The Tripartite Auction Folk Theorem☆

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Abstract

We formally study two bidder first price, second price, and all-pay auctions with known values, deriving the equilibrium payoffs and strategies and showing when all three yield the same equilibrium payoffs to the bidders. This latter result, the tripartite auction theorem, does not hold for all auctions, in particular it can fail for symmetric auctions with high stakes and in auctions with very low stakes.

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1. Introduction

By the tripartite auction theorem we mean the proposition that with two bidders and known values first price, second price, and all-pay auctions are equivalent from the point of view of the bidders. These auctions are relevant in political economy because they provide a simple model of two groups competing over a political prize. In voting this contest is usually an all-pay auction while in lobbying for political favor this contest is usually a winner-pay auction, either first or second price. The key point is that - under appropriate conditions - the structure of the auction does not matter for the utility of the bidders. This result has additional interest because it is known that for a variety of contests with random outcomes, such as the Tullock contest, the utility of the bidders is the same as in the all-pay auction.[4]

We study only two party auctions. While auctions have sometimes been used to model multi-party political contests, and while it is by no means true that all political contests involve only two parties, multi-party contests are complicated by the fact that the prize can ordinarily be shared by several contestants, and that coalitions can be, and often are, formed either prior to or after the contest. In addition we treat each of the two parties as single decision makers although in political economy parties are typically made up of many individuals. There is a long tradition in political economy of treating groups as individuals, and modern models such as those of ethical voters and social mechanisms provide a theoretical underpinning for this approach.[5] Social mechanism theory, in particular, shows how particular cost functions for effort provision arise from the underlying mechanism design problem faced by a group that must overcome the public good problem of inducing individual members to provide effort.[6] Here we abstract from that and take the cost of effort provision as given. Hence, the all-pay auctions models here apply to two parties or coalitions competing in an election and the winner-pays auction to two coalitions lobbying for or against

[4]See Ewerhart (2017) and Levine, Mattozzi and Modica (2022).

[5]See See Feddersen and Sandroni (2006), Coate and Conlin (2004) and Levine and Mattozzi (2020) .

[6]See Levine, Mattozzi and Modica (2022).

some particular legislation.

As we are interested in auctions arising in political economy with effort provision costs that arise from an underlying public goods problem for each group, we study general cost functions. We allow bidding caps to reflect the possibility that the parties have limited resources and we allow head starts (see Siegel (2014)) to reflect the possibility that parties may have committed or expressive members who will provide effort regardless of strategic considerations. Both of these are common in the literature on auctions. We also allow the less commonly studied possibility in which there is a fixed cost of entering into the auction. This arises naturally in the theory of social mechanisms and is essentially the opposite of head starts. We give precise conditions under which the tripartite auction does hold, and conditions under which it does not. In particular the tripartite auction theorem always holds in the generic case of what we call standard auctions: either one group has a higher willingness to pay or both an have equal willingness to pay and the bidding caps do not bind. Che and Gale (1998) argue that it does not hold in the symmetric case when there are binding bidding caps and linear cost. We extend their results to general cost functions - and show in addition that this case is the only important one in which the tripartite auction theorem fails.

Additionally, we study the revenue generation of the different types of auctions and the implications for welfare. In the case of voting the effort has no social value, but in the case of lobbying the effort may be in the form of transfer payments to politicians, so revenue generation is of interest in that case. Here we show that with convex cost and asymmetry the winner pay auctions generate more revenue than the all-pay auction, and that this result continues to hold provided cost is not "too concave." This forms a sharp contrast to the results for the symmetric case with linear cost and symmetric uncertain values where Krishna and Morgan (1997) show that the all-pay auction generates more revenue. In the political economy setting, where the value of the prize to the parties is not easily kept secret, with linear cost it is only when values are symmetric (or one party is unwilling to bid) that the all-pay auction does as well as the winner pays auctions.

This paper is dedicated to the memory of Konrad Mierendorff. Konrad is noted for his work on mechanism design and auctions in particular. He was particularly interested in the types of constraints, such as deadlines, that are crucial in applied work. He was extremely precise and focused in his work and always aimed to produce general results and not to simply study special cases. Our goal in writing this paper is to follow in those footsteps providing precise, focused, and general results and we hope this is a paper he would have appreciated.

2. The Model

Two bidders indexed by $k \in \{1, 2\}$ compete for a prize worth $V_k > 0$ to contestant k. Each bidder chooses a bid $b_k \geq 0$. We define $c_k(b_k)$ the cost of b_k relative to the value of the prize V_k and without loss of generality we divide the objective function by V_k so that the value of the prize is normalized to 1 and so that the $c_k(b_k)$ is the cost of bidding b_k.

We assume that $c_k(b_k) \geq 0$ and that that $c_k(0) = 0$. We assume that $c_k(b_k)$ is continuous for $b_k > 0$ and that it is strictly increasing for $c_k(b_k) > 0$. This allows for head starts where $c_k(b_k) = 0$ for some initial interval of bids and for a fixed cost of entry where $c_k(b_k)$ is discontinuous at $b_k = 0$. We define $c_k(0^+) \equiv \lim_{b_k \downarrow 0} c_k(b_k)$. If $c_k(0^+) = 0$, c_k is clearly continuous. In the discontinuous case where $c_k(0^+) > 0$ we allow a bid of 0^+ which beats 0 and costs $c_k(0^+)$ - this corresponds to an infinitesimal bid. We assume $c_k(0^+) < 1$ for at least one k - otherwise no bidding takes place. To avoid a horde of uninteresting special cases we also make the generic assumption that $c_k(0^+) \neq 1$.

On the upper end of $c_k(b_k)$, we assume that large enough bids are more costly than the prize, that is, for some b_k we have $c_k(b_k) > 1$. In addition there are bidding caps: k cannot bid more than $\overline{b}_k$ where $c_k(\overline{b}_k) > 0$. Note that there is no lack of generality in this: if $c_k(\overline{b}_k) > 1$ the bidding caps will not bind.

We define $\underline{b}_k \equiv \inf\{b_k | c_k(b_k) > 0\}$: this is the lowest bid that is not weakly dominated. We define the *desire to bid* as B_k satisfying $c_k(B_k) = 1$; in the discontinuous case when $c_k(0^+) > 1$ we take $B_k = 0$. We define the *willingness to bid* as $W_k \equiv \min\{B_k, \overline{b}_k\}$ and say that bidder k is advantaged if $W_k > W_{-k}$.

We use the letter d for disadvantaged, and write $W_{-d} > W_d$; we may also write W_d for $\min W_k$. Note that if $c_k(0^+) > 0$ then $\underline{b}_k = 0$ and observe that $\overline{b}_k > \underline{b}_k$, $W_k \geq \underline{b}_k$ and if either c_k is continuous or $c_k(0^+) < 1$ then $W_k > \underline{b}_k$. Observe also that if $W_{-d} > W_d$ it must be $c_{-d}(0^+) < 1$ (for $c_k(0^+) \geq 1$ implies $W_k = 0$).

We will study three types of auctions. In each both bidders submit bids. We will assume that if both cost functions are discontinuous and both submit a bid of 0 neither wins the prize. The first two auctions are *winner pays auctions*. In a *second price auction* the high bid wins and pays the low bid. In a *first price auction* the high bid wins and pays their own bid. In the *all-pay auction* both pay their bid and the high bid wins.

At this point we digress briefly to discuss modeling choices. The strategy spaces for these auctions are infinite and payoffs are fundamentally discontinuous since shifting a bid from slightly below to slightly above an opponent's bid turns a certain loss into a certain win. Our base concept is Nash equilibrium, and there is a satisfactory general theory of Nash equilibrium for these games - that of Simon and Zame (1990). However, for existence of equilibrium this theory requires that in cases of equal bids the tie breaking rule cannot be arbitrary but must be correctly chosen. Moreover, Nash equilibrium is not always adequate for our purposes. We can illustrate the issues in the first price auction where one bidder is willing to bid less than the other. Here it is a Nash equilibrium for the disadvantaged bidder to bid any amount greater than their value but less than the value of the advantaged bidder and for the advantaged bidder to bid the same with the endogenous tie-breaking rule that the advantaged bidder wins. These equilibria make little sense as it is weakly dominated for the disadvantaged bidder to bid more than their value, and we wish to rule them out. Indeed much of Bernheim and Whinston (1986)'s modeling of menu auctions revolves around doing exactly this. There is, however, no general theory of Nash equilibrium in weakly undominated strategies for discontinuous games with a continuum of actions, and as indicated below we will make several modeling choices to rule out "enough" weakly dominated strategies, while still assuring existence of equilibrium.

Concerning tie-breaking rules, in both the first-price and all-pay auction we introduce them at the *top* and at the *bottom:* except in these cases, in the event of a tie each bidder has a 50% chance of winning. For the first-price and all-pay auctions the tie-breaking rule at the top specifies that if $W_{-d} > W_d$ and there is a tie at $b_{-d} = b_d = W_d$ then $-d$ wins for sure. This tie-breaking rule reflects the fact that $-d$ could bid a little higher and win for sure while d would not wish to do so. At the bottom it specifies that if $c_k(0^+) > 0$ and $c_{-k}(0^+) = 0$ and both bid 0 then k loses for sure since $-k$ can raise the bid at minimal cost and $-k$ cannot.

A strategy for bidder k is is a cdf G_k on $[0, \infty)$. Corresponding this is a probability measure and if $\mathscr{B}$ is a measurable set we will write $G_k[\mathscr{B}]$ for the probability of the set according to that measure.

Turning to weak dominance we mean with respect to all opponents bids, feasible or not. As indicated, our equilibrium notion is Nash in which no bidder plays any weakly dominated strategy, but we make one exception: a bidder k may always bid W_k. The reason for this exception is that weak dominance is not well behaved in games with a continuum of actions. Bids lower than a headstart $b_k < \underline{b}_k$ are weakly dominated by $\underline{b}_k$ and this is fine because the set of undominated bids is closed. However, bidding $B_k > 0$ in a first price auction is weakly dominated by bidding a bit less since bidding your value guarantees getting nothing. This leads to an open set of undominated bids and that creates an existence problem. In particular in a first price auction where bidding caps do not bind and both bidders have the same value $B_k = B_{-k}$ we would like it to be an equilibrium for both the bid their value and get nothing. In order to allow this we must allow bidding B_k as the limit of weakly undominated strategies, although it is itself weakly dominated.

We study both equilibrium strategies - in the winner pays auctions these are relatively simple, but not so in the all-pay auction - and *utility equivalence* meaning that the utilities of both bidders are the same in the auctions. In the final section we also study revenue.

While we do not want to limit attention to "generic" parameters, and in particular we want to allow the non-generic but important case of symmetry, we do not study all auctions. As indicated above we always assume $c_k(0^+) \neq 1$.

Moreover, we say that an auction is *special* if for one bidder j we have $c_j(\overline{b}_j) \leq 1$, so that the bidding cap binds, and for the other we have $\overline{b}_{-j} \geq \overline{b}_j$ and $c_{-j}(\overline{b}_j) = 1$. Special auctions are *weakly symmetric* in the sense that $W_j = W_{-j}$ (because both are equal to $\overline{b}_j$ in this case). Special first price auctions are badly behaved: it is an equilibrium for j to bid W_j and for $-j$ to bid W_j with probability $1 \geq \pi > 0$ and $W_j - \epsilon$ with probability $1-\pi$ where ϵ (dependent on π) is chosen so that for j bidding W_j is at least as good as bidding slightly more than $W_j - \epsilon$, that is $(1-\pi)(1-c_j(W_j-\epsilon)) \leq (1-\pi/2)(1-c_j(W_j))$, equivalently

$$\frac{1-\pi}{1-\pi/2} \leq \frac{1-c_j(W_j)}{1-c_j(W_j-\epsilon)}.$$

Hence $-j$ gets zero while j gets $(1-\pi/2)(1-c_j(\overline{b}_j))$, that is any amount between $(1/2)(1-c_j(\overline{b}_j))$ and $1-c_j(\overline{b}_j)$. By contrast in the second price auction the only equilibrium is for both to bid W and for j to get $(1/2)(1-c_j(\overline{b}_j))$, so utility equivalence fails rather badly.

As it would be a terrible coincidence if one bidder happened to be indifferent between winning and staying out at the other's bidding cap we study only auctions that are not special. It is in the same spirit that we have ruled out the case $c_k(0^+) = 1$ in which a bidder happens to be indifferent between staying out and paying the fixed cost and winning for certain.

Roadmap

We say that an auction is *standard* if either one bidder is advantaged $W_{-d} > W_d$ or if they have equal willingness to pay $W_k = W_{-k} = W$ and the constraints do not bind so that $W_k = W_{-k} = B_k = B_{-k}$. An auction is *weakly symmetric with high stakes* if both bidders have the same strictly binding bidding cap $\overline{b}_k = \overline{b}_{-k}$ with $c_j(\overline{b}_j) < 1$ for both j. While weakly symmetric with high stakes auctions are not generic, they are important. For example, in the theory of voting, bidding caps are naturally interpreted as party size and Downsian platform competition prior to the election may force equality of party sizes. In the case of all-pay lobbying, as in Che and Gale (1998), the bidding caps are equal because they are established by law and apply equally to each lobbying group.

We first show that if an auction is not special it is either standard or weakly symmetric with high stakes.

Lemma 1. *If an auction is neither standard nor special it is weakly symmetric with high stakes.*

Proof. Since the auction is not standard the bidders must have equal willingness to pay $W_k = W_{-k} = W$ and the constraint must strictly bind for one of them, that is, for one j we have $c_j(\overline{b}_j) < 1$. This establishes that $W = \overline{b}_j$. Observe that if $c_{-j}(\overline{b}_j) > 1$ then $W_{-j} < \overline{b}_j$ so that weak symmetry is violated. Hence we can have weak symmetry and $c_j(\overline{b}_j) < 1$ only when $c_{-j}(\overline{b}_j) \leq 1$. Moreover, we cannot have $\overline{b}_{-j} < \overline{b}_j$ as this would violate weak symmetry. Hence, since the auction is not special, $c_{-j}(\overline{b}_j) \neq 1$, so $c_{-j}(\overline{b}_j) < 1$. This means in addition that if $\overline{b}_{-j} > \overline{b}_j$ weak symmetry is violated. Hence $\overline{b}_{-j} = \overline{b}_j = W$, so the auction is weakly symmetric with high stakes. □

In standard auctions we will show that the first price and all-pay auctions are utility equivalent, and the first and second price auctions are utility equivalent unless both cost functions are discontinuous and the disadvantaged bidder is unwilling to bid, in which case the second price auction is better for the advantaged bidder.

In weakly symmetric auctions with high stakes we show that the first and second price auctions are utility equivalent but the all-pay auction is not. We say that a weakly symmetric high stakes auction has *very high stakes* if $c_k(\overline{b}_k) < 1/2$ for both k: in this case the all-pay auction gives lower utility than the winner-pays. To further study the all-pay auction in the remaining case where for at least one bidder j we have $c_j(\overline{b}_j) > 1/2$ - which we call *moderately high stakes* - we make the additional generic assumption that $c_k(\overline{b}_k) \notin \{1/2, (1 + c_k(0^+))/2\}$ for either k. In this case one bidder gets zero, less than in the winner-pays auctions, but the other bidder may get either more or less.

We turn now to the details. We assume throughout, without further stating, that auctions are not special. By Lemma 1 this means auctions that are either standard or weakly symmetric with high stakes. We always assume that $c_k(0^+) \neq 1$ for both k.

3. Equilibrium

The characterization of second price auctions follows from the usual considerations of weak dominance:

Theorem 1. *In the second price auction unique equilibrium each bidder bids her willingness to bid W_k. An advantaged bidder $-d$ with $W_{-d} > W_d$ gets $1 - c_{-d}(W_d)$ and the other bidder d gets nothing. If $W_k = W_{-k} = W > 0$ then bidder k gets $(1/2)(1 - c_k(W))$. If $W_k = W_{-k} = 0$ both get 0.*

Proof. The strategies follow from the fact that in a second price auction bidding the willingness to bid weakly dominates all other strategies. The payoffs follow directly; in particular $W_k = W_{-k} = 0$ implies both have discontinuous cost so when bidding zero the prize is not awarded. Note that $c_d(W_d)$ might not be equal to $c_{-d}(W_d)$ if the constraint $\overline{b}_{-d}$ binds. □

We turn to first-price auctions.

Theorem 2. *In any equilibrium of a first-price standard auction:*

1. If $W_d = 0$ then the disadvantaged bidder d bids 0 and gets nothing while the advantaged bidder $-d$ bids 0^+ and gets $1 - c_{-d}(0^+)$. If $-d$'s cost is also discontinuous this is not equivalent to the second price auction where $-d$ gets $1 - c_{-d}(0) > 1 - c_{-d}(0^+)$, otherwise it is.

2. If $W_{-d} > W_d$ then $-d$ bids W_d and d loses for sure and chooses G_d with support in $[\underline{b}_d, W_d]$ such that it is optimal for $-d$ to bid W_d. One such strategy is to bid W_d for certain. The advantaged bidder $-d$ gets $1 - c_{-d}(W_d)$. Utilities are equivalent to the second price auction.

3. If $W_k = W_{-k} = B_k = B_{-k}$ one k bids $\min W_k$ and the other k chooses G_k with support in $[\underline{b}_k, W_k]$ such that it is optimal for $-k$ to bid $\min W_k$, and both get zero. Utilities are equivalent to the second price auction.

Proof. We start from the first case: For $W_d = 0$ it must be that $c_d(0^+) > 1$ (since we ruled out it being equal to 1), and by assumption then $1 > c_{-d}(0^+)$. Hence d must bid 0. If $-d$ bids 0 then $-d$ loses for sure because the prize is not awarded. As $1 > c_{-d}(0^+)$ it would be better to bid 0^+ and needlessly costly to bid more, so this is the equilibrium. The payoffs follow directly.

The other two cases we prove making use of a Lemma. We let $\underline{G}_k$ be the inf of the support of G_k. $\square$

Lemma 2. *In any equilibrium of a first price auction:*

1. Bids by k are in the range $[\underline{b}_k, \min W_k]$.

2. If $\min W_k > \min \underline{G}_k$ then one bidder gets zero and the other bidder bids $\min W_k$.

Proof. By weak dominance $b_k \leq W_k$ and $b_k \geq \underline{b}_k$. In no case does either bid more than $\min W_k$. If $W_{-d} > W_d$ then the tie-breaking rule means it is better for $-d$ to bid W_d than higher because this guarantees a win. Note that without the tie-breaking rule d might not have an optimal bid. If $W_{-d} = W_d$ this follows from $b_k \leq W_k$. This proves (1).

Suppose $\min W_k > \min \underline{G}_k$. If there is a k such that $c_k(b_k)$ is discontinuous and k plays 0 with positive probability, since a 0 bid yields zero for sure (either because $c_{-k}(0^+) = 0$ or because both are discontinuous and if both bid zero the prize is not awarded) then k gets 0. Suppose on the contrary that a k with discontinuous c_k (if any) does not play zero with positive probability. If $\min \underline{G}_k = 0$ it cannot be that both have an atom at 0^+ since it would be better to bid a bit more. For the same reason, if $\min \underline{G}_k > 0$ it cannot be that both have an atom at $\min \underline{G}_k > 0$. Suppose that $-k$ has no atom at 0^+ if $\min \underline{G}_k = 0$ or at $\min \underline{G}_k > 0$. If $\underline{G}_k > \min \underline{G}_k$ then $-k$ gets zero. If $\underline{G}_k = \min \underline{G}_k$ then k bidding down to $\underline{G}_k$ and $-k$ having no atom there implies that k gets zero. The reason is that k is bidding with positive probability in any interval $(\underline{G}_k, b_k]$ and those bids win with probability at most $G_{-k}(b_k) \to 0$ as $b_k \to \underline{G}_k$. Finally, suppose that k gets zero. If $-k$ bids less than $\min W_k$ then k would have a bid giving a positive payoff, so $-k$ must bid $\min W_k$ with probability 1. $\square$

Next we prove Theorem 2.

Proof. As we have already dealt with case (1) we may assume $W_d > 0$. If both bidders bid $\min W_k$ this is an equilibrium and we are done. Suppose instead that one bidder bids G_k with support in $[\underline{b}_k, W_k]$. If so Lemma 2 implies $\min \underline{G}_k < \min W_k$ so from Lemma 2 one bidder gets zero and the other bids $\min W_k$ with probability 1. If $W_{-d} > W_d$ then $-d$ does not get zero, so $-d$ is bidding $\min W_k$ which means by the tie-breaking rule that d loses for sure. If $W_k = W_{-k} = B_k = B_{-k}$ then whichever k bids $\min W_k$ also gets zero. $\square$

We now consider the case of a standard all-pay auction.

Theorem 3. *In any equilibrium of a standard all-pay auction:*

1. If $W_d = 0$ the strategies and payoffs are exactly as in the first price auction.

2. If $W_d \leq \underline{b}_{-d}$ bids are $b_d = \underline{b}_d$ and $b_{-d} = \underline{b}_{-d}$, hence d gets 0, and $-d$ gets $1 - c_{-d}(W_d) = 1 - c_{-d}(\underline{b}_{-d}) = 1$. Utilities are equivalent to the first price auction.

3. If $W_d > \underline{b}_{-d}$ the advantaged bidder $-d$ gets $1 - c_{-d}(W_d)$ and the disadvantaged bidder gets 0. The range $(\max \underline{b}_k, \min W_k)$ is nonempty, and in that open interval the strategies are given by $G_d(b_d) = 1 - c_{-d}(W_d) + c_{-d}(b_d)$ and $G_{-d}(b_{-d}) = c_d(b_{-d})$ while $G_k(\min W_k) = 1$. All remaining probability is on $\{\underline{b}_d, \underline{b}_{-d}, 0^+\}$. The disadvantaged bidder d has an atom at $\underline{b}_{-d}$ of size $G_d^0 = 1 - c_{-d}(W_d) + \lim_{b_d \downarrow \max \underline{b}_k} c_{-d}(b_d)$. The advantaged bidder $-d$ has an atom at $\underline{b}_{-d}$ if $c_{-d}(b_{-d})$ is continuous and at 0^+ if not. The size of the atom is $G_{-d}^0 = \lim_{b_d \downarrow \max \underline{b}_k} c_d(b_d)$. Utilities are equivalent to the first price auction.

While the detailed proof of the crucial third case is complex the idea which dates back to Hillman and Riley (1989) is not. They studied the case of linear cost and no bidding caps, but the case of strictly increasing continuous cost with $W_d < W_{-d}$, which is in Levine and Mattozzi (2020) is no more difficult. The idea is to deal first with low bids then with high bids. Low bids have to be very near zero, for if not someone is losing almost for sure and bidding a positive amount and would do better to bid zero. The near zero bidder must be earning zero, and it must be the disadvantaged bidder since the advantaged bidder can insure a positive utility by bidding a bit more than W_d. This is the first half of equivalence: the disadvantaged bidder gets nothing. Then we turn to the high bids. These have to be near W_d for if not the disadvantaged bidder can bid close to W_d and get positive utility. However, the disadvantaged bidder cannot actually bid W_d with positive probability since then it would get negative utility. Hence the advantaged bidder must be indifferent to bidding at W_d and winning for sure, which is exactly what they do in the winner-pays auctions, hence the equivalence.

Proof. In the first case for $W_d = 0$ it must be that $c_d(0^+) > 1$ (since we ruled out it being equal to 1). Hence d must bid 0. Given this, the auction now becomes a first price auction for $-d$.

In the second case by weak dominance neither bids more than $\min W_k$. If $W_{-d} > W_d$ then for $-d$ the tie-breaking rule means it is better to bid W_d rather than higher because this guarantees a win. Note that here again without the tie-breaking rule d might not have an optimal bid. Since $\underline{b}_{-d} \geq W_d > 0$ then c_{-d} cannot be discontinuous for that would imply $\underline{b}_{-d} = 0$. Suppose then that $W_d \leq \underline{b}_{-d}$ and $c_{-d}(b_{-d})$ is continuous. The unique equilibrium is $b_d = \underline{b}_d$ and $b_{-d} = \underline{b}_{-d}$, hence d gets 0, and $-d$ gets 1. The third case we prove through a series of Lemmas in Appendix I. □

4. High Stakes in Symmetric Auctions

We turn now to the non-standard case. As we are not studying special auctions, this means by Lemma 1 that we study symmetric auctions with high stakes in which the bidding caps are identical and strictly bind on both bidders.

Theorem 4. *In weakly symmetric high stakes first or second price auction there is a unique equilibrium and both bid $b_k = W$ and utility for k is $(1/2)(1-c_k(W))$.*

Proof. In the second price auction the equilibrium strategies are given by 1.

Turning to the first price auction, notice that both k must get positive utility since by bidding W they get at least $1/2 - (1/2)c_k(\overline{b}_k)$. In a weakly symmetric high stakes auction this is strictly positive. Hence Lemma 2 shows that this implies $W \leq \min \underline{G}_k$, that is, neither can bid less than W. From the equilibrium strategies each has a $1/2$ chance of winning so the payoffs follow. □

We next turn to the all-pay auction. Our treatment generalizes that of Che and Gale (1998) who study only linear cost functions. Recall that a weakly symmetric high stakes auction has *very high stakes* if $c_k(\overline{b}_k) < 1/2$ for both k.

Theorem 5. *In a weakly symmetric very high stakes all-pay auction there is a unique equilibrium, both bid $b_k = W$ and utility for k is $1/2 - c_k(W)$.*

Proof. Notice that both k must get positive utility since by bidding W they get at least $1/2 - c_k(\bar{b}_k) > 0$. Lemma 3 shows that then neither can bid less than W. From the equilibrium strategies each has a $1/2$ chance of winning so the payoffs follow. $\square$

In these auctions, while the all-pay strategies are the same as in the winner pays auctions, utility is strictly less since the bid has to be paid even when the auction is lost.

We next study the remaining weakly symmetric high stakes case with *moderate stakes* in the sense that for one bidder j we have $c_j(\bar{b}_j) > 1/2$. In Lemmas 5, 6 and 7 we characterize the equilibria and payoffs for the moderately high stakes case. For one part of the result we need the additional generic assumption that $c_k(\bar{b}_k) \neq (1 + c_k(0^+))/2$.

Theorem 6. *In a weakly symmetric moderately high stakes auction with $c_k(\bar{b}_k) \neq 1/2$ for either k, if $c_k(0^+) < 2c_k(W) - 1$ we may define $\tilde{b}_k$ as the unique solution to $c_k(b_k) = 2c_k(W) - 1$ and otherwise set $\tilde{b}_k = 0$.*

1. if $\max \tilde{b}_k > \max \underline{b}_k$ there is a unique equilibrium. Choose $z \in \{1, 2\}$ so that $\tilde{b}_z \geq \tilde{b}_{-z}$. Then z gets zero and $-z$ gets $\hat{u}_{-z} = c_{-z}(\max \tilde{b}_k) - (2c_{-z}(W) - 1)$. At W there are atoms $G_k[\{W\}] = 2(1 - c_{-k}(W) - \hat{u}_{-k})$. In $(\max \underline{b}_k, \max \tilde{b}_k)$ the equilibrium strategies are given by $G_z(b_z) = c_{-z}(b_z) + \hat{u}_z$ *and* $G_{-z}(b_{-z}) = c_z(b_z)$. *All remaining probability is on $\{\underline{b}_z, \underline{b}_{-z}, 0^+\}$. Bidder z has an atom at $\underline{b}_{-z}$. Bidder $-z$ has an atom at $\underline{b}_{-z}$ if $c_{-z}(b_{-z})$ is continuous and at 0^+ if not. The size of the atoms are $G^0_{-k} = \lim_{b_k \downarrow \max \underline{b}_\ell} c_k(b_k) + \hat{u}_k$.*

If $\max \tilde{b}_k \leq \max \underline{b}_k$ but $c_k(\bar{b}_k) \neq (1 + c_k(0^+))/2$ for both k then

2. if $c_{-j}(\bar{b}_{-j}) > 1/2$ there are three equilibria. In one both bidders get zero and have an atom at W of $G_k[\{W\}] = 2(1 - c_{-k}(W))$, with the remaining probability at 0. For each bidder z there is an equilibrium in which z gets 0 and $-z$ gets $\hat{u}_{-z} = c_{-z}(0^+) - (2c_{-z}(W) - 1)$. Bidder $-z$ has $G_{-z}[\{W\}] = 2(1 - c_z(W))$ with the remaining probability at 0^+ while $G_z[\{W\}] = 2(c_{-z}(W) - c_{-z}(0^+)$ with the remaining probability at 0.

3. if $c_{-j}(\bar{b}_{-j}) < 1/2$ there is a unique equilibrium in which j gets 0 and $-j$ gets $\hat{u}_{-j} = c_{-j}(0^+) - (2c_{-j}(W) - 1)$. Bidder $-j$ has $G_{-j}[\{W\}] = 2(1 - c_j(W))$ with the remaining probability at 0^+ while $G_j[\{W\}] = 2(c_{-j}(W) - c_{-j}(0^+)$ with the remaining probability at 0. This is the same as the second type of equilibrium in case (2) in which $z = j$.

The proof can be found in Appendix II, and an immediate implication is

Corollary 1. *In a weakly symmetric moderately high stakes all-pay auction with $c_k(\bar{b}_k) \notin \{1/2, (1 + c_k(0^+))/2\}$ for either k, a bidder z that gets 0 in the all-pay auction gets strictly less than in the winner-pays auctions. If $\max \tilde{b}_k > \max \underline{b}_k$ then $-z$ gets $c_{-z}(\max \tilde{b}_k) - (2c_{-z}(W) - 1)$, otherwise $-z$ gets $c_{-z}(0^+) - (2c_{-z}(W) - 1)$. This can be greater than the payoff in the winner-pays auctions $(1/2)(1 - c_{-z}(W))$, if for example $c_{-z}(W)$ is close to zero; it can also be less, for example, if $\tilde{b}_z = \tilde{b}_{-z}$ and both get zero.*

5. Revenue and Welfare Considerations

We turn now to the more standard question in auction theory, that of revenue equivalence. That is, so far we have been considering the utility of the parties. What happens with the bids? Even for elections politicians and some others seem to feel that high turnout, that is, high revenue as measured by the number of votes, is a vindication of democratic ideals or something like that, or, in the case of politicians, they simply view it in much the same way as athletes who like a larger audience. In the case of bribes, whether in the form of lavish dinners or high paying low responsibility jobs either for relatives or after the fact, the bids are to an extent a transfer payment, so the revenue is not entirely lost. Hence, from an efficiency point of view, given that the parties are indifferent between the different types of auctions, higher expected revenue is welfare improving. Hence we now take the point of the auctioneer and ask which auction yields the highest expected revenue?

The first price auction and second price auction are easily seen yield the same revenue - this is the standard revenue equivalence result in the simplest case of known values. If $1 > c_{-d}(0^+) > 0$ and $W_d = 0$ the winner incurs a greater cost but still pays nothing to the auctioneer; in the other cases the winning bid is the same for both auctions, so in all cases the auctioneer gets $\min W_k$. Note that the second price auction is more efficient than the first price auction when it avoids an unnecessary fixed cost. What about the all pay auction?

To get a bit of intuition recall from Theorem 3 that the equilibrium cdfs in the all pay auction are roughly given by the opponents cost plus their utility. If the cost - and so the cdf - is convex then the density is downwards sloping meaning that bids tend to be low, while if it is concave then the density is upwards sloping meaning that bids tend to be high. Hence we might expect that convexity also means low revenue, while concavity means high revenue. Our first result addresses the convex case and shows that this intuition is exact.

Theorem 7. *In a standard auction*

1. *if $W_d = 0$ or $W_d = W_{-d}$ and $c_k(b_k)$ is linear for both k then the all-pay auction is expected revenue equivalent to the first price auction. Otherwise*
2. *if $c_k(b_k)$ is convex for both k then the all pay auction yields strictly less expected revenue than the first price auction.*

Proof. If $W_d = 0$ we already observed in Theorem 3 that the all pay auction is the same as the first price auction so certainly yields the same expected revenue. We treat the remaining cases.

Let $\tilde{b}_k$ be the random variable on $[0, W_d] \cup \{0^+\}$ that is the equilibrium bid of k in the all pay auction and let p_k represent k's equilibrium chance of winning. From Theorem 3 $-d$ gets $1 - c_{-d}(W)$ so $1 - c_{-d}(W) = p_{-d} - Ec_{-d}(\tilde{b}_{-d})$. Similarly as d gets 0 we have $0 = p_d - Ec_d(\tilde{b}_d)$. Adding these together we see that in equilibrium $c_{-d}(W_d) = Ec_{-d}(\tilde{b}_{-d}) + Ec_d(\tilde{b}_d)$. Dividing through by $c_{-d}(W_d)$ as this is certainly positive we can write this as

$$\frac{Ec_{-d}(\tilde{b}_{-d})}{c_{-d}(W)} W + \frac{c_d(W_d)}{c_{-d}(W_d)} \frac{Ec_d(\tilde{b}_d)}{c_d(W_d)} W_d = W_d$$

where we know that W_d is the revenue from the first price auction. Moreover, if $c_k(b_k)$ is (weakly) convex since $c_k(0) = 0$ it follow that $c_k(b_k) \geq c_k(W_d) b_k / W_d$ including for $b_k = 0^+$ with strict inequality unless $c_k(b_k)$ is linear. We may write this as

$$b_k \leq \frac{c_k(b_k)}{c_k(W_d)} W_d \tag{5.1}$$

so that

$$E\tilde{b}_{-d} + \frac{c_d(W_d)}{c_{-d}(W_d)} E\tilde{b}_d \leq W_d \tag{5.2}$$

with strict inequality if either $c_k(b_k)$ fails to be linear. Recalling that this is a standard auction, in the symmetric case $c_d(W_d) = c_{-d}(W_d)$ and with linear cost this holds with equality which is the second part of (1). Otherwise the inequality is strict. □

What about the concave case? To start with, the reverse result is not true. The inequality 5.1 is reversed so the revenue inequality 5.2 is reversed reading

$$E\tilde{b}_{-d} + \frac{c_d(W_d)}{c_{-d}(W_d)} E\tilde{b}_d \geq W_d$$

but while concavity pushes revenue in favor of the all pay auction, this is not enough because of the term $c_d(W_d)/c_{-d}(W_d)$ which is less than one unless the auction is symmetric. Roughly speaking the more asymmetric is the auction the greater the concavity needed in cost for the all pay auction to generate more revenue that the first price auction. In one important special case we can make this trade-off explicit.

We say the $-d$ has a homogeneous cost advantage over d if $c_{-d}(b_{-d}) = \nu c_d(b_{-d})$ with $\nu < 1$. Define $\Omega = (1/W_d) \int_0^{W_d} c_d(b_d) db_d$. This is a measure of the convexity of $c_d(b_d)$. In fact, $\Omega = 1/2$ if $c_d(b_d)$ is linear, $\Omega > 1/2$ if $c_d(b_d)$ is strictly convex, and $<1/2$ if $c_d(b_d)$ is strictly concave.

Theorem 8. *In a standard auction if $-d$ has a homogeneous cost advantage, $\underline{b}_d = 0$ and $c_k(b_k)$ is concave for both k, the all pay auction generates more expected revenue than the first price auction if and only if*

$$\Omega < \frac{\nu}{1+\nu} c_d(W_d).$$

Note that the RHS is no greater than 1/2. We see from this that there are two forces working against revenue in the all pay auction: the RHS is increasing in ν so less symmetry, meaning smaller ν requires greater concavity meaning smaller Ω. Second, the RHS is increasing in $c_d(W_d)$ so that when the constraint binds on d and $c_d(W_d) < 1$ greater concavity is also required.

Proof. With a homogeneous cost advantage $\underline{b}_d = \underline{b}_{-d}$ so both are zero. Concavity implies $c_k(0^+) = 0$. Hence from Theorem 3 $G_d[\{0\}] = 1 - c_{-d}(W_d)$ and $1 - G_{-d}[\{W_d\}] = c_d(W_d)$ and these are the only atoms. Moreover in $(0, W_d)$ we have $G_{-d}(b_{-d}) = c_d(b_{-d})$ and $G_d(b_d) = c_{-d}(b_d) + 1 - c_{-d}(W_d)$. Integrating by parts we have $E\tilde{b}_{-d} = \int_0^{W_d} [1 - c_d(b_{-d})]\, db_{-d} = W_d - \Omega W_d$ and $E\tilde{b}_d = \int_0^{W_d} (c_{-d}(W_d) - c_{-d}(b_d))db_d = W_d c_{-d}(W_d) - \nu\Omega W_d$. Adding up we get

$$E\tilde{b}_{-d} + E\tilde{b}_d = (1 - \Omega + \nu c_d(W_d) - \nu\Omega)\, W_d$$

Hence the all pay auction generates more expected revenue than the first price auction exactly as stated. $\square$

6. Conclusion

In the spirit of Konrad Mierendorff this paper is a theory paper: it is not about a "killer-app" but rather provides set of tools for analyzing the important case of two bidder auctions under complete information. The intention, of course, is that these results will be used in applications, perhaps in ways that we cannot foresee.

Although this is not the purpose of this paper there are economic conclusions to be drawn from these results and we conclude by mentioning some of these. First, there is a long literature about the fact that small groups have an advantage in lobbying[7] - while the opposite is the case in voting.[8] Payments to politicians, when they are not direct cash payments, are typically in the form of employment contracts after leaving office, book deals, employment for spouses, and so forth[9] - and these are only paid by the winner. Empirically, then, lobbying is typically a winner pays auction, while, of course, voting is an

[7] See Olson (1965).
[8] See Levine and Mattozzi (2020)
[9] See Levine, Mattozzi and Modica (2022)

all pay auction. In principle this difference in mechanism might favor either larger or smaller groups: but the results here show that this is not the case - we have shown that only in very special circumstances do the consequences of the auction mechanism make a difference to the utility of the bidders. Hence we must look elsewhere to explain why small groups excel at lobbying and large groups in elections. Second: the reason for the difference in mechanisms should be clear - again, except under special circumstances, the winner pays auctions generate more revenue than the all-pay auction, so naturally politicians have an incentive to employ the former rather than the latter.

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Appendix I: All-Pay Auction Proofs

We now develop the key properties of the all-pay auction that lead to Theorem 3.

Lemma 3. *In an all pay auction with* $\underline{b}_{-d} < W_d \leq W_{-d}$

1. Bids are either $\min \underline{b}_k, 0^+$ *or in the range* $[\max \underline{b}_k, \min W_k]$ *and in particular* $G_k(\min W_k) = 1$.

2. In the non-empty range $(\max \underline{b}_k, \min W_k)$ *there can be no atoms and bidder* k *with* $\underline{b}_k < \underline{b}_{-k}$ *cannot have an atom at* $\underline{b}_{-k}$.

3. Unless both have an atom of size 1 *at* $\min W_k$ *one of the two bidders must get zero and there is a* $\overline{G}$ *such that there can be no open interval with zero probability for either bidder in* $(\max \underline{b}_k, \overline{G})$*, and* $[\overline{G}, \min W_k)$ *has zero probability. If one does not have an atom at* $\min W_k$ *then* $\overline{G} = \min W_k$ *and in particular each bidder must bid arbitrarily close to* $\max \underline{b}_k$ *and* $\min W_k$.

4. Suppose that $W_d = W_{-d}$ *and for one* k *we have* $c_k(\overline{b}_k) > 1/2$*. Then both do not have an atom of size* 1 *at* $\min W_k$*. If the auction is a standard one then both do not have an atom at* $\min W_k$.

Proof. 1. The hypothesis $\underline{b}_{-d} < W_d \leq W_{-d}$ implies that $W_k > 0$ for both k. This implies $c_k(0^+) < 1$ so $W_k > \underline{b}_k$. By weak dominance we may assume there are no bids $b_k \in [0, \underline{b}_k)$ as these are weakly dominated by $\underline{b}_k$. By weak dominance we may assume that $b_k \leq W_k$ since $b_k > W_k$ is weakly dominated by bidding 0.

After applying weak dominance we are free to apply iterated strict dominance as this does not eliminate any equilibrium strategies. By strict dominance we may assume that $b_k \leq W_{-k}$ since $b_k > W_{-k}$ is strictly dominated by $b_k - (b_k - W_{-k})/2$. In particular $G_k(\min W_k) = 1$ as asserted. By strict dominance we may assume there are no bids bid b_k for which $\underline{b}_k < b_k < \underline{b}_{-k}$ since $b_{-k} \geq \underline{b}_{-k}$ so that such bids are costly but losing.

Putting this together, we may restrict bids b_k to be either $\min \underline{b}_k, 0^+$ or in the range $[\max \underline{b}_k, \min W_k]$. By assumption $W_k > \underline{b}_{-k}$ for both bidders. Since $W_k > \underline{b}_k$ this implies $(\max \underline{b}_k, \min W_k)$ is nonempty.

2. In the range $(\max \underline{b}_k, \min W_k)$ there can be no atoms by the usual argument for all-pay auctions: if there was an atom at b_k then bidder $-k$ would

prefer to bid a bit more than b_k rather than a bit less, and since consequently there are no bids by $-k$ immediately below b_k bidder k would prefer to choose the atom at a lower bid. It is also the case that a bidder k with $\underline{b}_k < \underline{b}_{-k}$ cannot have an atom at $\underline{b}_{-k}$. If $-k$ has an atom there, then k should increase its atom slightly to break the tie. If $-k$ does not have an atom there, then k should shift its atom to $\underline{b}_k$ since it does not win either way.

3. Assume it is not the case that both bidders have an atom of size 1 at $\min W_k$.

Let $\overline{G}_k \equiv \inf\{b_k | G_k\left((b_k, \min W_k)\right) = 0\}$ - this is basically the highest bid by k with positive probability - and $\overline{G} = \max_k \overline{G}_k$. We observe that in $(\max_k \underline{b}_k, \overline{G})$ there can be no open interval with zero probability from either bidder. If bidder k has such an interval, then bidder $-k$ will not submit bids in that interval since the cost of the bid is strictly increasing so it would do strictly better to bid at the bottom of the interval. Hence there would have to be an interval in which neither bidder submits bids. But then, for the same reason, it would be strictly better to lower the bid for bids slightly above the interval. This implies that if $\overline{G} > \max \underline{b}_k$ each bidder must bid arbitrarily close to $\max_k \underline{b}_k$.

We can now show that one of the two must get zero. Denote by $\mathscr{B} \equiv \{\underline{b}_d, \underline{b}_{-d}, 0^+\}$. If $\overline{G} > \max \underline{b}_k$ both must bid arbitrarily close to $\max \underline{b}_k$. If $\overline{G} = \max \underline{b}_k$ since both do not have an atom of size one at $\min W_k$ one must put positive weight on the set $\mathscr{B}$. If only one does so they get zero, so we may assume both do so.

Suppose first that $\max \underline{b}_k > 0$ or both have continuous cost. From (2) a bidder k with $\underline{b}_k < \underline{b}_{-k}$ cannot have an atom at $\underline{b}_{-k}$. If $\underline{b}_k = \underline{b}_{-k} > 0$ or both have continuous cost both cannot have an atom at $\underline{b}_k$ since both would like to bid a bit more.

If $\overline{G} > \max \underline{b}_\ell$ since one k has an opponent without an atom at $\max \underline{b}_\ell$ and $(\overline{G}_k, \min W_k)$ has zero probability, then bidding down to $\max \underline{b}_\ell$ bidder k can get more than zero only if $-k$ has positive probability of playing less than $\max \underline{b}_\ell$; this implies that $\max \underline{b}_\ell = \underline{b}_k$ and that $-k$ gets zero since her bids below $\underline{b}_k$ lose for sure and have positive probability.

If $\overline{G} = \max \underline{b}_k$ then both must have a positive probability of playing $\mathscr{B}$ so for one k it must be that $\underline{b}_k = \max \underline{b}_k$ so k has an atom there. This means that $-k$ does not so loses for sure and gets zero.

Suppose now that $\max \underline{b}_k = 0$ and that k has a discontinuous cost. If k bids 0 with positive probability then k gets zero, so we may assume this is not the case. Hence if $-k$ bids 0 with positive probability then $-k$ gets 0 so we may assume neither has an atom at 0. They cannot both have an atom at 0^+ so one ℓ has an opponent without an atom there. If $\overline{G} = 0$ then ℓ should not bid 0^+ since this loses for sure. This implies that ℓ has an atom of size 1 at $\min_k W_k$ and since $-\ell$ does not $-\ell$ has a bid that loses for sure, so cannot get more than 0 so $-\ell$ must get 0. If $\overline{G} > 0$ then ℓ bidding down to zero must get zero.

This establishes that unless both have an atom of size 1 at $\min W_k$ one must get zero.

Suppose that one does not have an atom at $\min W_k$. If neither has an atom and $\overline{G} < \min W_k$ then each can get can get a positive utility by bidding $(\min W_k + \overline{G})/2$, contradicting the fact that one must get zero. If k has an atom and $-k$ does not and $\overline{G} < \min W_k$ then k should move their atom to a lower bid.

4. Suppose in addition that either $W_{-d} > W_d$ or if $W_d = W_{-d}$ then for one k we have $c_k(\overline{b}_k) > 1/2$. Then both do not have an atom of size 1 at $\min W_k$. If in fact the auction is a standard one then both to not have an atom at $\min W_k$.

Suppose that $W_d = W_{-d}$ and for one k we have $c_k(\overline{b}_k) > 1/2$. If both have an atom of size one at $\min W_k$ then k has a negative utility. So this is ruled out.

If $W_{-d} > W_d$ and $-d$ has an atom at $\min W_k$ then d loses for sure so has negative utility. The other standard auction case is $W_k = W_{-k} = B_k = B_{-k}$ so if both have an atom both get negative utility because the probability of winning B_k is less than one, while the probability of paying B_k is one. This shows that in the standard case both do not have an atom. □

Next we prove Theorem 3.

Proof. In both cases from Lemma 3 (3) and (4) $\overline{G} = \min W_k$so both must bid arbitrarily close to $\min W_k$.

If $W_{-d} > W_d$ then $-d$ can get $\hat{u}_{-d} = 1 - c_{-d}(W_d) > 0$ by bidding W_d. Hence it must be $-d$ that gets zero. On the other hand $-d$ cannot get more than this as they must bid arbitrarily close to W_d so must get less than or equal this amount. In the symmetric case each k must bid arbitrarily close to W_k so cannot get a positive amount.

We now find the equilibrium strategies. From the absence of zero probability open intervals in $(\max \underline{b}_k, \min W_k)$ it follows that the indifference condition for the advantaged bidder $-d$ is

$$G_d(b_{-d}) - c_{-d}(b_{-d}) = 1 - c_{-d}(W_d)$$

must hold for at least a dense subset. For the disadvantaged bidder we have

$$G_{-d}(b_d) - c_d(b_d) = 0$$

for at least a dense subset. This uniquely defines the cdf for each bidder in $(\max \underline{b}_k, \min W_k)$:

$$G_d(b_d) = 1 - c_{-d}(W_d) + c_{-d}(b_d)$$

and

$$G_{-d}(b_{-d}) = c_d(b_d)$$

As these are differentiable they can be represented by continuous density functions which are by taking the derivative.

The remaining probability mass must be on $\mathscr{B} = \{\underline{b}_d, \underline{b}_{-d}, 0^+\}$. If d has an atom at 0^+then $-d$ does not.

If $-d$ gets positive then $-d$ does not have an atom at 0, In this case d must have an atom at $\underline{b}_d$ which must lose for sure. This means that for $-d$ the mass is on either $\underline{b}_{-d}$ or if $c_{-d}(b_{-d})$ is discontinuous, on 0^+. Note that in the case where $\underline{b}_{-d} < \underline{b}_d$ so the advantaged bidder has less of a head start advantage than d it could only be the case that $-d$ had an atom at $\underline{b}_{-d}$ if $-d$ was also getting zero. However, in this case we see that $G_{-d}(\max \underline{b}_k) = G_{-d}(\underline{b}_d) = c_d(\underline{b}_d) = 0$ so in fact $-d$ places no probability on $\mathscr{B}$.

If both get 0 and $\underline{b}_\ell > 0$ for some ℓ then each k must put their mass on $\underline{b}_k$.

Finally if both get 0 and $\underline{b}_k = \underline{b}_{-k} = 0$ then $G_k(0^+) = c_{-k}(0^+)$ each must put their mass on zero, otherwise the other would strictly prefer 0^+.

We may compute the size of these atoms from the excess probability mass from G_k as $G^0_d = 1 - c_{-d}(W_d) + c_{-d}(\max \underline{b}_k)$ and $G^0_{-d} = c_d(\max \underline{b}_k)$. In particular if $\max \underline{b}_k = \underline{b}_d$ then $G^0_{-d} = 0$, otherwise $G^0_{-d} = c_d(\underline{b}_{-d})$ which means if d bids $\underline{b}_{-d}$ and wins for sure that d gets 0. Moreover if $c_{-d}(b_{-d})$ is discontinuous so that $\underline{b}_{-d} = 0$ then $\max \underline{b}_k = \underline{b}_d$ so there is no atom. □

Appendix II: Weakly Symmetric Moderately High Stakes Auctions

Here we prove Theorem 6.

Lemma 4. *In a weakly symmetric moderately high stakes auction with* $c_k(b_k) \neq 1/2$ *for either* k*, both have an atom at* W *of size less than one, one bidder,* z*, gets zero and there is a* $\overline{G} < W$ *such there can be no open interval with zero probability for either bidder in* $(\max b_k, \overline{G})$ *and* $[\overline{G}, W)$ *has zero probability. If* $\hat{u}_k$ *are the equilibrium utilities the size of the atoms are given by* $G_k[\{W]\} = 2(1 - c_{-k}(W) - \hat{u}_{-k})$.

Proof. By definition for some j we have $c_j(W) > 1/2$. The parts that do not follow directly from 3 and 4 are that both must have an atom, the size of the atoms, and that $\overline{G} < W$. Observe that the utility to $-k$ from bidding W is $\hat{u}_{-k} = 1 - G_k[\{W\}] + G_k[\{W\}]/2 - c_{-k}(W) = 1 - G_k[\{W\}]/2 - c_{-k}(W)$. We may write this as $G_k[\{W]\} = 2(1 - c_{-k}(W) - \hat{u}_{-k})$, the result for the size of the atom. Since $\hat{u}_z = 0$ it follows that $G_{-z}[\{W]\} = 2(1 - c_z(W)) > 0$ so that $-z$ has an atom. If z does not have an atom then $\overline{G} = W$ otherwise $-z$ would lower their atom a bit. The result will follow from $\overline{G} < W$.

The intuition for $\overline{G} < W$ is this. In the asymmetric case where the constraints bind $z = d$ the disadvantaged bidder. Although $-z$ has an atom at W if z were to try to bid $\min W_k$ then the tie-breaking rule means that z would lose for sure reflecting the fact that $-d$ is willing to bid a bid more than $\min W_k$ and d is not. Here, however, neither is able to bid more than W, so if z bids W they win with probability $1 - G_{-z}\{[W]\}/2 > 1/2$ and this is a substantially higher probability than bidding just below W.

Specifically if $\overline{G} = W$ there must be a sequence of bids by z approaching W with zero utility. That is, these bids have cost nearly $c_z(W)$ and have very little chance of losing except to the atom by $-z$ at W. Specifically as $b_z \uparrow W$ it must be that $1 - G_{-z}[\{W\}] - c_z(b_z) \to 0$. Since c_z is continuous at $W > 0$ it follows that $1 - G_{-z}[\{W\}] - c_z(W) = 0$. Hence for bidding W we find that z gets

$$1 - G_{-z}[\{W\}]/2 - c_z(W) = 1 - (1 - c_z(W))/2 - c_z(W) = (1/2)(1 - c_z(W)) > 0$$

which contradicts the fact that z must not get more than zero from any bid. It follows that $\overline{G} < W$. This in turn shows that $-z$ has an atom at W. □

Lemma 5. *In a weakly symmetric moderately high stakes auction with $c_k(b_j) \neq 1/2$ for either k, the equation $c_k(b_k) = 2c_k(W) - 1$ has a unique solution $\tilde{b}_k > \underline{b}_k$ if and only if $c_k(W) > 1/2$ and $c_k(0^+) < 2c_k(W) - 1$.*

Proof. If $c_k(W) < 1/2$ then $c_k(b_k) = 2c_k(W) - 1$ has no solution. Otherwise, the LHS is strictly increasing and continuous for $b_k > \underline{b}_k$ and $\lim_{b_k \downarrow \underline{b}_k} c_k(b_k) = \max\{c_k(0^+), c_k(\underline{b}_k)\}$. Certainly $c_k(\underline{b}_k) = 0 < 2c_k(W) - 1$, while $c_k(W) > 2c_k(W) - 1$, so the former is the condition for a solution. □

Lemma 6. *A weakly symmetric moderately high stakes auction with $c_k(b_j) \neq 1/2$ for either k has an equilibrium with $\overline{G} > \max \underline{b}_k$ if and only if $\max \tilde{b}_k > \max \underline{b}_k$, in which case it is unique, there is a bidder z satisfying $\tilde{b}_z \geq \tilde{b}_{-z}$ who gets zero and $\hat{u}_{-z} = c_{-z}(\max \tilde{b}_k) - (2c_{-z}(W) - 1)$. At W there are atoms $G_k[\{W]\} = 2(1 - c_{-k}(W) - \hat{u}_{-k})$. In $(\max \underline{b}_k, \overline{G})$ the equilibrium strategies are given by $G_z(b_z) = c_{-z}(b_z) + \hat{u}_z$ and $G_{-z}(b_{-z}) = c_z(b_z)$. All remaining probability is on $\{\underline{b}_z, \underline{b}_{-z}, 0^+\}$. bidder z has an atom at $\underline{b}_{-z}$. bidder $-z$ has an atom at $\underline{b}_{-z}$ if $c_{-z}(b_{-z})$ is continuous and at 0^+ if not. The size of the atoms are $G^0_{-k} = \lim_{b_k \downarrow \max \underline{b}_\ell} c_k(b_k) + \hat{u}_k$. If $\max \tilde{b}_k > \max \underline{b}_k$ there is no other equilibrium. In case $c_{-j}(\overline{b}_{-j}) < 1/2$ then $z = j$.*

Proof. First we show that an equilibrium with $\overline{G} > \max \underline{b}_k$ also has $\overline{G} = \max \tilde{b}_k$, then finish the proof by constructing the unique equilibrium when $\max \tilde{b}_k > \max \underline{b}_k$.

Assume that $\overline{G} > \max \underline{b}_k$. Observe by Lemma 3 there are no atoms in $(\max \underline{b}_k, W)$, and since $\overline{G} > \max \underline{b}_k$ both must bid up to $\overline{G}$. In particular when z bids at $\overline{G}$ then z gets $(1 - G_{-z}[\{W]\}) - c_z(\overline{G})) = 0$ while by Lemma

4 $G_{-z}[\{W\}] = 2(1 - c_z(W))$, so $(2c_z(W) - 1) - c_z(\overline{G})) = 0$, and in particular $\overline{G} = \tilde{b}_z$. Notice this shows that the bidder z that gets zero must be one for whom $c_z(W) > 1/2$. Moreover, at $\overline{G}$ we have that $(1 - G_z[\{W\}]) - c_{-z}(\overline{G})) = \hat{u}_{-z}$ and $G_z[\{W\}] = 2(1 - c_{-z}(W) - \hat{u}_{-z})$ giving $(2c_{-z}(W) + 2\hat{u}_{-z} - 1) - c_{-z}(\overline{G})) = \hat{u}_{-z}$ or $\hat{u}_{-z} = -(2c_{-z}(W) - 1) + c_{-z}(\overline{G}) \geq 0$. Hence it must be that $c_{-z}(\overline{G}) \geq 2c_{-z}(W) - 1$ which since $c_{-z}(b_{-z})$ is strictly increasing in b_{-z} for $b_z > \underline{b}_z$ means that $\overline{G} \geq \tilde{b}_{-z}$. Note that if $-z$ has $c_{-z}(W) < 1/2$ then $c_{-z}(\overline{G}) \geq 2c_{-z}(W) - 1$ is always satisfied and in this case by definition we have $\tilde{b}_{-z} = 0$. Hence indeed the bidder getting zero must satisfy $\tilde{b}_z \geq \tilde{b}_{-z}$ and $\hat{u}_{-z} = -(2c_{-z}(W) - 1) + c_{-z}(\overline{G})$ as asserted.

Now assume that $\overline{G} = \tilde{b}_z \geq \tilde{b}_{-z}$. The construction of equilibrium proceeds much as in the proof of Theorem 3. The atoms at W are given by Lemma 4. Between $[\overline{G}, W)$ the cdfs are flat. In $(\max \underline{b}_k, \overline{G})$ the indifference condition for $-z$ is

$$G_z(b_{-z}) - c_{-z}(b_{-z}) = \hat{u}_{-z}$$

must hold for at least a dense subset. For bidder z we have

$$G_{-z}(b_z) - c_z(b_z) = 0$$

for at least a dense subset. This uniquely defines the cdf for each bidder in $(\max \underline{b}_k, \min W_k)$ as given in the result.

The argument concerning $\mathscr{B} = \{\underline{b}_z, \underline{b}_{-z}, 0^+\}$ is exactly as in the proof of Theorem 3 replacing d with z.

Finally, we show that there is no other equilibrium if $\max \tilde{b}_k > \max \underline{b}_k$. Observe that if $\overline{G} = \max \underline{b}_k$ then ℓ bidding $b_\ell > \max \underline{b}_k$ earns $(2c_\ell(W\} - 1 + 2\hat{u}_\ell) - c_\ell(b_\ell))$ which is greater than $\hat{u}_\ell$ for $\max \underline{b}_k < b_\ell < \tilde{b}_\ell$. Hence $\overline{G} > \max \underline{b}_k$. □

Lemma 7. *In a weakly symmetric moderately high stakes auction with* $c_k(\overline{b}_k) \notin \{1/2, (1 + c_k(0^+))/2\}$ *for either* k*, suppose that* $\max \tilde{b}_k \leq \max \underline{b}_k$*. Then there are three possible types of equilibria. In one both get zero, have an atom at* W *of* $G_k[\{W\}] = 2(1 - c_{-k}(W))$ *with the remaining probability at* 0*. For each* z *there is an equilibrium in which* z *gets* 0 *and* $-z$ *gets* $\hat{u}_{-z} = c_{-z}(0^+) - (2c_{-z}(W) - 1)$*.*

bidder $-z$ *has* $G_{-z}[\{W\}] = 2(1 - c_z(W))$ *with the remaining probability at* 0^+*while* $G_z[\{W\}] = 2(c_{-z}(W) - c_{-z}(0^+)$ *with the remaining probability at* 0*. If* $c_{-j}(\overline{b}_{-j}) > 1/2$ *then all three types co-exist. If* $c_{-j}(\overline{b}_{-j}) < 1/2$ *the only the latter type exists, and only for* $z = j$*, so it is unique.*

Proof. The only case in which $\max \tilde{b}_k > \max \underline{b}_k$ fails is if $c_k(0^+) > 2c_k(W) - 1$ for both k so $\max \underline{b}_k = 0$. In this case $\overline{G} = 0$ from Lemma 6.

Each k faces probability $1 - G_{-k}[\{W\}] = 2c_k(W) - 1 + 2\hat{u}_k)$ of $-k$ playing in $\{0, 0^+\}$. Bidder z therefore cannot bid 0^+ since even if $-z$ was not bidding 0^+ it would still create a loss for k to bid 0^+. This implies that if $c_{-j}(\overline{b}_{-j}) < 1/2$ then $z = j$.

There are now two possibilities. If $c_{-j}(\overline{b}_{-j}) > 1/2$ it is an equilibrium for $-z$ also to get zero and bid zero for the same reason.

There is also an equilibrium where $\hat{u}_{-z} > 0$ in which case $-z$ must bid 0^+ but not 0. In this case we must have $2c_{-z}(W) - 1 + 2\hat{u}_{-z} - c_{-z}(0^+) = \hat{u}_{-z}$ giving $\hat{u}_{-z} = c_{-z}(0^+) - (2c_{-z}(W) - 1)$ and $G_z[\{W\}] = 2(1 - c_{-z}(W) - \hat{u}_{-z}) = 2(c_{-z}(W) - c_{-z}(0^+)$. □