosyncratic shocks are much larger than common shocks. This in turn results in agents choosing to have better information on the idiosyncratic shocks rather than the common shocks. This implies that the response to the common shock is sluggish as the information on it is poor.

Both papers have very contrasting results on the response of the economy to common shocks, although the underlying assumptions are quite similar. Note that not only the strategic environment is similar, but also the fact that agents have little information on common valued shocks. Their contrasting analysis thus indicates the extent to which the response to the fundamentals may be driven by the informational composition of the signals.

The Role of Heterogeneous Interactions and the Dimensions of Uncertainty In a recent paper Angeletos and La'O (2013) demonstrate how informational frictions can generate aggregate fluctuations in a economy even without any aggregate uncertainty. They consider a neoclassical economy with trade, which in its reduced form can be interpreted with linear best responses as in (32) with $r_A = 0$ and $\rho_{\theta\theta} = 0$. In their environment, each agent is also assumed to know his own payoff state. In such a model, but with aggregate rather than pairwise interaction, there would be a unique equilibrium as the agents would face no uncertainty at all. Yet, given the pairwise interaction, there is a dimension of uncertainty, which enables aggregate fluctuations to arise in the absence of aggregate payoff uncertainty. If each agent observes the payoff state of his partners through a noisy signal with a common error term, then the aggregate volatility will not be equal to 0 as each agent responds to same common error term in the individual signal.

Our analysis suggest the emergence of aggregate volatility from the pairwise trading does not hinge on any particular interaction structure. The only necessary condition for the aggregate volatility to emerge is the presence of some residual uncertainty. Moreover, even though one might be skeptical about fluctuations arising merely from common noise error terms, it should be clear that the same mechanism could explain how the response to a small aggregate fundamental could be amplified. Thus, it is not necessary to have strong aggregate fluctuations in the fundamentals or strong aggregate interactions to generate large aggregate fluctuations, but these can arise from strong local interactions, as in the model of Angeletos and La'O (2013) with pairwise interactions. More generally, the present analysis also demonstrates that different sources of uncertainty and their impact on the volatility of outcomes cannot be analyzed independently in the absence of complete information.

In this paper we have focused on the information structures that are completely exogenous to the action taken by players, that is, all information agents get is concerning the fundamentals. We could have also considered information structures in which each agent gets a signal that is informative about the average action taken by other players, and then compute the set of outcomes that are consistent with a rational expectations equilibrium. The set of feasible outcomes that are consistent with a rational expectation equilibrium would constitute a subset of the outcomes when only information on the fundamentals is considered, yet we leave the characterization of such outcomes as work for future research. A recent contribution by Benhabib, Wang, and Wen (2013) (and related Benhabib, Wang, and Wen (2012)) consider a model with this kind of information structure. Their model has in its reduced form a linear structure like the one we have analyzed, yet agents can get a noisy signal on the average action taken by other players. They show that the equilibria can display non-fundamental uncertainty, which can be thought of as sentiments. From a purely game theoretic perspective, their model is equivalent to our benchmark model with common values, but agents only receive a (possibly noisy) signal on the average action taken by all players.

The Drivers of Co-Movement in Actions One of the first results presented here is the fact the co-movement in actions is not only driven by fundamentals, but also by the specific information structure that the agents have access to. Of course, under a fixed information structure, in which agents receive possibly multi-dimensional signals, the co-movement in the actions can only change with the interaction parameters. But more generally, by changes in the informational environment would also lead to changes in the co-movement of actions. Moreover, we showed that any feasible correlations structure can be rationalized for some information structure, independent of the interaction parameters. The importance of the information structure has been recognized in macroeconomics to offer explanations to open empirical puzzles. For example, Veldkamp and Wolfers (2007) provide an explanation of why there is more co-movement in output across industries than the one that could be predicted by co-movement in TFP shocks and complementarities in production. They suggest that the sectoral industries might be better informed about the common shocks than about the idiosyncratic shocks, which in turn would generate the excess co-movement. They suggest that this information structure might be prevalent since information has a fixed cost for acquiring but no cost of replicating and thus different industries may typically have more information on common shocks than on idiosyncratic shocks.

8 Appendix

The appendix collects the omitted proofs from the main body of the text.

Proof of Lemma 1. Clearly, we have that $\mu_a = \mu_A$, as it follows from the law of iterated expectations. By the previous definition (and decomposition) of the idiosyncratic state θ_i, we observe that the expectations of the following products all agree: $\mathbb{E}_i[a_i\bar{\theta}] = \mathbb{E}_i[A\theta_i] = \mathbb{E}_i[A\bar{\theta}]$. This can be easily seen as follows:

$$\mathbb{E}[\theta_i A] = \mathbb{E}[\bar{\theta}A] + \mathbb{E}[\Delta\theta_i A] = \mathbb{E}[\bar{\theta}A] + \mathbb{E}[A \cdot \underbrace{\mathbb{E}[\Delta\theta_i|A]}_{=0}] = \mathbb{E}[\bar{\theta}A],$$

where we just use the law of iterated expectations and the fact that the expected value of a idiosyncratic variable conditioned on an aggregate variable must be 0. Thus:

$$cov(a_i, \bar{\theta}) = cov(A, \theta_i) = cov(A, \bar{\theta}) = cov(a_i, \theta_j) = \mathbb{E}[a_i\theta_j] - \mu_a\mu_\theta = \rho_{a\phi}\sigma_\theta\sigma_a.$$

Similarly, since we consider a symmetric Bayes correlated equilibrium, the covariance of the actions of any two individuals, a_i and a_j, which is denoted by $\rho_{aa}\sigma_a^2$, is equal to the aggregate variance. Once again, this can be easily seen as follows,

$$\mathbb{E}[a_i a_j] = \mathbb{E}[A^2] + \mathbb{E}[A\Delta a_j] + \mathbb{E}[\Delta a_i A] + \mathbb{E}[\Delta a_i \Delta a_j] = \mathbb{E}[A^2],$$

where in this case we need to use that the equilibrium is symmetric and thus $\mathbb{E}[\Delta a_i \Delta a_j] = 0$. Thus, we have

$$\sigma_A^2 = cov(a_i, a_j) = cov(A, a_i) = \rho_{aa}\sigma_a^2. \blacksquare$$

Proof of Proposition 3. From the first order conditions we know that:

$$a_i = \mathbb{E}[rA + \theta_i | a_i],$$

and in this case we know that $a_i = \nu(\lambda)((1-\lambda)\bar{\theta} + \lambda\Delta\theta_i)$, thus $A = \nu(\lambda)((1-\lambda)\bar{\theta})$. Multiplying by a_i and appealing to the law of iterated expectations:

$$a_i^2 = \mathbb{E}[rAa_i + \theta_i a_i | a_i],$$

implies that:

$$\nu(\lambda)((1-r)(1-\lambda)^2\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta})) = ((1-\lambda)\rho_{\theta\theta} + \lambda(1-\rho_{\theta\theta})),$$

and solving for $\nu(\lambda)$ yields the expression in (22).

We can compute the variance and correlation coefficients by inserting (22) and obtain:

$$\sigma_a^2 = \nu(\lambda)^2((1-\lambda)^2\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta}));$$

and from

$$\rho_{aa}\sigma_a^2 = \text{cov}(a_i, a_j) = \nu(\lambda)^2\mathbb{E}[((1-\lambda)\theta + \lambda\Delta\theta_i)((1-\lambda)\theta + \lambda\Delta\theta_j)] = \nu(\lambda)^2(1-\lambda)^2\rho_{\theta\theta}),$$

we obtain

$$\rho_{aa} = \frac{(1-\lambda)^2\rho_{\theta\theta}}{(1-\lambda)^2\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta})};$$

similarly from

$$\rho_{a\theta}\sigma_a\sigma_\theta = \text{cov}(a_i, \theta_i) = \nu(\lambda)\mathbb{E}[((1-\lambda)\theta + \lambda\Delta\theta_i)\theta_i] = \nu(\lambda)(1-\lambda)\rho_{\theta\theta} + \lambda(1-\rho_{\theta\theta}),$$

we obtain:

$$\rho_{a\theta} = \frac{(1-\lambda)\rho_{\theta\theta} + \lambda(1-\rho_{\theta\theta})}{\sqrt{(1-\lambda)^2\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta})}};$$

and finally from

$$\rho_{a\phi}\sigma_a\sigma_\theta = \text{cov}(a_i, \bar{\theta}) = \nu(\lambda)\mathbb{E}[((1-\lambda)\theta + \lambda\Delta\theta_i)\bar{\theta}] = \nu(\lambda)(1-\lambda)\rho_{\theta\theta}$$

we obtain:

$$\rho_{a\theta} = \frac{(1-\lambda)\rho_{\theta\theta}}{\sqrt{(1-\lambda)^2\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta})}}.\blacksquare$$

Proof of Proposition 4. ($\Leftarrow$) We first prove that if the variables $(\theta_i, \bar{\theta}, a_i, A)$ form a BNE for some information structure $\mathcal{I}_i$ (and associated signals), then the variables $(\theta_i, \bar{\theta}, a_i, A)$ also form a BCE. Consider the case in which agents receive normally distributed signals through the information structure $\mathcal{I}_i$, which by minor abuse of notation also serves as conditioning event. Then in any BNE of the game, we have that the actions of the agents are given by:

$$a_i = r\mathbb{E}[A|\mathcal{I}_i] + \mathbb{E}[\theta_i|\mathcal{I}_i], \quad \forall i, \forall \mathcal{I}_i, \tag{40}$$

and since the information is normally distributed, the variables $(\theta_i, \bar{\theta}, a_i, A)$ are jointly normal as well. By taking the expectation of (40) conditional on the information set $\mathcal{I}_i' = \{\mathcal{I}_i, a_i\}$ we get:

$$\begin{aligned} \mathbb{E}[a_i|\mathcal{I}_i, a_i] = a_i &= r\mathbb{E}[\mathbb{E}[A|\mathcal{I}_i]|\mathcal{I}_i, a_i] + \mathbb{E}[\mathbb{E}[\theta_i|\mathcal{I}_i]|\mathcal{I}_i, a_i] \\ &= r\mathbb{E}[A|\mathcal{I}_i, a_i] + \mathbb{E}[\theta_i|\mathcal{I}_i, a_i]. \end{aligned} \tag{41}$$

In other words, agents know the recommended action they are supposed to take, and thus, we can assume that the agents condition on their own actions. By taking expectations of (41) conditional on $\{a_i\}$ we get:

$$\begin{aligned}\mathbb{E}[a_i|a_i] = a_i &= r\mathbb{E}[\mathbb{E}[A|\mathcal{I}_i, a_i]|a_i] + \mathbb{E}[\mathbb{E}[\theta_i|\mathcal{I}_i, a_i]|a_i] \\ &= r\mathbb{E}[A|a_i] + \mathbb{E}[\theta_i|a_i], \end{aligned} \quad (42)$$

where we used the law of iterated expectations. In other words, the information contained in $\{a_i\}$ is a sufficient statistic for agents to compute their best response, and thus the agents compute the same best response if they know $\{\mathcal{I}_i, a_i\}$ or if they just know $\{a_i\}$. Yet, looking at (42), by definition $(\theta_i, \bar{\theta}, a_i, A)$ form a BCE.

($\Rightarrow$) We now prove that if $(\theta_i, \bar{\theta}, a_i, A)$ form a BCE, then there exists an information structure $\mathcal{I}_i$ such that the variables $(\theta_i, \bar{\theta}, a_i, A)$ form a BNE when agents receive this information structure. We consider the case in which the variables $(\theta_i, \bar{\theta}, a_i, A)$ form a BCE, and thus the variables are jointly normal and

$$a_i = r\mathbb{E}[A|a_i] + \mathbb{E}[\theta_i|a_i]. \quad (43)$$

Since the variables are jointly normal we can always find $\nu \in \mathbb{R}$ and $\lambda \in [-1, -1]$, such that:

$$a_i = \nu(\lambda\Delta\theta_i + (1-\lambda)\bar{\theta} + \varepsilon_i),$$

where the variables (λ, ν) and the random variable ε are defined by the following equations of the BCE equilibrium distribution:

$$\nu\lambda = cov(a_i, \Delta\theta_i), \qquad \nu(1-|\lambda|) = cov(a_i, \bar{\theta}),$$

and

$$\varepsilon = \frac{a_i - cov(a_i, \Delta\theta_i)\Delta\theta_i - cov(a_i, \bar{\theta})\bar{\theta}}{\nu}.$$

Now consider the case in which agents receive a one-dimensional signal

$$s_i \triangleq \frac{a_i}{\nu} = (\lambda\Delta\theta_i + (1-\lambda)\bar{\theta} + \varepsilon_i).$$

Then, by definition, we have that:

$$a_i = \nu s_i = r\mathbb{E}[A|a_i] + \mathbb{E}[\theta_i|a_i] = r\mathbb{E}[A|s_i] + \mathbb{E}[\theta_i|s_i],$$

where we use the fact that conditioning on a_i is equivalent to conditioning on s_i. Thus, when agent i receives information structure (and associated signal s_i): $\mathcal{I}_i = \{s_i\}$, then agent i taking action $a_i = \nu s_i$ constitutes a Bayes Nash equilibrium, as it complies with the best response condition. Thus, the distribution $(\theta_i, \bar{\theta}, a_i, A)$ forms a BNE when agents receive signals $\mathcal{I}_i = \{s_i\}$. ■

Proof of Proposition 5. By rewriting the constraints (14) of Proposition 1 we obtain:

1. $\rho_{\theta\theta}\rho_{aa} - (\rho_{a\phi})^2 \geq 0$;

2. $(1-\rho_{aa})(1-\rho_{\theta\theta}) - (\rho_{a\theta} - \rho_{a\phi})^2 \geq 0$.

If $\psi(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$ is strictly increasing, then in the optimum the above inequality (2) must bind. Moreover, if the constraint (1) does not bind, then we can just increase $\rho_{a\theta}$ and $\rho_{a\phi}$ in equal amounts, without violating (2) and increasing the value of ψ. Thus, in the maximum of ψ we must have that both bind. Moreover, since $\psi(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$ is strictly increasing in $\rho_{a\theta}$ and weakly increasing in $\rho_{a\phi}$, it is clear that the maximum will be achieved positive root of (18). ■

Proof of Proposition 6. If $\rho_{\theta\theta} = 1$, then we must solve:

$$\max_{\rho_{a\theta}, \rho_{aa}} \left\{ \rho_{aa} \cdot \left(\frac{\rho_{a\theta}\sigma_\theta}{1 - r\rho_{aa}} \right)^2 \right\},$$

such that $\rho_{aa} \geq \rho_{a\theta}^2$. We observe that the objective function is increasing in ρ_{aa} and $\rho_{a\theta}$, and thus we must have that the constraint is satisfied as an equality. Thus the problem is equivalent to:

$$\max_{\rho_{aa} \in [0,1]} \left\{ \frac{\rho_{aa}\sigma_\theta}{1 - r\rho_{aa}} \right\}.$$

Since $|r| < 1$, we once again have that the objective function is increasing in ρ_{aa}, thus the maximum is achieved at $\rho_{aa} = \rho_{a\theta} = 1$, which is the complete information equilibrium. ■

Proof of Proposition 7. Given a noise-free equilibrium parametrized by λ we have that:

$$\partial \mathbb{E}[a_i|\bar{\theta}] = \nu(\lambda, \rho_\theta)(1-\lambda)\bar{\theta} = \frac{((1-\lambda)\rho_\theta + \lambda(1-\rho_\theta))}{((1-r)(1-\lambda)^2\rho_\theta + \lambda^2(1-\rho_{\theta\theta}))}(1-\lambda)\bar{\theta},$$

$$\partial \mathbb{E}[a_i|\Delta\theta_i] = \nu(\lambda, \rho_\theta)\lambda\Delta\theta_i = \frac{((1-\lambda)\rho_\theta + \lambda(1-\rho_\theta))}{((1-r)(1-\lambda)^2\rho_\theta + \lambda^2(1-\rho_\theta))}\lambda\Delta\theta_i.$$

But, note that if $\lambda < \lambda^*$, then $\frac{\lambda}{(1-r)} < (1-\lambda)$, but then

$$\begin{aligned} \frac{\partial \mathbb{E}[a_i|\bar{\theta}]}{\partial \bar{\theta}} &= \frac{((1-\lambda)\rho_{\theta\theta}\lambda + \lambda^2(1-\rho_\theta))}{((1-r)(1-\lambda)^2\rho_{\theta\theta} + \lambda^2(1-\rho_\theta))}\Delta\theta_i \\ &\geq \frac{((1-\lambda)^2\rho_\theta + \frac{\lambda^2}{1-r}(1-\rho_\theta))}{((1-r)(1-\lambda)^2\rho_\theta + \lambda^2(1-\rho_\theta))} = \frac{1}{1-r} \end{aligned}$$

with strict inequality if $\lambda > 0$. Thus, the response to the idiosyncratic component is greater than in the complete information equilibrium if $\lambda \in (\lambda^*, 1)$. For the second part we repeat the same argument. Note that if $\lambda > \lambda^*$, then $\lambda > (1-\lambda)(1-r)$, but then

$$\begin{aligned} \frac{\partial \mathbb{E}[a_i|\Delta\theta_i]}{\partial \Delta\theta_i} &= \frac{((1-\lambda)^2\rho_{\theta\theta} + (1-\lambda)\lambda(1-\rho_\theta))}{((1-r)(1-\lambda)^2\rho_\theta + \lambda^2(1-\rho_\theta))}\Delta\theta_i \\ &\geq \frac{((1-\lambda)^2\rho_\theta + \lambda^2(1-\rho_{\theta\theta}))}{((1-r)(1-\lambda)^2\rho_\theta + \lambda^2(1-\rho_\theta))} = 1 \end{aligned}$$

with strict inequality if $\lambda < 1$. ■

Proof of Proposition 8. The comparative statics with respect to the maximum are direct from the envelope theorem. The comparative statics with respect to the argmax are shown by proving that the quantities have a unique maximum, which is interior, and then use the sign of the cross derivatives (the derivative with respect to λ and r). Finally, the last part is proved by comparing the derivatives.

(1.) and **(2.)** We begin with the individual variance, and using (22) we can write it in terms of λ:

$$\begin{aligned}\sigma_a^2 &= \left(\frac{((1-\lambda)\rho_{\theta\theta}+\lambda(1-\rho_{\theta\theta}))}{((1-r)(1-\lambda)^2\rho_{\theta\theta}+\lambda^2(1-\rho_{\theta\theta}))}\right)^2((1-\lambda)^2\rho_{\theta\theta}+\lambda^2(1-\rho_{\theta\theta}))\sigma_\theta^2 \\ &= \rho_{\theta\theta}\frac{(1+yx)^2}{((1-r)+x^2)^2}(1+x^2)\sigma_\theta^2,\end{aligned}$$

where

$$x \triangleq \frac{\sqrt{(1-\rho_{\theta\theta})}\lambda}{\sqrt{\rho_{\theta\theta}}(1-\lambda)}, \quad y \triangleq \frac{\sqrt{1-\rho_{\theta\theta}}}{\sqrt{\rho_{\theta\theta}}}. \tag{44}$$

Note that x is strictly increasing in λ, and if $\lambda \in [0,1]$ then $x \in [0,\infty]$, and thus maximizing with respect to $x \in [0,\infty]$ is equivalent to maximizing with respect to $\lambda \in [0,1]$. Finding the derivative we get:

$$\frac{\partial \sigma_a^2}{\partial x} = -\frac{2(xy+1)\left(x^3+(2r-1)yx^2+(r+1)x-(1-r)y\right)}{\left(x^2+1-r\right)^3}\sigma_\theta^2$$

It is easy to see that $\frac{d\sigma_a^2}{dx}$ is positive at $x=0$ and negative if we take a x large enough, and thus the maximum must be in $x \in (0,\infty)$. We would like to show that the polynomial:

$$\left(x^3+(2r-1)yx^2+(r+1)x-(1-r)y\right)$$

has a unique root in $x \in (0,\infty)$. If $r > 1/2$ or $r < -1$, we know it has a unique root in $x \in (0,\infty)$. For $r \in [-1, 1/2]$ we define the determinant of the cubic equation:

$$\Delta = 18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2.$$

We know that if $\Delta < 0$ then the polynomial has a unique root. Replacing by the respective values of the cubic polynomial we get:

$$\Delta = 4y^4(2r-1)^3(1-r) + y^2((2r-1)^2(1+r)^2 - 18(1-r^2)(2r-1) - 27(1-r)^2)) - 4(1+r)^3$$

using the fact that for $r \in [-1, 1/2]$ we have that $(2r-1) \leq 0$ and $1+r \geq 0$, we know that the term with y^4 and without y are negative. We just need to check the term with y^2, but this is also negative for $r \in [-1, 1/2]$. Thus, $\Delta < 0$, and thus for $r \in [-1, 1/2]$ the polynomial has a unique root.

Thus, we have that there exists a unique λ that maximizes σ_a^2. Finally, we have that:

$$\frac{\partial \sigma_a^2}{\partial r} = 2\frac{(1-\lambda)^2\rho_{\theta\theta}}{((1-r)(1-\lambda)^2\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta}))}\sigma_a^2.$$

Note that

$$\frac{\partial}{\partial \lambda}\frac{(1-\lambda)^2\rho_{\theta\theta}}{((1-r)(1-\lambda)^2\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta}))} < 0,$$

and thus at the maximum:

$$\frac{\partial^2 \sigma_a^2}{\partial r \partial \lambda} = 2\sigma_a^2\frac{\partial}{\partial \lambda}\frac{(1-\lambda)^2\rho_{\theta\theta}}{((1-r)(1-\lambda)^2\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta}))} < 0,$$

and thus $\text{argmax}_\lambda\, \sigma_a^2$ is decreasing in r. Finally, we know that $\lambda^* = \frac{1-r}{2-r}$ and thus $x^* = y(1-r)$, thus we have that:

$$\frac{\partial \sigma_a^2}{\partial x}|_{x=x^*} = -\frac{2(y^2(1-r)+1)\,(y^3(1-r)^2 r + y(1-r)r)}{(y^2(1-r)^2+1-r)^3}\sigma_\theta^2 \leq 0$$

thus, $\text{argmax}_\lambda\, \sigma_a^2 \leq \lambda^*$ if and only if $r \geq 0$.

Next, we consider the aggregate variance $\rho_{aa}\sigma_a^2$, and write it in terms of λ:

$$\begin{aligned}\rho_{aa}\sigma_a^2 &= \left(\frac{((1-\lambda)\rho_{\theta\theta} + \lambda(1-\rho_{\theta\theta}))}{((1-r)(1-\lambda)^2\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta}))}\right)^2(1-\lambda)^2\rho_{\theta\theta}\sigma_\theta^2 \\ &= \rho_{\theta\theta}\frac{(1+yx)^2}{((1-r)+x^2)^2}\sigma_\theta^2, \end{aligned} \quad (45)$$

where x and y are defined as in (44). Maximizing with respect to $x \in [0, \infty]$ is equivalent to maximizing with respect to $\lambda \in [0, 1]$. Finding the derivative we get:

$$\frac{\partial \rho_{aa}\sigma_a^2}{\partial x} = -\frac{2(xy+1)\,(2x + (x^2+r-1)\,y)}{(x^2+1-1r)^3}\sigma_\theta^2. \quad (46)$$

Again, we have that $(2x + (x^2+r-1)\,y)$ has a unique root in $(0, \infty)$ Thus, we have that there exists a unique λ that maximizes $\rho_{aa}\sigma_a^2$. Finally, we have that:

$$\frac{\partial \rho_{aa}\sigma_a^2}{\partial r} = 2\frac{(1-\lambda)^2\rho_{\theta\theta}}{((1-r)(1-\lambda)^2\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta}))}\rho_{aa}\sigma_a^2.$$

Note that,

$$\frac{\partial}{\partial \lambda}\frac{(1-\lambda)^2\rho_{\theta\theta}}{((1-r)(1-\lambda)^2\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta}))} < 0,$$

and thus at the maximum $\frac{\partial^2 \sigma_a^2}{\partial r \partial \lambda} < 0$, and thus $\text{argmax}_\lambda\ \rho_{aa}\sigma_a^2$ is decreasing in r. Finally, we know that $\lambda^* = \frac{1-r}{2-r}$ and thus $x^* = y(1-r)$, there we have that:

$$\frac{\partial \rho_{aa}\sigma_a^2}{\partial x}|_{x=x^*} = -\frac{2(y^2(1-r)+1)\,(y(1-r)+y^3(1-r))}{(y^2(1-r)+1-r)^3}\sigma_\theta^2 \leq 0;$$

thus, $\text{argmax}_\lambda\ \rho_{aa}\sigma_a^2 \leq \lambda^*$.

Finally, we consider the dispersion, $(1-\rho_{aa})\,\sigma_a^2$, expressed in terms of λ:

$$\begin{aligned}(1-\rho_{aa})\sigma_a^2 &= (\frac{((1-\lambda)\rho_{\theta\theta}+\lambda(1-\rho_{\theta\theta}))}{((1-r)(1-\lambda)^2\rho_{\theta\theta}+\lambda^2(1-\rho_{\theta\theta}))})^2\lambda^2(1-\rho_{\theta\theta})\sigma_\theta^2 \\ &= \rho_{\theta\theta}\frac{(1+yx)^2}{((1-r)+x^2)^2}x^2\sigma_\theta^2,\end{aligned}$$

where x and y are defined in (44). As before, maximizing with respect to $x \in [0,\infty]$ is equivalent to maximizing with respect to $\lambda \in [0,1]$. Finding the derivative we get:

$$\frac{\partial(1-\rho_{aa})\sigma_a^2}{\partial x} = -\frac{2x(xy+1)\,(x^2+2(r-1)yx+r-1)}{(x^2+1-r)^3}\sigma_\theta^2.$$

Again, we have that $(2x+(x^2+r-1)\,y)$ has a unique root in $(0,\infty)$ Thus, there exists a unique λ that maximizes $(1-\rho_{aa})\,\sigma_a^2$. Finally, we have that:

$$\frac{\partial(1-\partial\rho_{aa})\sigma_a^2}{\partial r} = 2\frac{(1-\lambda)^2\rho_{\theta\theta}}{((1-r)(1-\lambda)^2\rho_{\theta\theta}+\lambda^2(1-\rho_{\theta\theta}))}(1-\rho_{aa})\sigma_a^2.$$

Note that

$$\frac{\partial}{\partial\lambda}\frac{(1-\lambda)^2\rho_{\theta\theta}}{((1-r)(1-\lambda)^2\rho_{\theta\theta}+\lambda^2(1-\rho_{\theta\theta}))} < 0,$$

and thus at the maximum $\frac{\partial^2 \sigma_a^2}{\partial r \partial \lambda} < 0$, and thus $\text{argmax}_\lambda\ (1-\rho_{aa})\sigma_a^2$ is decreasing in r. Finally, we know that $\lambda^* = \frac{1-r}{2-r}$ and thus at $x^* = y(1-r)$ we have that:

$$\frac{\partial \sigma_a^2}{\partial x}|_{x=x^*} = \frac{2y(1-r)(y^2(1-r)+1)\,(y^2(1-r)^2+(1-r))}{(y^2(1-r)+1-r)^3}\sigma_\theta^2 \geq 0,$$

thus, $\text{argmax}_\lambda\ (1-\rho_{aa})\sigma_a^2 \geq \lambda^*$.

(3.) Finally, we want to show that $\text{argmax}_\lambda\ (1-\rho_{aa})\sigma_a^2 > \text{argmax}_\lambda\ \sigma_a^2 > \text{argmax}_\lambda\ \rho_{aa}\sigma_a^2$. These inequalities follows from comparing the derivatives of $(1-\rho_a)\sigma_a^2$, σ_a^2 and $\rho_{aa}\sigma_a^2$ with respect to λ. Since the derivatives satisfy the previous inequalities, the maximum must also satisfy the same inequalities. ■

Proof of Proposition 9. We first solve for $\max_\lambda\{\rho_{aa}\sigma_a^2\}$. By setting (46) equal to 0, we have that the aggregate volatility is maximized at,

$$x = \frac{\sqrt{1+y^2(1-r)}-1}{y}.$$

Substituting the solution in (45) and using the definitions of x and y we get that the maximum volatility is equal to:

$$\frac{\sigma_{\theta}^2(1-\rho_{\theta\theta})^2}{4(\sqrt{\rho_{\theta\theta}}-\sqrt{\rho_{\theta\theta}+(1-r)(1-\rho_{\theta\theta})})^2}.$$

Using the definition of $\sigma_{\bar{\theta}}$ and σ_{θ_i} we get (**??**). It follows from (**??**) that:

$$\lim_{\sigma_{\bar{\theta}}^2\to 0}\max_{\lambda}\{\rho_{aa}\sigma_a^2\}=\frac{\sigma_{\theta_i}^2}{4(1-r)}.$$

It is also immediate that $\sigma_{\bar{\theta}}^2/(1-r)^2$ is the aggregate volatility in the complete information equilibrium. Thus, we are only left with proving that,

$$\lim_{\sigma_{\theta_i}^2\to 0}\max_{\lambda}\{\rho_{aa}\sigma_a^2\}=\sigma_{\bar{\theta}}^2/(1-r)^2.$$

The limit can be easily calculated using L'Hopital's rule. That is, just note that as $\sigma_{\theta_i}^2\to 0$ we have that:

$$4(\sigma_{\bar{\theta}}-\sqrt{\sigma_{\bar{\theta}}^2+(1-r)\sigma_{\theta_i}^2})^2\approx\sigma_{\theta_i}^4(1-r)^2/\sigma_{\bar{\theta}}^2+o(\sigma_{\theta_i}^6),$$

and hence we get the result. ■

Proof of Proposition 10. (1.) The solution can be found in Bergemann and Morris (2013b). The BCE that maximizes the individual variance is given by:

$$\rho_{a\theta}^2=\rho_{aa}=\begin{cases}1, & r\geq -1;\\ \frac{-1}{r}, & r<-1.\end{cases}$$

To find the signal that decentralizes such a BCE, just note that when the signal is of the form $s_i=\bar{\theta}+\varepsilon_i$, where ε_i is a purely idiosyncratic noise, then the correlation coefficient of the actions is given by:

$$\rho_{aa}=\frac{\sigma_{\theta}^2}{\sigma_{\theta}^2+\sigma_{\varepsilon}^2}.$$

Thus, we have that:

$$\sigma_{\varepsilon}^2=\begin{cases}0, & r\geq -1;\\ -r-1, & r<-1.\end{cases}$$

(2.) The solution is provided by the following problem (where we take the limit $\rho_{\theta\theta}\to 0$ of the (14) in Proposition 1):

$$\max_{\rho_{a\theta},\rho_{aa}}\left\{\frac{\rho_{a\theta}}{1-r\rho_{aa}}\right\},$$

subject to

$$(1-\rho_{aa}) \geq \rho_{a\theta}^2, \quad \rho_{aa} \geq 0.$$

The constraints will clearly bind at the optimum, thus we have that we need to solve:

$$\max_{\rho_{aa}} \left\{ \frac{\sqrt{1-\rho_{aa}}}{1-r\rho_{aa}} \right\},$$

with

$$\rho_{aa} \geq 0.$$

This yields the following solution:

$$\rho_{a\theta}^2 = 1-\rho_{aa} = \begin{cases} 1, & r \leq \frac{1}{2}; \\ \frac{1-r}{r}, & \frac{1}{2} < r < 1. \end{cases}$$

It is clear that for $r < 1/2$, the complete information equilibrium is the solution, while for $r > 1/2$ the solution will not be the complete information equilibrium. We find the signal that decentralizes such a BCE with signal $s_i = \Delta\theta_i + \varepsilon$, where ε is a common noise, and as the correlation coefficient across actions is given by:

$$\rho_{aa} = \frac{\sigma_\varepsilon^2}{\sigma_\theta^2 + \sigma_\varepsilon^2},$$

we find that:

$$\sigma_\varepsilon^2 = \begin{cases} 0, & r \leq 1/2; \\ \frac{2r-1}{1-r}, & r < 1/2; \end{cases}$$

which completes the argument. ■

Proof of Proposition 11. Clearly all constraints from the BCE with unrestricted information structure must continue to hold. We first prove that in any BCE in which agents know their own payoff state constraint (29) must hold. The proof is just multiplying (28) by θ_i, in which case we get:

$$a_i\theta_i = \mathbb{E}[A\theta_i|a_i,\theta_i] + \mathbb{E}[\theta_i^2|a_i,\theta_i],$$

and taking expectations and dividing by σ_θ, we get the result.

We now prove sufficiency. That is, we assume that all conditions of Proposition 11 are satisfied, but

$$a_i \neq \mathbb{E}[\theta_i|a_i,\theta_i] + r\mathbb{E}[A|a_i,\theta_i],$$

where we note that we already proved that the conditions of Proposition 11 imply that the variance-covariance matrix of the random variables $(\theta_i, \bar{\theta}, a_i, A)$ is positive semidefinite. If $\rho_{a\theta} = 1$ then by definition $a_i = \theta_i$, which implies that we can use the same argument as in Proposition 1. Thus we assume $\rho_{a\theta} < 1$ and proceed by contradiction. In this case, since $(\theta_i, \bar{\theta}, a_i, A)$ are normally distributed we know that there exists constants c_a and c_θ such that,

$$c_a a_i + c_\theta \theta_i = r\mathbb{E}[A|a_i, \theta_i] + \mathbb{E}[\theta_i|a_i, \theta_i],$$

We need to prove that if $c_\theta \neq 0$ or $c_a \neq 1$ then either (29) or (13) (or both) do not hold. Multiplying the previous equation by a_i and taking expectation and later doing the same for θ_i we get,

$$c_\theta \rho_{a\theta}\sigma_\theta + c_a\sigma_a = \rho_{a\theta}\sigma_\theta + r\rho_{aa}\sigma_a,$$

$$c_\theta\sigma_\theta + c_a\rho_{a\theta}\sigma_a = \sigma_\theta + r\sigma_{a\phi}\sigma_a,$$

and as the system of equations is invertible for c_a and c_θ, we have a unique solution for these variables. We know that the solution when (29) and (13) hold is $c_a = 1$ and $c_\theta = 0$, thus if $c_\theta \neq 0$ or $c_a \neq 1$ then either (39) or (13) do not hold. Thus, we obtain sufficiency as well. ■

Proof of Proposition 12. From the best response conditions. we have that,

$$a_i = \theta_i + r\mathbb{E}[A|\mathcal{I}_i],$$

and multiplying by θ_i and taking expectations (note that because θ_i is in $\mathcal{I}_i$, we have that $\theta_i\mathbb{E}[A|\mathcal{I}_i] = \mathbb{E}[\theta_i A|\mathcal{I}_i]$), we find that

$$\rho_{a\theta}\sigma_a = \sigma_\theta + r\rho_{a\phi}\sigma_a. \tag{47}$$

We also use the fact that

$$\sigma_a = \rho_{a\theta}\sigma_\theta + r\rho_{aa}\sigma_a,$$

and hence inserting in (47) we obtain:

$$\rho_{a\phi} = \frac{1}{r}(\rho_{a\theta} - \frac{1 - r\rho_{aa}}{\rho_{a\theta}}). \tag{48}$$

Thus, the inequalities in (14) can be written as follows,

$$(1 - \rho_{aa})(1 - \rho_{\theta\theta}) \geq \frac{1}{r^2}((1 - r)\rho_{a\theta} - \frac{1 - r\rho_{aa}}{\rho_{a\theta}})^2, \tag{49}$$

$$\rho_{aa}\rho_{\theta\theta} \geq \frac{1}{r^2}(\rho_{a\theta} - \frac{1 - r\rho_{aa}}{\rho_{a\theta}})^2. \tag{50}$$

For both of the previous inequalities the right hand side is a convex function of $\rho_{a\theta}$. Thus, for a fixed ρ_{aa}, inequalities (49) and (50) independently constraint the set of feasible $\rho_{a\theta}$ to be in a convex interval with non-empty interior for all $\rho_{aa} \in [0,1]$ (it is easy to check that there exists values of $\rho_{a\theta}$ such that either inequality is strict). Thus, if we impose both inequalities jointly, we will find the intersection of both intervals which is also a convex interval. We first find the set of feasible ρ_{aa} and prove that it is always case that one of the inequalities provides the lower bound and the other inequality provides the upper bound on the set of feasible $\rho_{a\theta}$ for each feasible ρ_{aa}. For this we make several observations.

First, when agents know their own payoff type there are only two noise-free equilibria, one in which agents know only their type and the complete information equilibria. This implies that there are only two pair of values for $(\rho_{aa}, \rho_{a\theta})$ such that inequalities (49) and (50) both hold with equality. Second, the previous point implies that there are only two pairs of values for $(\rho_{aa}, \rho_{a\theta})$ such that the bound of the intervals imposed by inequalities (49) and (50) are the same. Third, the previous two points imply that there are only two possible ρ_{aa} such that the set of feasible $\rho_{a\theta}$ is a singleton. These values are $\rho_{aa} \in \{\rho_{\theta\theta}, \rho^*_{aa}\}$, which corresponds to the ρ_{aa} of the complete information equilibria and the equilibria in which agents only know their type. Fourth, it is clear that there are no feasible BCE with $\rho_{aa} \in \{0,1\}$. Fifth, the upper and lower bound on the feasible $\rho_{a\theta}$ that are imposed by inequalities (49) and (50) move smoothly with ρ_{aa}. Sixth, this implies that the set of feasible ρ_{aa} is bounded by the values of ρ_{aa} in which the set of feasible $\rho_{a\theta}$ is a singleton. Thus, the set of feasible ρ_{aa} is in $[\min\{\rho_{\theta\theta}, \rho^*_{aa}\}, \max\{\rho_{\theta\theta}, \rho^*_{aa}\}]$. Moreover, it is easy to see that for all ρ_{aa} in the interior of this interval one of the inequalities provides the upper bound for $\rho_{a\theta}$ while the other inequality will provide the lower bound.

We now provide the explicit functional forms for the upper and lower bounds. To check which of the inequalities provides the upper and lower bound respectively we can just look at the equilibria in which agents know only their own type, and thus $\rho_{aa} = \rho_{\theta\theta}$. In this case we have the following inequalities for $\rho_{a\theta}$,

$$(1-\rho_{\theta\theta})^2 \geq \frac{1}{r^2}((1-r)\rho_{a\theta} - \frac{1-r\rho_{\theta\theta}}{\rho_{a\theta}})^2,$$

$$\rho_{\theta\theta}^2 \geq \frac{1}{r^2}(\rho_{a\theta} - \frac{1-r\rho_{\theta\theta}}{\rho_{a\theta}})^2.$$

As expected it is easy to check that $\rho_{a\theta} = 1$ satisfies both inequalities. Moreover, it is easy to see that if $r > 0$ then $\rho_{a\theta} = 1$ provides a lower bound on the set of $\rho_{a\theta}$ that satisfies the first inequality while $\rho_{a\theta} = 1$ provides an upper bound on the set of $\rho_{a\theta}$ that satisfies the second inequality. If $r < 0$ we get the opposite, $\rho_{a\theta} = 1$ provides a upper bound on the set of $\rho_{a\theta}$ that satisfies the first inequality while $\rho_{a\theta} = 1$ provides a lower bound on the set of $\rho_{a\theta}$ that satisfies the second inequality. Thus, if $r > 0$, then the inequality (49) provides the lower bound for $\rho_{a\theta}$ and the inequality (50) provides an upper bound on the set of feasible $\rho_{a\theta}$ for all ρ_{aa}. If $r < 0$ we get the opposite result. Note that the conclusions about the bounds when $\rho_{aa} = \rho_{\theta\theta}$ can be extended for all feasible ρ_{aa} because we know that the bounds of the intervals are different for all $\rho_{aa} \notin \{\rho_{\theta\theta}, \rho^*_{aa}\}$, and they move continuously, thus the relative order is preserved. Finally, by taking the right roots (just check which one yields $\rho_{a\theta} = 1$ when $\rho_{aa} = \rho_{\theta\theta}$), we define implicitly the functions $\rho^i_{a\theta}$ and $\rho^c_{a\theta}$,

$$\frac{1}{r}((1-r)\rho^c_{a\theta} - 1 - r\rho_a) + \rho^c_{a\theta}\sqrt{(1-\rho_a)(1-\rho_\theta)} = 0,$$

$$\frac{1}{r}((\rho^i_{a\theta})^2 - (1 - r\rho_a)) - \rho^i_{a\theta}\sqrt{\rho_a\rho_\theta} = 0.$$

Therefore, for all feasible BCE in which agents know their own type $\rho_{aa} \in [\min\{\rho_{\theta\theta}, \rho^*_{aa}\}, \max\{\rho_{\theta\theta}, \rho^*_{aa}\}]$, while the set of $\rho_{a\theta}$ is bounded by the functions $\rho^i_{a\theta}$ and $\rho^c_{a\theta}$. If $r > 0$ then function $\rho^i_{a\theta}$ provides the upper bound while $\rho^c_{a\theta}$ provides the lower bound. If $r < 0$, then the function $\rho^c_{a\theta}$ provides the upper bound while $\rho^i_{a\theta}$ provides the lower bound. ■

Proof of Proposition 13. We first prove that (36) and (37) is equivalent to $\mathbb{V}$ being positive semi-definite, where now $\mathbb{V}$ is given by:

$$\mathbb{V} = \begin{pmatrix} \sigma^2_\theta & \rho_{\varphi\varphi}\sigma^2_\theta & \rho_{\theta\theta}\sigma^2_\theta & \rho_{a\theta}\sigma_\theta\sigma_a & \rho_{a\varphi}\sigma_\theta\sigma_a & \rho_{a\phi}\sigma_\theta\sigma_a \\ \rho_{\varphi\varphi}\sigma^2_\theta & \sigma^2_\theta & \rho_{\theta\theta}\sigma^2_\theta & \rho_{a\varphi}\sigma_\theta\sigma_a & \rho_{a\theta}\sigma_\theta\sigma_a & \rho_{a\phi}\sigma_\theta\sigma_a \\ \rho_{\theta\theta}\sigma^2_\theta & \rho_{\theta\theta}\sigma^2_\theta & \rho_{\theta\theta}\sigma^2_\theta & \rho_{a\phi}\sigma_\theta\sigma_a & \rho_{a\phi}\sigma_\theta\sigma_a & \rho_{a\phi}\sigma_\theta\sigma_a \\ \rho_{a\theta}\sigma_\theta\sigma_a & \rho_{a\varphi}\sigma_\theta\sigma_a & \rho_{a\phi}\sigma_\theta\sigma_a & \sigma^2_a & \rho_{aa}\sigma^2_a & \rho_{aA}\sigma^2_a \\ \rho_{a\varphi}\sigma_\theta\sigma_a & \rho_{a\theta}\sigma_\theta\sigma_a & \rho_{a\phi}\sigma_\theta\sigma_a & \rho_{aa}\sigma^2_a & \sigma^2_a & \rho_{aA}\sigma^2_a \\ \rho_{a\phi}\sigma_\theta\sigma_a & \rho_{a\phi}\sigma_\theta\sigma_a & \rho_{a\phi}\sigma_\theta\sigma_a & \rho_{aA}\sigma^2_a & \rho_{aA}\sigma^2_a & \rho_{aA}\sigma^2_a \end{pmatrix}. \tag{51}$$

A symmetric BCE with 2 agents is given by a distribution:

$$\begin{pmatrix} \theta_i \\ \theta_j \\ a_i \\ a_j \end{pmatrix} \sim \mathcal{N}\left(0, \begin{pmatrix} \sigma^2_\theta & \rho_{\varphi\varphi}\sigma^2_\theta & \rho_{a\theta}\sigma_\theta\sigma_a & \rho_{a\varphi}\sigma_\theta\sigma_a \\ \rho_{\varphi\varphi}\sigma^2_\theta & \sigma^2_\theta & \rho_{a\varphi}\sigma_\theta\sigma_a & \rho_{a\theta}\sigma_\theta\sigma_a \\ \rho_{a\theta}\sigma_\theta\sigma_a & \rho_{a\varphi}\sigma_\theta\sigma_a & \sigma^2_a & \rho_{aa}\sigma^2_a \\ \rho_{a\varphi}\sigma_\theta\sigma_a & \rho_{a\theta}\sigma_\theta\sigma_a & \rho_{aa}\sigma^2_a & \sigma^2_a \end{pmatrix}\right),$$

where the variance-covariance matrix is positive semidefinite and the best response conditions. The conditions (37) come from the condition that the variance-covariance matrix is positive semidefinite. To get a simple expression for the conditions that the variance-covariance matrix is positive semidefinite we make some change of variables. We first use the definitions for θ_i^+ and θ_i^- and note that:

$$\begin{pmatrix}\theta_i^+\\ \theta_i^-\\ a_i^+\\ a_i^-\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & \frac{1}{2} & 0 & 0\\ \frac{1}{2} & \frac{-1}{2} & 0 & 0\\ 0 & 0 & \frac{1}{2} & \frac{1}{2}\\ 0 & 0 & \frac{1}{2} & \frac{-1}{2}\end{pmatrix} \cdot \begin{pmatrix}\theta_i\\ \theta_j\\ a_i\\ a_j\end{pmatrix},$$

using the linearity normally distributed random variables,

$$\begin{aligned}
\text{var}\begin{pmatrix}\theta_i^+\\ \theta_i^-\\ a_i^+\\ a_i^-\end{pmatrix} &= \begin{pmatrix}\frac{1}{2} & \frac{1}{2} & 0 & 0\\ \frac{1}{2} & \frac{-1}{2} & 0 & 0\\ 0 & 0 & \frac{1}{2} & \frac{1}{2}\\ 0 & 0 & \frac{1}{2} & \frac{-1}{2}\end{pmatrix} \begin{pmatrix}\sigma_\theta^2 & \rho_{\varphi\varphi}\sigma_\theta^2 & \rho_{a\theta}\sigma_\theta\sigma_a & \rho_{a\varphi}\sigma_\theta\sigma_a\\ \rho_{\varphi\varphi}\sigma_\theta^2 & \sigma_\theta^2 & \rho_{a\varphi}\sigma_\theta\sigma_a & \rho_{a\theta}\sigma_\theta\sigma_a\\ \rho_{a\theta}\sigma_\theta\sigma_a & \rho_{a\varphi}\sigma_\theta\sigma_a & \sigma_a^2 & \rho_{aa}\sigma_a^2\\ \rho_{a\varphi}\sigma_\theta\sigma_a & \rho_{a\theta}\sigma_\theta\sigma_a & \rho_{aa}\sigma_a^2 & \sigma_a^2\end{pmatrix} \begin{pmatrix}\frac{1}{2} & \frac{1}{2} & 0 & 0\\ \frac{1}{2} & \frac{-1}{2} & 0 & 0\\ 0 & 0 & \frac{1}{2} & \frac{1}{2}\\ 0 & 0 & \frac{1}{2} & \frac{-1}{2}\end{pmatrix}\\
&= \begin{pmatrix}\frac{\sigma_\theta^2}{2}(1+\rho_{\varphi\varphi}) & 0 & (\rho_{a\theta}+\rho_{a\varphi})\frac{\sigma_\theta\sigma_a}{2} & 0\\ 0 & \frac{\sigma_\theta^2}{2}(1-\rho_{\varphi\varphi}) & 0 & (\rho_{a\theta}-\rho_{a\varphi})\frac{\sigma_\theta\sigma_a}{2}\\ (\rho_{a\theta}+\rho_{a\varphi})\frac{\sigma_\theta\sigma_a}{2} & 0 & \frac{\sigma_a^2}{2}(1+\rho_{aa}) & 0\\ 0 & (\rho_{a\theta}-\rho_{a\varphi})\frac{\sigma_\theta\sigma_a}{2} & 0 & \frac{\sigma_a^2}{2}(1-\rho_{aa})\end{pmatrix}.
\end{aligned}$$

Thus, the joint distribution of $(\theta_i^+, \theta_i^-, a_i^+, a_i^-)'$ is given by:

$$\begin{pmatrix}\theta_i^+\\ \theta_i^-\\ a_i^+\\ a_i^-\end{pmatrix} \sim \mathcal{N}\left(0, \begin{pmatrix}\frac{\sigma_\theta^2}{2}(1+\rho_{\varphi\varphi}) & 0 & (\rho_{a\theta}+\rho_{a\varphi})\frac{\sigma_\theta\sigma_a}{2} & 0\\ 0 & \frac{\sigma_\theta^2}{2}(1-\rho_{\varphi\varphi}) & 0 & (\rho_{a\theta}-\rho_{a\varphi})\frac{\sigma_\theta\sigma_a}{2}\\ (\rho_{a\theta}+\rho_{a\varphi})\frac{\sigma_\theta\sigma_a}{2} & 0 & \frac{\sigma_a^2}{2}(1+\rho_{aa}) & 0\\ 0 & (\rho_{a\theta}-\rho_{a\varphi})\frac{\sigma_\theta\sigma_a}{2} & 0 & \frac{\sigma_a^2}{2}(1-\rho_{aa})\end{pmatrix}\right),$$

and we note that the variance-covariance matrix is positive semidefinite if and only if,

$$(1+\rho_{\varphi\varphi})(1+\rho_{aa}) \geq (\rho_{a\theta}+\rho_{a\varphi})^2,$$

$$(1-\rho_{\varphi\varphi})(1-\rho_{aa}) \geq (\rho_{a\theta}-\rho_{a\varphi})^2,$$

subject to $\rho_{\theta\theta}, \rho_{aa} \in [-1, 1]$. Note that,

$$\text{cov}(A, a^+) = \rho_{aA}\sigma_a^2 \;\; ; \;\; \text{cov}(A, \theta^+) = \rho_{a\phi}\sigma_a\sigma_\theta \;\; ; \;\; \text{cov}(\bar{\theta}, \theta^+) = \rho_{\theta\theta}\sigma_\theta^2 \;\; ; \;\; \text{cov}(\bar{\theta}, a^+) = \rho_{a\phi}\sigma_a\sigma_\theta,$$

and that the orthogonal terms yield:

$$\text{cov}(A, a^-) = 0 \;\; ; \;\; \text{cov}(A, \theta^-) = 0 \;\; ; \;\; \text{cov}(\bar{\theta}, \theta^-) = 0 \;\; ; \;\; \text{cov}(\bar{\theta}, a^-) = 0,$$

where we note that $\text{cov}(A, a^+) = \text{cov}(A, \frac{a_i+a_j}{2}) = \frac{1}{2}(\text{cov}(A, a_i) + \text{cov}(A, a_j)) = \rho_{aA}\sigma_a^2$ and $\text{cov}(A, a^-) = \text{cov}(A, \frac{a_i-a_j}{2}) = \frac{1}{2}(\text{cov}(A, a_i) - \text{cov}(A, a_j)) = 0$. Thus the variance-covariance matrix of $\left(\theta^-, a^-, \theta^+, a^+, \bar{\theta}, A\right)$ is given by:

$$\begin{pmatrix} \frac{\sigma_\theta^2}{2}(1-\rho_{\varphi\varphi}) & (\rho_{a\theta}-\rho_{a\varphi})\frac{\sigma_\theta\sigma_a}{2} & 0 & 0 & 0 & 0 \\ (\rho_{a\theta}-\rho_{a\varphi})\frac{\sigma_\theta\sigma_a}{2} & \frac{\sigma_a^2}{2}(1-\rho_{aa}) & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sigma_\theta^2}{2}(1+\rho_{\varphi\varphi}) & (\rho_{a\theta}+\rho_{a\varphi})\frac{\sigma_\theta\sigma_a}{2} & \rho_{\theta\theta}\sigma_\theta^2 & \rho_{a\phi}\sigma_a\sigma_\theta \\ 0 & 0 & (\rho_{a\theta}+\rho_{a\varphi})\frac{\sigma_\theta\sigma_a}{2} & \frac{\sigma_a^2}{2}(1+\rho_{aa}) & \rho_{a\phi}\sigma_a\sigma_\theta & \rho_{aA}\sigma_a^2 \\ 0 & 0 & \rho_{\theta\theta}\sigma_\theta^2 & \rho_{a\phi}\sigma_a\sigma_\theta & \rho_{\theta\theta}\sigma_\theta^2 & \rho_{a\phi}\sigma_a\sigma_\theta \\ 0 & 0 & \rho_{a\phi}\sigma_a\sigma_\theta & \rho_{aA}\sigma_a^2 & \rho_{a\phi}\sigma_a\sigma_\theta & \rho_{aA}\sigma_a^2 \end{pmatrix}$$

With one more change of variable, and defining:

$$\Delta a^+ \triangleq a^+ - A \quad ; \quad \Delta\theta^+ \triangleq \theta^+ - \bar{\theta},$$

we find that:

$$\begin{pmatrix} \Delta\theta^+ \\ \Delta a^+ \\ \bar{\theta} \\ A \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \theta^+ \\ a^+ \\ \bar{\theta} \\ A \end{pmatrix}.$$

Thus, we find that the variance-covariance matrix of $\left(\theta^+, a^+, \bar{\theta}, A\right)$ is given by:

$$\begin{pmatrix} \frac{\sigma_a^2}{2}(-2\rho_{\theta\theta}+\rho_{\varphi\varphi}+1) & \frac{\sigma_a\sigma_\theta}{2}(\rho_{a\theta}-2\rho_{a\phi}+\rho_{a\varphi}) & 0 & 0 \\ \frac{\sigma_\theta\sigma_a}{2}(\rho_{a\theta}-2\rho_{a\phi}+\rho_{\alpha\varphi}) & \frac{\sigma_\theta^2}{2}(-2\rho_{aA}+\rho_{aa}+1) & 0 & 0 \\ 0 & 0 & \sigma_\theta^2\rho_{\theta\theta} & \sigma_a\sigma_\theta\rho_{a\phi} \\ 0 & 0 & \sigma_a\sigma_\theta\rho_{a\phi} & \sigma_a^2\rho_{aA} \end{pmatrix}.$$

This allows us to conclude that the variance-covariance matrix of $\left(\theta^-, a^-, \theta^+, a^+, \bar{\theta}, A\right)$ is given by:

$$\begin{pmatrix} \frac{\sigma_\theta^2}{2}(1-\rho_{\varphi\varphi}) & (\rho_{a\theta}-\rho_{a\varphi})\frac{\sigma_\theta\sigma_a}{2} & 0 & 0 & 0 & 0 \\ (\rho_{a\theta}-\rho_{a\varphi})\frac{\sigma_\theta\sigma_a}{2} & \frac{\sigma_a^2}{2}(1-\rho_{aa}) & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sigma_\theta^2}{2}(-2\rho_{\theta\theta}+\rho_{\varphi\varphi}+1) & \frac{\sigma_a\sigma_\theta}{2}(\rho_{a\theta}-2\rho_{a\phi}+\rho_{a\varphi}) & 0 & 0 \\ 0 & 0 & \frac{\sigma_a\sigma_\theta}{2}(\rho_{a\theta}-2\rho_{a\phi}+\rho_{\alpha\varphi}) & \frac{\sigma_a^2}{2}(-2\rho_{aA}+\rho_{aa}+1) & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_\theta^2\rho_{\theta\theta} & \sigma_a\sigma_\theta\rho_{a\phi} \\ 0 & 0 & 0 & 0 & \sigma_a\sigma_\theta\rho_{a\phi} & \sigma_a^2\rho_{aA} \end{pmatrix} \tag{52}$$

Since all change of variables made are invertible, we know that the variance/covariance matrix defined by (51) is invertible if and only if (52) is invertible. Yet, since the above matrix is defined blockwise it is easy to see that it is positive-semidefinite if and only if,

$$(1-\rho_{\varphi\varphi})(1-\rho_{aa}) \geq (\rho_{a\theta}-\rho_{a\varphi})^2,$$

$$(1 - 2\rho_{\theta\theta} + \rho_{\varphi\varphi})(1 + \rho_{aa} - 2\rho_{aA}) \geq (\rho_{a\theta} - 2\rho_{a\phi} + \rho_{a\varphi})^2,$$

and

$$\rho_{aA}\rho_{\theta\theta} \geq \rho_{a\phi}^2,$$

which yields the desired result.

We now prove that, given random variables $(\theta_i, \theta_j, \bar{\theta}, a_i, a_j, A)$, Proposition 13.1 holds if and only if

$$a_i = \mathbb{E}[\theta_i + r_a a_j + r_A A | a_i].$$

As usual, the "only if" part is proven by multiplying the best response conditions by a_i and taking expectations. As before, the "if" part is just proven by contradiction. That is, by the normality of the random variables there exists c_a such that:

$$c_a a_i = \mathbb{E}[\theta_i + r_a a_j + r_A A | a_i].$$

Multiplying by a_i and taking expectations we get,

$$c_a \sigma_a = \rho_{a\theta}\sigma_\theta + \rho_{aa}\sigma_a + \rho_{aA}\sigma_a.$$

But, if Proposition 13.1 is satisfied, then obviously we must have that $c_a = 1$. Hence, we get the result. ■

Proof of Proposition 14. The proof is completely analogous to the proof of Proposition 11. As before, (39) is a necessary condition for the first order conditions to be satisfied, and this can be obtained by multiplying (35) by θ_i and taking expectations. The "if" part can also be done in exactly the same way by contradiction.■

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