Strong Bubbles and Common Expected Bubbles in a Finite Horizon Model*

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Abstract

An (A) expected (strong) bubble is said to exist if it is mutual knowledge that the price of the asset is higher than the expected (possible) dividend. By requiring common knowledge instead of mutual knowledge, the new concept of common expected bubble (common strong bubble) is developed. In a simple finite horizon model with asymmetric information and short sale constraints, which follows Allen, Morris and Postlewaite (1993), it is showed that two results hold true for any finite number of agents: First, common strong bubbles never exist in any rational expectations equilibrium; Second, it is possible to have a bubble, which is both a strong bubble and a common expected bubble, in a rational expectations equilibrium, even with common knowledge of trades. Furthermore, the first result crucially depends on the implicit assumption of perfect memory, hence an example of common strong bubbles can be constructed in case that agents are forgetful. Based on these results, this paper, as well as Conlon (2004), provides a partial answer to what properties rational bubbles can have and cannot have in a rational expectations equilibrium.

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1 Introduction

Bubbles exist in many markets, not only with those assets whose fundamental values are hard to be determined or to be observed (stocks, for instance), but also with some assets whose fundamental values are known to be less than the prices (fiat money, for instance). How can bubbles be explained and what must be true for the existence of bubbles? Though claiming that most bubbles are irrational is much easier than interpreting bubbles in a rational way, economists have made and are still making efforts to deal with the latter.

Among the huge literature on the existence of bubbles, one strand has developed the models based on the existence of some irrational agents, often called as noise traders in the literature (see, for example, De Long, Shleifer, et al. (1990), Abreu and Brunnermeier (2003), and Zurita (2004)). Papers in this strand interpret the bubbles by the interaction between the rational and the irrational.[1]

Another strand of the literature, has tried to model the bubbles under the more traditional assumption that all agents are rational.[2] In such settings, an asset bubble can be explained either by the assumption of infinite horizon or by the infinite presence of new agents (see Tirole (1982) and Tirole (1985) for example). However, in order to interpret the existence of a finite horizon bubble[3] in a rational expectations equilibrium with a finite number of agents, either change of standard assumptions (for instance, symmetric information) or introduction of specific settings (for instance, short sale constraints) has to be made. Thus the question comes: What is the minimum requirement for the existence of such a rational bubble?

By the well-known no-trade theorem by Milgrom and Stokey (1982), under the standard setting, if the initial allocation is efficient relative to each agent's belief, then the common knowledge of feasibility of and voluntary participation in trade will give agents no incentive to trade, no matter whether they have private information or not. If there is no trade in a finite horizon economy, there is certainly no bubble. Hence the ex ante inefficiency of endowment allocation, or the potential gains from trade, should be one necessary condition for such a bubble to exist.[4]

By Allen, Morris and Postlewaite (1993) (AMP (1993) henceforth), private information about the states and short sale constraints for all agents are another two necessary conditions for the existence of strong bubbles. Under all these three together with a fourth requirement that the agents' trade should not be common knowledge, an example of strong bubbles in a rational expectations equilibrium with three agents and

[1]Though the rational agents have incentive to take advantage of the irrational, it is possible that noise traders may actually earn a higher expected return than rational investors do. For details, see De Long, Shleifer, et al. (1990).

[2]In fact it is assumed that the rationality of the agents is common knowledge in most paper of this strand. Under the assumption of rational expectations, these two are equivalent.

[3]Among all the bubble phenomena, finite horizon bubbles have occurred most frequently and are probably most puzzling.

[4]For a complete proof, see Tirole (1982).

three periods[5] is presented in that paper. This model captures the "greater fools" dynamic pretty well in the sense that due to the asymmetric information agents may hold a worthless asset at a positive price in the first period (hence the strong bubble), in hopes of selling it in the second period to someone else who thinks it may be worth something. In short, a rational bubble can exist under this setting because even though everyone knows that the asset is overpriced they may still hold it with the belief that others might think that it is valuable.

Given the success of Allen, Morris and Postlewaite model, economists are somewhat not satisfied with the last assumption, the one requiring no common knowledge of trades, since many bubbles do exist in reality with the public information of agents' actions. Conlon (2004), another paper quite related to the issue we addressed above, according to my knowledge, was the first to give a strong bubble example in a similar setting[6] where there are only two agents. Since trades are automatically common knowledge for the two-agent case, this result has questioned the necessity of assumption of no common knowledge of trades for the existence of a finite horizon bubble in a rational expectations equilibrium. Another contribution of Conlon (2004) is that the bubble in the model is not only strong but also robust to nth order knowledge, that is (all agents know that)^n the price is higher than any possible dividend agents will receive.

Based on the fact of the existence of nth order bubbles, one may naturally ask whether a bubble can be robust to common knowledge. In this paper, by requiring common knowledge instead of mutual knowledge, we develop two new concepts of bubbles: common expected bubble and common strong bubble. The concept of common strong bubble is so "strong" that it can be showed never to exist in any rational expectations equilibrium under the standard assumption of perfect memory. However, we are able to show that within the same framework as AMP (1993) model but with common knowledge of trades, a strong bubble can exist in the case of two agents, and this bubble can still exist even when it is common knowledge that the price is higher than the expected dividend agents will receive (hence a common expected bubble). Moreover, such a bubble, both strong and common expected, is robust to symmetric perturbation, and can exist for any finite number of agents.[7] This result, on one hand, weakens the assumptions of the models of bubbles by reducing the four necessary conditions to three, and hence improves these models' applicability and powers

[5] It has been shown in that paper that there is no expected bubble in the last two period under their framework, which will be described in section 2, hence the minimum number of periods for the existence of a bubble is 3.

[6] The setting of Conlon (2004) model differs from AMP (1993) model in the sense that agents' information structures are determined both by the private signals they receive at the beginning of period 1 and by the public signals they receive at the beginning of every period. The information structures are chosen so that prices reveal no additional information.

[7] I will assume that each agent is distinguished from another in the sense that either their beliefs are heterogeneous or their information structures are different, or both. Otherwise, this result would hold trivially since each agent can be "divided" according to endowments into any finite number of subagents.

in interpretation. On the other hand, the surprising result of the existence of common expected bubbles is somewhat counterintuitive but captures the idea that agents do not rush in face of bubbles since, given the common knowledge of the heterogeneous beliefs and the information structures, they believe that they can take advantage of it in a later period. Furthermore, it is worth noting that the implicit assumption of perfect memory plays a key role in the nonexistence result for common strong bubbles: if people might forget some information they knew before, then a common strong bubble may happen in equilibrium.

The next section of the paper introduces the basic framework following AMP (1993), gives four concepts of bubbles, and shows the nonexistence of common strong bubbles in any rational expectations equilibrium. Section 3 presents a simple example of rational bubbles with two agents, which is both a strong bubble and a common expected bubble. Section 4 gives a counterexample of the nonexistence result of common strong bubbles if we allow for imperfect memory. Section 5 shows the general results for any finite number of agents, and some concluding remarks and discussions are made in the last section.

2 The Model

2.1 Basic Setup

The same framework is established here as in AMP (1993), except that the requirement that the trades should not be common knowledge is removed.

In the pure exchange economy under study, there are I (≥ 2) risk neutral[8] agents $(i = 1, 2, \cdots, I)$, T (≥ 3) periods $(t = 1, 2, \cdots, T)$ and N (≥ 2) states of the world represented by $\omega \in \Omega$. Only 2 assets exist in the market: one riskless (money) and the other risky. There is no discount between any two periods. Each share of the risky asset will only pay a state-dependent dividend denoted by $d(\omega)$ at the end of period T.

Agent i is endowed with m_i units of money and e_i shares of the risk asset at the beginning of period 1. In each period t and in each realized state ω, agents can exchange claims on the risky asset at a state-and-period-dependent price $P_t(\omega)$. Agent i's net trade in period t when state ω is realized is denoted by $x_{it}(\omega)$, and we write $x_i = (x_{i1}, x_{i2}, \cdots, x_{iT})$, $x_t = (x_{1t}, x_{2t}, \cdots, x_{It})$ and $x = (x_1, x_2, \cdots, x_I)$. Hence agent i's final consumption in state ω with net trades x_i at price $P(\omega) = (P_1(\omega), P_2(\omega), \cdots, P_T(\omega))$, denoted by $y_i(\omega, P(\omega), x_i)$, is equal to $m_i + e_i P_T(\omega) + \sum_{t=1}^{T} x_{it}(\omega)[P_{t+1}(\omega) - P_t(\omega)]$, where $P_{T+1}(\omega) = d(\omega)$. Under the assumption of identity utility function, agent i's

[8] Agents are assumed to be either risk averse or risk neutral in AMP (1993). Here for simplicity, I only consider the case of risk neutrality. All the results will remain valid for the risk averse case as long as the potential gain from trade is high enough.

payoff in state ω with net trades x_i at price $P(\omega)$, is just his final consumption, hence $u_i(\omega, P(\omega), x_i) = y_i(\omega, P(\omega), x_i)$.

Each agent i has a subjective belief about the probability distribution of the state, denoted by $\pi_i(\omega)$.[9] $\forall i = 1, 2, \cdots, I, \forall \omega \in \Omega, \pi_i(\omega) > 0$.

2.2 Information Structure

At the beginning of each period t, before observing the current price and making the trade, agent i's information about the state is represented by S_{it}, a partition of the space Ω, and his price–and-trade-refined information is represented by S_{it}^{PX}.[10] We denote by $s_{it}(\omega)$ ($s_{it}^{PX}(\omega)$) the partition member in S_{it} (S_{it}^{PX}) containing the state ω. In other words, $s_{it}(\omega)$ consists of all the possible states agent i believes he might be in when the state ω is realized. For example, $s_{i1}(\omega_1) = \{\omega_1, \omega_2\}$ means that at period 1 agent i believes he might be either in ω_1 or ω_2 when ω_1 is realized.

S_{it}^{PX} is determined by (S_{it}, P_t, x_t) such that

$$\forall \omega \in \Omega, s_{it}^{PX}(\omega) = s_{it}(\omega) \cap \{\omega' | P_{t'}(\omega') = P_{t'}(\omega) \text{ and } x_{t'}(\omega') = x_{t'}(\omega) \ \forall t' \leq t\}$$

Obviously $\forall i = 1, 2, \cdots, I, \forall t = 1, 2, \cdots, T, \forall \omega \in \Omega, \{\omega\} \subseteq s_{it}^{PX}(\omega) \subseteq s_{it}(\omega)$.

We assume agents have perfect memory so that

$$\forall i = 1, 2, \cdots, I, \forall \omega \in \Omega, \forall t > t', s_{it}(\omega) \subseteq s_{it'}(\omega)$$

Obviously this implies that

$$\forall i = 1, 2, \cdots, I, \forall \omega \in \Omega, \forall t > t', s_{it}^{PX}(\omega) \subseteq s_{it'}^{PX}(\omega)$$

It should be noted that when agents make trades to optimize their payoffs, the information they based on is $s_{it}^{PX}(\omega)$ instead of S_{it}, since it is assumed that rational agents should make use of all the information they can obtain. As we will see, the assumption of perfect memory plays an important role in Proposition 1, which we will state at the end of this section.

2.3 Rational Expectations Equilibrium

Before we come to the definition of rational expectations equilibrium, in order to be consistent with the AMP (1993) model, two concepts have to be introduced first.

[9] We assume that π_i's are heterogeneous in order to give agents incentive to trade. For other approaches to induce trade, see AMP (1993) for details.

[10] In the AMP (1993) model, they only focus on the price-refined information S_{it}^{P}. This is because they assume that the trades are not common knowledge and hence agents cannot get additional information from trades.

Definition 1 (Information Feasibility) *Agent i's net trades x_i are information feasible if in each period t, x_{it} is measurable with respect to player i's price–and-trade-refined information, S_{it}^{PX}. Formally, x_i are information feasible if*

$$\forall t = 1, 2, \cdots, T, \forall \omega \in \Omega, s_{it}^{PX}(\omega) \subseteq \{\omega' : x_{it}(\omega') = x_{it}(\omega)\}$$

The last part of the above expression is equivalent to $\forall \omega', \omega'' \in s_{it}^{PX}(\omega), x_{it}(\omega') = x_{it}(\omega'')$, which might capture more intuition than the one used in the definition. Basically, information feasibility rules out the possibility of acting differently given the same information.

Definition 2 (No Short Sales) *Agent i's net trades x_i satisfy no short sales if in each period t and in each state ω agent i's holdings of the risky asset are non-negative. Formally, x_i satisfy no short sales if*

$$\forall t = 1, 2, \cdots, T, \forall \omega \in \Omega, e_i + \sum_{s=0}^{t} x_{it}(\omega) \geq 0$$

As shown in AMP (1993), this no short sales condition is necessary for the existence of a bubble in a rational expectations equilibrium. It should be noted that there is no constraint on the short sales of money.

Denote by $j_t(\omega)$ the join of $s_{1t}(\omega), s_{2t}(\omega), \cdots, s_{It}(\omega)$,[11] and by $m_t(\omega)$ the meet of $s_{1t}(\omega), s_{2t}(\omega), \cdots, s_{It}(\omega)$.[12]

Now we are ready to give the definition of a Rational Expectations Equilibrium in this pure exchange economy.

Definition 3 (Rational Expectations Equilibrium) $(P, x) \in R_+^{NT} \times R^{INT}$ *is a Rational Expectations Equilibrium if*

(C1) $\forall i = 1, 2, \cdots, I$, *$x_i$ are information feasible and satisfy no short sales. Denote the set of all such x_i's by $F_i(e_i, P, x_{-i}, S_i)$, where $S_i = (S_{i1}, S_{i2}, \cdots, S_{iT})$.*[13]

(C2) $\forall i = 1, 2, \cdots, I$, $x_i \in \arg\max_{x_i' \in F_i(e_i, P, x_{-i}, S_i)} \sum_{\omega \in \Omega} \pi_i(\omega) u_i(\omega, P, x_i')$ [14]

(C3) $\forall t = 1, 2, \cdots, T, \forall \omega \in \Omega, \sum_{i=1}^{I} x_{it}(\omega) = 0.$

[11] The join $j_t(\omega)$ of $s_{1t}(\omega), s_{2t}(\omega), \cdots, s_{It}(\omega)$ is such that (1) $\forall i = 1, 2, \cdots, I, j_t(\omega) \subseteq s_{it}(\omega)$ and (2) for all $j_t'(\omega)$ satisfying (1), $j_t'(\omega) \subseteq j_t(\omega)$.

[12] The meet $m_t(\omega)$ of $s_{1t}(\omega), s_{2t}(\omega), \cdots, s_{It}(\omega)$ is such that (1) $\forall i = 1, 2, \cdots, I, s_{it}(\omega) \subseteq m_{it}(\omega)$ and (2) for all $m_t'(\omega)$ satisfying (1), $m_t(\omega) \subseteq m_t'(\omega)$.

[13] Since $\forall x_i \in F_i$, x_i are information feasible, F_i depends on the information structure S_i, the prices P, and other agents' trades x_{-i}. Since x_i satisfy no short sales, F_i depends on the endowment e_i. That's why it is written as $F_i(e_i, P, x_{-i}, S_i)$.

[14] Another way, perhaps a more intuitive way, to express (C2) is (C2') $\forall i = 1, 2, \cdots, I, x_i \in \arg\max_{x_i' \in F_i(e_i, P, x_{-i}, S_i)} E_i\left[u_i(\omega, P, x_i') | S_{i1}^{PX}\right]$. It is easy to see that (C2') is equivalent to (C2).

(C4) $\forall t = 1, 2, \cdots, T, P_t(\cdot)$ *is measurable with respect to* $j_t(\omega)$. *Formally,* $\forall t = 1, 2, \cdots, T, \forall \omega \in \Omega, j_t(\omega) \subseteq \{\omega' : P_t(\omega') = P_t(\omega)\}$.

Basically, (C1) describes the feasible set of trade for each agent, (C2) says that each agent maximizes his expected utility given his price-and-trade-refined information, (C3) requires that the market should clear in equilibrium, and (C4) implies that all the information contained in price is from the join of the individual information.

2.4 Different Concepts of Bubbles

Different definitions of bubbles will lead to different results even within the same framework. As a base line, we use the concept of expected bubbles, defined in AMP (1993). As we will see, the stronger the concept of a bubble become, the harder for it to exist in equilibrium.

Definition 4 (Expected Bubble) *As in AMP (1993), an expected bubble is said to exist in state* ω *at period* t *if in state* ω *it is mutual knowledge that the price of the risky asset at* t *is higher than the expected dividend an agent will receive, that is*

$$\forall i = 1, 2, \cdots, I, P_t(\omega) > \frac{1}{\sum_{\omega' \in s_{it}^{PX}(\omega)} \pi_i(\omega')} \sum_{\omega' \in s_{it}^{PX}(\omega)} \pi_i(\omega') d(\omega')$$

Definition 5 (Strong Bubble) *As in AMP (1993), a strong bubble is said to exist in state* ω *at period* t *if in state* ω *it is mutual knowledge that the price of the risky asset at* t *is higher than the possible dividend agents will receive, that is*

$$\forall i = 1, 2, \cdots, I, \forall \omega' \in s_{it}^{PX}(\omega), P_t(\omega) > d(\omega')$$

As is seen above, the concept of strong bubbles strengthens the concept of expected bubble in a way that it requires the asset price be higher than the maximal possible dividend. As will be seen below, another way to strengthen the concept of expected bubble is to require common knowledge instead of mutual knowledge. This requirement is reasonable since in the real world people's behaviors do not only depend on their own beliefs, but also depend on others' beliefs, others beliefs on their own beliefs, and so on. Therefore, we might expect to see something different when common knowledge is introduced into the concept of bubbles.

Definition 6 (Common Expected Bubble) *A common expected bubble is said to exist in state* ω *at period* t *if in state* ω *it is common knowledge that the price of the risky asset at* t *is higher than the expected dividend agents will receive, that is*

$$\forall i = 1, 2, \cdots, I, \forall \omega' \in m_t^{PX}(\omega), P_t(\omega) > \frac{1}{\sum_{\omega'' \in s_{it}^{PX}(\omega')} \pi_i(\omega'')} \sum_{\omega'' \in s_{it}^{PX}(\omega')} \pi_i(\omega'') d(\omega'')$$ [15]

.

[15] $m_t^{PX}(\omega)$ is the meet of $s_{1t}^{PX}(\omega), s_{2t}^{PX}(\omega), \cdots, s_{It}^{PX}(\omega)$.

Definition 7 (Common Strong Bubble) *A common strong bubble is said to exist in state ω at period t if in state ω it is common knowledge that the price of the risky asset at t is higher than the possible dividend agents will receive, that is*

$$\forall \omega' \in m_t^{PX}(\omega), P_t(\omega) > d(\omega')$$

2.5 Nonexistence of Common Strong Bubble in Equilibrium

Among the 4 definitions above, clearly common strong bubble is the strongest one. One may wonder if there can exist such a bubble in a rational expectations equilibrium. The answer is NO, due to the following proposition.

Proposition 1 *Under perfect memory assumption, $\forall \omega \in \Omega, \forall t = 1, 2, \cdots, T$,it is impossible for a common strong bubble to exist in state ω at period t in any rational expectations equilibrium.*

Proof. Suppose it is possible and $\exists \omega, \exists t$ such that a common strong bubble exists in state ω at period t in a rational expectations equilibrium. Then $m_t^{PX}(\omega)$ is the set of states where there is common knowledge among agents when ω is realized. Thus we have $\forall \omega' \in m_t^{PX}(\omega), P_t(\omega) = P_t(\omega') > d(\omega')$. By the feature of rational expectations equilibrium, there must exist some agent i for whom buying is at least as good as selling, which implies that $P_t(\omega) \leq E_i\left[P_{t+1}(\omega') | \omega' \in s_{it}^{PX}(\omega)\right]$. Therefore, $P_t(\omega) \leq \max_i \max_{\omega' \in s_{it}^{PX}(\omega)} P_{t+1}(\omega') \leq \max_{\omega' \in m_t^{PX}(\omega)} P_{t+1}(\omega')$. Since agents have perfect memory, we have $\forall i = 1, 2, \cdots, I, s_{i(t+1)}^{PX}(\omega) \subseteq s_{it}^{PX}(\omega)$, which implies $m_{t+1}^{PX}(\omega) \subseteq m_t^{PX}(\omega)$. By induction we have $P_t(\omega) \leq \max_{\omega' \in m_t^{PX}(\omega)} P_{T+1}(\omega') = \max_{\omega' \in m_t^{PX}(\omega)} d(\omega')$. Thus $\exists \omega^* \in m_t^{PX}(\omega)$ such that $d(\omega^*) \geq P_t(\omega)$, which causes a contradiction. ■

The intuition behind the nonexistence of common strong bubble is that if it is common knowledge that the price today is higher than the highest dividend agents may receive, then agents might be better off by selling the asset instead of holding it, no matter what kind of heterogeneous beliefs they may have. Since everyone wants to sell, there cannot be a rational expectations equilibrium any more. It is worth noting that the result of Proposition 1 is independent of the assumption of common knowledge of trades. In the case of no common knowledge of trades, the result is still true. The only modification is replacing the price–and-trade-refined information by the price-refined information. It is also worth noting that the result of Proposition 1 crucially depends on the perfect memory assumption. If we allow for agents to forget some information they knew before, a common strong bubble may exist in a rational expectations equilibrium. Such a counterexample is presented in Section 6.

Though under the standard assumption of perfect memory there is no common strong bubble in any rational expectations equilibrium, an expected bubble, which is both strong and common expected, as shown in the next section, can exist in a rational expectations equilibrium of a three-period two-agent economy.

3 Simple Example: Strong Bubbles and Common Expected Bubbles with Two Agents

3.1 Exogenous Setting

AMP (1993) has shown a strong bubble in a rational expectations equilibrium of a three-period three-agent economy with the assumption of no common knowledge of trades. In this section, I will provide a simple example of the existence of strong bubbles with two agents where trades become automatically common knowledge. Moreover, as will be shown, the bubble in the example will also be robust to common knowledge in the expected sense, hence a common expected bubble.

There are 2 agents (A and B), 3 periods (1, 2, and 3) and 8 states (ω_1, ω_2, ω_3, ω_4, ω_5, ω_6, ω_7 and ω_8). Only 2 assets exist in the market: one is money and the other is called a risky asset. Each share of the risky asset will pay a dividend of amount 4 at the end of period 3 if the state is either ω_1 or ω_4, and will pay nothing otherwise, as shown in the table below.

State	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
$d(\omega)$	4	0	0	4	0	0	0	0

Each agent is endowed with m_i unit of money and 1 share of the risk asset at the beginning of period 1. Agents can trade in each of period 1, 2, and 3. At period 3, after the trade is made, the dividend is realized, and then the consumption takes place.

Keeping in mind that the asymmetric information is the key to generate bubbles, we achieve this goal by giving agents different information structures. Remind that agent i's ($i = A, B$) information about the state in period t ($t = 1, 2, 3$) is represented by S_{it}, a partition of the space Ω. The specific structures of S_{it}'s are given below.

$$
\begin{aligned}
S_{A1} &= \{\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_8\}, \{\omega_6, \omega_7\}\} \\
S_{B1} &= \{\{\omega_1, \omega_2, \omega_4, \omega_5, \omega_6, \omega_8\}, \{\omega_3, \omega_7\}\} \\
S_{A2} &= \{\{\omega_1, \omega_2, \omega_3\}, \{\omega_4, \omega_5\}, \{\omega_6, \omega_7\}, \{\omega_8\}\} \\
S_{B2} &= \{\{\omega_4, \omega_5, \omega_6\}, \{\omega_1, \omega_2\}, \{\omega_3, \omega_7\}, \{\omega_8\}\} \\
S_{A3} &= S_{B3} = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}, \{\omega_5\}, \{\omega_6\}, \{\omega_7\}, \{\omega_8\}\}
\end{aligned}
$$

At first glance, this particular structure of information may seem complicated, but as our analysis goes on, the reason why it is set in this form will become clear. So far, there are at least three observations: First, at period 3, each agent is perfectly informed of what the realized state is and hence there is no asymmetric information then; Second, at period 2, agent A only receives more information when he observed $\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_8\}$ at period 1 and agent B only receives more information when he observed $\{\omega_1, \omega_2, \omega_4, \omega_5, \omega_6, \omega_8\}$ at period 1; Third, at period 1, if the state ω_7 is realized, each agent knows that he will receive no dividend for sure.[16] Hence if the price is positive at period $t = 1$ in state $\omega = \omega_7$, there will be a strong bubble, and that is part of what we are going for.

In order to generate potential gains from trade, we let each agent have a heterogeneous belief about the probability distribution of the state, as shown in the table below with weight $W = \frac{1}{16}$.

State	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
π_A	2	1	1	1	2	1	1	7
π_B	1	2	1	2	1	1	1	7

Also, the structure of the beliefs may seem complicated for now, but it will become clear why it serves for the existence of a bubble in a rational expectation equilibrium. So far, it is easy to observe that within the two states where there will be a dividend of 4, agent A puts a higher weight on state ω_1, agent B puts a higher weight on state ω_4. They put the same weight on state ω_7, and state ω_8, respectively. The weights they put on events $\{\omega_1, \omega_2, \omega_3\}$ and $\{\omega_4, \omega_5, \omega_6\}$ are also the same, respectively.

3.2 A Rational Expectations Equilibrium with a Bubble

Recall the standard definition given in the last section, and in our example a rational expectations equilibrium will be a vector $(P, x) \in R_+^{3\times 8} \times R^{2\times 3\times 8}$ such that

(C1) $\forall i = A, B$, x_i are information feasible and satisfy no short sales.

(C2) $\forall i = A, B$, x_i maximizes player i's expected payoff with respect to his own price-and-trade-refined information.

(C3) $\forall t = 1, 2, 3, \forall n = 1, \cdots, 8, x_{At}(\omega_n) + x_{Bt}(\omega_n) = 0$.

(C4) $\forall t = 1, 2, 3, \forall n, m = 1, \cdots, 8, j_t(\omega_n) \subseteq \{\omega_m : P_t(\omega_m) = P_t(\omega_n)\}$.

Although there are multiple rational expectations equilibria for this example, the one with the equilibrium prices and trades given in the following two tables is what we are interested in - the one in which there is a strong bubble and a common expected bubble.

State	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
$P_1(\omega)$	1	1	1	1	1	1	1	1
$P_2(\omega)$	2	2	2	2	2	2	0	0
$P_3(\omega)$	4	0	0	4	0	0	0	0

[16] Take agent A into consideration for example. When ω_7 is realized, agent A will have observed the event $\{\omega_6, \omega_7\}$. Since in either state ω_6 or ω_7, there is no dividend payment, agent A knows that he will receive no dividend with probability 1.

$\forall \omega \in \Omega, x_{A1}(\omega) = x_{B1}(\omega) = x_{A3}(\omega) = x_{B3}(\omega) = 0$

State	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
$x_{A2}(\omega)$	1	1	1	-1	-1	-1	0	0
$x_{B2}(\omega)$	-1	-1	-1	1	1	1	0	0
$x_{A2}(\omega) + x_{B2}(\omega)$	0	0	0	0	0	0	0	0

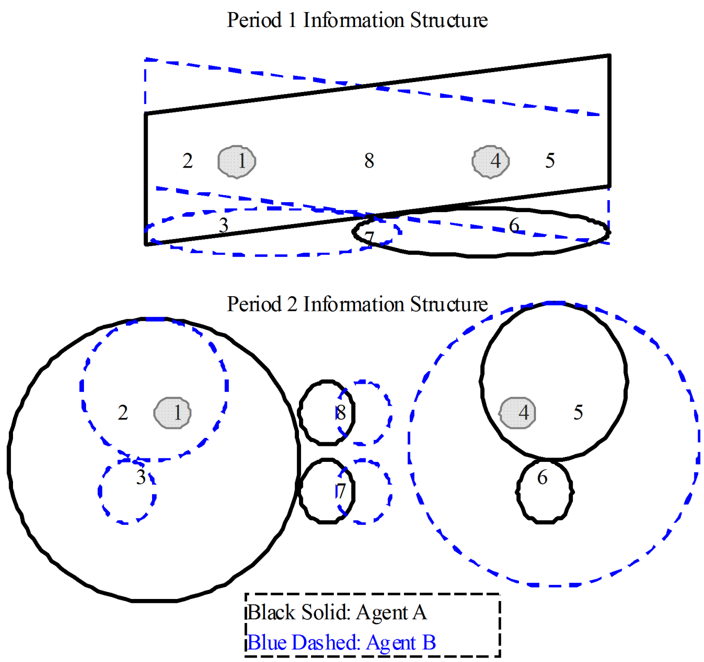
3.2.1 Price-and-Trade-Refined Information

First derive the price–and-trade-refined information for each agent in each period. It is easy to observe from the price table that $P_1(\omega) = 1\ \forall \omega \in \Omega$ and from the trade table that $x_{A1}(\omega) = x_{B1}(\omega) = 0\ \forall \omega \in \Omega$. This implies that the prices and trades in period 1 reveal no information. Hence $S_{A1}^{PX} = S_{A1}$, $S_{B1}^{PX} = S_{B1}$. Since in period 3, all agents already have full information about the state before observing the prices and making the trades,[17] the prices and trades in period 3 again, reveal no information. Hence $S_{A3}^{PX} = S_{A3}$, $S_{B3}^{PX} = S_{B3}$. The only new information revealed by prices and trades at period 2 is that agents know where they are for sure when the state ω_7 is realized. Hence agents' price–and-trade-refined information at period 2 is the following, with the original information structure attached below for comparison.

$$
\begin{aligned}
S_{A2}^{PX} &= \{\{\omega_1, \omega_2, \omega_3\}, \{\omega_4, \omega_5\}, \{\omega_6\}, \{\omega_7\}, \{\omega_8\}\} \\
S_{B2}^{PX} &= \{\{\omega_4, \omega_5, \omega_6\}, \{\omega_1, \omega_2\}, \{\omega_3\}, \{\omega_7\}, \{\omega_8\}\} \\
S_{A2} &= \{\{\omega_1, \omega_2, \omega_3\}, \{\omega_4, \omega_5\}, \{\omega_6, \omega_7\}, \{\omega_8\}\} \\
S_{B2} &= \{\{\omega_4, \omega_5, \omega_6\}, \{\omega_1, \omega_2\}, \{\omega_3, \omega_7\}, \{\omega_8\}\}
\end{aligned}
$$

The following graph may give more intuition about the information structure than the mathematical expression does. In the graph, agent A's information sets are described by the black solid curves, agent B's information sets are described by the blue dotted curves, and dividend paying states are emphasized in gray color.

[17] Actually there is no trade in period 3 in the equilibrium under study.



It is worth noting that at period 2, with the price-and-trade-refined information, agent A is better informed than agent B when event $\{\omega_4, \omega_5, \omega_6\}$ happens, and agent B is better informed than agent A when event $\{\omega_1, \omega_2, \omega_3\}$ happens. We will see soon that the subgroup of states $\{\omega_4, \omega_5, \omega_6\}$ is where agent A takes advantage of agent B by selling the asset he believes is overpriced to agent B, and similarly, the subgroup of states $\{\omega_1, \omega_2, \omega_3\}$ is where agent B takes advantage of agent A.

3.2.2 The Existence of Strong and Common Expected Bubbles

Second note that there is a strong bubble at period 1 in state ω_7 since for agent A, $s_{A1}^{PX}(\omega_7) = \{\omega_6, \omega_7\}$, $P_1(\omega_7) = 1 > 0 = d(\omega_6) = d(\omega_7)$, and for agent B, $s_{B1}^{PX}(\omega_7) = \{\omega_3, \omega_7\}$, $P_1(\omega_7) = 1 > 0 = d(\omega_3) = d(\omega_7)$. In short, the strong bubble exists at period 1 in ω_7 because at that state every agent knows the asset is worthless but with a positive current price.

In this example, $m_1^{PX}(\omega_7) = \Omega$. To see that this bubble is robust to common knowledge in the expected sense, we need to check that $\forall i = A, B, \forall \omega \in \Omega, 1 > \frac{1}{\sum_{\omega' \in s_{i1}^{PX}(\omega)} \pi_i(\omega')} \sum_{\omega' \in s_{i1}^{PX}(\omega)} \pi_i(\omega') d(\omega')$. There are four cases:

1 $\omega = \omega_7$: Agent A observes the event $\{\omega_6, \omega_7\}$, agent B observes the event $\{\omega_3, \omega_7\}$, each of them will induce that the expected dividend in period 3 will be $\frac{1}{2}0 + \frac{1}{2}0 = 0$, which is less to the current price.

2 $\omega = \omega_6$: Agent A observes the event $\{\omega_6, \omega_7\}$, and his expected dividend in period 3 is 0, less than the current price. Agent B observes $\Omega \backslash \{\omega_3, \omega_7\}$, and his expected dividend in period 3 is $\frac{3}{14}4 + \frac{11}{14}0 = \frac{6}{7}$, less than the current price.

3 $\omega = \omega_3$: Agent B observes the event $\{\omega_3, \omega_7\}$, and his expected dividend in period 3 is 0, less than the current price. Agent A observes $\Omega \backslash \{\omega_6, \omega_7\}$, and his expected dividend in period 3 is $\frac{3}{14}4 + \frac{11}{14}0 = \frac{6}{7}$, less than the current price.

4 $\omega_n \in \Omega \backslash \{\omega_3, \omega_6, \omega_7\}$, Agent A observes the event $\Omega \backslash \{\omega_6, \omega_7\}$, agent B observes the event $\Omega \backslash \{\omega_3, \omega_7\}$, each of them will induce that the expected dividend in period 3 will be $\frac{3}{14}4 + \frac{11}{14}0 = \frac{6}{7}$, which is less to the current price.

Therefore, the bubble at period 1 in state ω_7 is a common expected bubble. Actually, the reader can check that in our example the common expected bubble exists at period 1, not only in state ω_7, but also in any other state.

3.2.3 Check of Equilibrium Conditions

Last check that the prices and trades described above constitute a rational expectations equilibrium. We check all the four conditions step by step.

Check (C1): We observe from the trade table the minimum amount of trade at period 2 is -1. By the fact that there is no trade in either period 1 or 3 and that each agent is endowed with 1 share of the risky asset, the no short sale condition is satisfied for x_A and x_B. To see if x_i are information feasible, it suffices to only look at period 2 since no trade occurs either in period 1 or 3. In period 2, actually each agent's action remains the same given the same price–and-trade-refined information.[18] This implies that x_A and x_B also satisfy the information feasibility condition.

Check (C2): Maximization of the expected payoff at the beginning of period 1 under the constraints of information feasibility and no short sales, is equivalent to maximization of the expected payoff in each period given the current price–and-trade-refined information under the same constraints. In period 3, each agent has no incentive to trade since the price is exactly equal to the dividend for every state. In period 2, there are in total 4 cases:

(p2-i) $\forall i \in \{A, B\}$, if agent i observes the event $\{\omega_7\}$ or $\{\omega_8\}$, he knows that with probability 1 the price in period 3 will be 0, which is equal to the current price, thus he is indifferent between trading or not at period 2, so the equilibrium trade of 0 maximizes his expected payoff in this case.

(p2-ii) If agent A observes the event $\{\omega_1, \omega_2, \omega_3\}$ (or if agent B observes the event $\{\omega_4, \omega_5, \omega_6\}$), he will induce that the expected price in period 3 will be $\frac{1}{2}4 + \frac{1}{4}0 + \frac{1}{4}0 = 2$, which is equal to the current price, thus he is indifferent between trading

[18] Take agent A for example.
$\forall \omega = \omega_6, s_{A2}^{PX}(\omega) = \{\omega_6\} \subset \{\omega_4, \omega_5, \omega_6\} = \{\omega' : x_{A2}(\omega') = x_{A2}(\omega)\}$,
$\forall \omega \in \{\omega_4, \omega_5\}, s_{A2}^{PX}(\omega) = \{\omega_4, \omega_5\} \subset \{\omega_4, \omega_5, \omega_6\} = \{\omega' : x_{A2}(\omega') = x_{A2}(\omega)\}$,
$\forall \omega \in \{\omega_1, \omega_2, \omega_3\}, s_{A2}^{PX}(\omega) = \{\omega_1, \omega_2, \omega_3\} = \{\omega' : x_{A2}(\omega') = x_{A2}(\omega)\}$,
$\forall \omega \in \{\omega_7, \omega_8\}, s_{A2}^{PX}(\omega) = \{\omega\} \subset \{\omega_7, \omega_8\} = \{\omega' : x_{A2}(\omega') = x_{A2}(\omega)\}$.

or not at period 2, so the equilibrium trade of 1 maximizes his expected payoff in this case.

(p2-iii) If agent A observes the event $\{\omega_4, \omega_5\}$ (or if agent B observes the event $\{\omega_1, \omega_2\}$), he will induce that the expected price in period 3 will be $\frac{1}{3}4 + \frac{2}{3}0 = \frac{4}{3}$, which is less the current price 2, thus he has incentive to sell any of the asset he owns at period 2, so under the short sale constraint and given there is no trade in period 1, the equilibrium trade of -1 maximizes his expected payoff in this case.

(p2-iv) If agent A observes the event $\{\omega_6\}$ (or if agent B observes the event $\{\omega_3\}$), he knows that with probability 1 the price in period 3 will be 0, which is less the current price 2, thus he has incentive to sell any of the asset he owns at period 2, so under the short sale constraint and given there is no trade in period 1, the equilibrium trade of -1 maximizes his expected payoff in this case.

In period 1, there are 2 cases:

(p1-i) If agent A observes the event $\{\omega_6, \omega_7\}$ (or if agent B observes the event $\{\omega_3, \omega_7\}$), he will induce that the expected price in period 2 will be $\frac{1}{2}2 + \frac{1}{2}2 = 1$, which is equal to the current price, thus he is indifferent between trading or not at period 1, so the equilibrium trade of 0 maximizes his expected payoff in this case.

(p1-ii) If agent i observes the event other than the one described in (1-i), he will induce that the expected price in period 2 will be $\frac{2\times2+1\times3}{14}2 + \frac{7}{14}0 = 1$, which is equal to the current price, thus he is indifferent between trading or not at period 1, so the equilibrium trade of 0 maximizes his expected payoff in this case.

The above analysis guarantees that the condition (C2) is satisfied.

Check (C3) and (C4): It is seen that the market clears in each period at each state from the table of trades, hence (C3) is satisfied. Note that $P_1(\omega) = 1$ $\forall \omega \in \Omega$ hence $P_1(\cdot)$ is measurable with respect to $j_1(\cdot)$ and that $j_3(\omega) = \{\omega\}$ $\forall \omega \in \Omega$ hence $P_3(\cdot)$ is measurable with respect to $j_3(\omega)$. To see $P_2(\cdot)$ is measurable with respect to $j_2(\omega)$, note that $\forall n = 1, \cdots, 6, j_2(\omega_n) \subseteq \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\} = \{\omega : P_2(\omega) = P_2(\omega_n) = 2\}$ and $\forall n = 7, 8, j_2(\omega_n) \subseteq \{\omega_7, \omega_8\} = \{\omega : P_2(\omega) = P_2(\omega_n) = 0\}$ This completes the check that the prices and trades given in the example constitute a rational expectations equilibrium.

3.3 Discussion

We have shown that, in a simple finite horizon model with asymmetric information and short sale constraints, a strong bubble and a common expected bubble can exist in the same period at the same state in a rational expectations equilibrium with common knowledge of trades, under the same basic setting as in AMP (1993).

It is worthwhile to make some remarks about this simple example.

(1) The initial distribution of the asset is not efficient. To see this, with zero-trade, each agent's expected payoff $m_i+\sum_{\omega\in\Omega}\pi_i(\omega)\left[e_iP_T(\omega)+\sum_{t=1}^{T}x_{it}(\omega)\left[P_{t+1}(\omega)-P_t(\omega)\right]\right]$ would have been $m_i+\frac{3}{4}$, while in the equilibrium, each agent's expected payoff is m_i+1. Thus our example does not violate the no-trade theorem and the necessary condition of ex ante inefficiency is satisfied here. In fact, as the analysis has shown, in our example the ones who gain from the trade are the sellers whenever the trade takes place.

(2) The social welfare is maximized in the rational expectation equilibrium with bubbles if there is no initial endowment of money. Note that in our example the social welfare is maximized when in every state the social planner gives all the assets to the agent who puts the highest weight on that state. Hence the maximal social welfare should be $\frac{9}{8}(m_1+m_2)+2$. When either agent has positive endowment of money, the social welfare of the equilibrium outcome is not maximized. However, if each agent is endowed with no money, then the social welfare is maximized in equilibrium. To put it in another way, if the social planner is only allowed to reallocate on the risky asset, then the equilibrium maximizes the sum of the utilities of the agents. This implies a surprising observation that the rational bubbles do not necessarily lead to inefficiency.

(3) The short sale constraints are binding at period 2 for the sellers whenever the trade takes place. In the cases of (p2-iii) and (p2-iv), where agents play the seller's role, since the expected price for the asset is higher than the current price, agents would like to take advantage of this and sell as much as they could. If there were no short sale constraints, an equilibrium would not have been reached under the current price. This is where the no short sales assumption plays its role.

(4) The asymmetric information functions in the way that even though all agents know that the asset is overpriced, they are still willing to hold the asset as long as the information of overpricing is not common knowledge in the strong sense. And it is this feature that makes a bubble possible in a rational expectations equilibrium.

(5) For simplicity, the example is constructed in a way that even though the trade is common knowledge, it reveals no additional information for each agent.

4 An Example of Common Strong Bubbles with Agents of Imperfect Memory

In Section 2, we have pointed out that the nonexistence result for common strong bubbles relies heavily upon the assumption of perfect memory. Once this standard assumption is relaxed, that is to say agents might forget some information they originally knew, then it is possible to have a common strong bubble in a rational expectations equilibrium. A simple counter example is constructed below.

4.1 Exogenous Setting

The same as before, there are 2 agents (A and B), 3 periods (1, 2, and 3) and 8 states (ω_1, ω_2, ω_3, ω_4, ω_5, ω_6, ω_7 and ω_8). There are only 2 assets: money and the risky asset. The dividend distribution over states for the risky asset remains exactly the same as in Section 3, and is shown in the table below.

State	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
$d(\omega)$	4	0	0	4	0	0	0	0

Agents' endownment are the same as before, that is m_i unit of money and 1 unit of the risky asset. What differs from the previous example is the information structure. In period 1, both agents receive the same information, represented by S_{A1} and S_{B1} respectively, where $S_{A1} = S_{B1}$. When it comes to period 2, both agents forget everything they knew in period 1, and then they get to reveive some new information, represented by S_{A2} and S_{B2} respectively. In this case, S_{i2} is no longer necessarily a finer partition than S_{i1} is, for $i = A, B$. In period 3, again as before, each agent is perfectly informed of what the realized state is. The structure for the information partiontions is shown in the table below.

$$
\begin{aligned}
S_{A1} &= S_{B1} = \{\{\omega_2, \omega_3, \omega_5, \omega_6, \omega_8\}, \{\omega_1, \omega_4, \omega_7\}\} \\
S_{A2} &= \{\{\omega_1, \omega_2, \omega_3\}, \{\omega_4, \omega_5\}, \{\omega_6, \omega_7\}, \{\omega_8\}\} \\
S_{B2} &= \{\{\omega_4, \omega_5, \omega_6\}, \{\omega_1, \omega_2\}, \{\omega_3, \omega_7\}, \{\omega_8\}\} \\
S_{A3} &= S_{B3} = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}, \{\omega_5\}, \{\omega_6\}, \{\omega_7\}, \{\omega_8\}\}
\end{aligned}
$$

The heterogeneous belief about the probability distribution of the state, for each agent, is shown in the table below with weight $W = \frac{1}{16}$.

State	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
π_A	2	1	1	1	2	1	3	5
π_B	1	2	1	2	1	1	3	5

4.2 A Rational Expectations Equilibrium with Common Strong Bubbles

A similar calculation and check procedure will show that the above economy has a rational expectations equilibrium that is the same as the one we studied in Section 3. It is characterized by the price table and the trade table below.

State	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
$P_1(\omega)$	1	1	1	1	1	1	1	1
$P_2(\omega)$	2	2	2	2	2	2	0	0
$P_3(\omega)$	4	0	0	4	0	0	0	0

$$\forall \omega \in \Omega, x_{A1}(\omega) = x_{B1}(\omega) = x_{A3}(\omega) = x_{B3}(\omega) = 0$$

State	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
$x_{A2}(\omega)$	1	1	1	-1	-1	-1	0	0
$x_{B2}(\omega)$	-1	-1	-1	1	1	1	0	0
$x_{A2}(\omega) + x_{B2}(\omega)$	0	0	0	0	0	0	0	0

Now it is time to look for the common strong bubbles in such an equilibrium. Observe that at period 1 in any state from the set $\{\omega_2, \omega_3, \omega_5, \omega_6, \omega_8\}$, it is common knowledge that the dividend at period 3 will be 0. Give a positive price 1, it is exactly the case that it is common knowledge that the price of the risky asset is higher than the possible dividend agents will receive, and hence there is a common strong bubble ar period 1 in any state from the set $\{\omega_2, \omega_3, \omega_5, \omega_6, \omega_8\}$.

This counterexample shows that perfect memory is an important necessary assumption for the nonexistence of common strong bubbles. In the real world, it is arguable that not all people have perfect memory. Therefore, a common strong bubble may exist in an economy of the real life. This seems to be a surprising result, and it provides an alternative explanation of the existence of bubbles by the assumption of imperfect memoery, in stead of the assumption of noise traders.

5 General Results

In Section 3, an example of rational bubbles, which is both a strong bubble and a common expected bubble, is presented in a rational expectations equilibrium with 2 agents. Furthermore, as will be shown next, the assumption of no common knowledge of trade is not necessary for the existence of bubbles for any finite number of players.

Let $S^F \equiv \{\{\omega\} | \omega \in \Omega\}$ and S^F is called the perfect information structure for Ω. Before constructing bubble examples, we shall make some restrictions on the agents' information structure so as to avoid trial bubbles from duplications.

Assumption 1 (Different Information Structure) $\forall i, j = 1, \cdots, I, \forall t = 1, \cdots, T$, $S_{it}, S_{jt} \neq S^F \Rightarrow S_{it} \neq S_{jt}$.

The assumption of Different Information Structure says that as long as agents don't have perfect information, there must be somewhere their information differs from each other. This assumption rules out the possibility of duplicating identical agents.

Assumption 2 (Distinct Information Everywhere) $\forall i, j = 1, \cdots, I, \forall t = 1, \cdots, T,$ $\forall \omega \in \Omega,$ $s_{it}(\omega), s_{jt}(\omega) \neq \{\omega\} \Rightarrow s_{it} \neq s_{jt}$.

The assumption of Distinct Information Everywhere says that as long as agents don't have perfect information, their information differs from each other everywhere. It is easy to know that Assumption 2 is much stronger than Assumption 1. Assumption 2 implies Assumption 1, but not vice versa.

Assumption 3 (Common Knowledge of Trades) $\forall i = 1, \cdots, I, \forall t = 1, \cdots, T,$ *x_{it} is common knowledge.*

Based on the assumptions above, two propositions can be made on the existence of strong bubbles in a rational expectations equilibrium.

Proposition 2 *Under Assumption 1 and 3, for any $I \geq 2$, there exists an economy under the framework described in Section 2, with I agents, 3 periods and $3I + 2$ states, presenting a bubble, both strong and common expected, in a rational expectations equilibrium.*

Proof. See the appendix. ■

Proposition 3 *Under Assumption 2 and 3, for any $I \geq 2$, there exists an economy under the framework described in Section 2, with I agents, 3 periods and $I \cdot \max\{3, I\} + 2$ states, presenting a bubble, both strong and common expected, in a rational expectations equilibrium.*

Proof. See the appendix. ■

The strong bubble part of the result is not new, and has been analyzed by AMP (1993) and Conlon (2004). However, by presenting a bubble, not only strong but also common expected, the above propositions provide a new answer to what properties of bubbles we can expect to have in a rational world. The common part of the result is surprising since it is somewhat counterintuitive that an expected bubble can be robust to common knowledge in a raitional expectations equilibrium. But actually it is the common knowledge of the heterogeneous beliefs and the information structures that guarantees that agents have no incentive to rush in face of bubbles, because by rational expectations they know that they can take advantage of it in a later period.

It should also be noted that the conclusions above are independent of the assumption of no common knowledge of trade. In Proposition 3 of AMP (1993) paper, the assumption of no common knowledge of trades was argued as a necessary condition for the existence of bubbles in a rational expectations equilibrium. The idea of the argument is the following: Geanakoplos (1992) has argued that with common knowledge of trades, agents would have behaved in the same way without the private part of their information (originally stated as "common knowledge of actions negates asymmetric information about events"), and then there would be no strong bubbles since there is no asymmetric information about the states. However, as was pointed out by Conlon (2004), the conclusion that there are no strong bubbles is only true for the new economy where every agent has the same information, which is the common part of their original information. The bubble may still exist in the original economy since in period 1 there is no trade and hence there is still private information.

6 Conclusion

Based on the work of AMP (1993), as well as Conlon (2004), this paper shows that for any finite number of agents, (1) there is no common strong bubble in any raitonal expectations equilibrium under the perfect memory assumption; and (2) there exists a three-period economy with asymmetric information and short sales constraints, where an expected bubble can exist in a rational expectations equilibrium, and moreover this bubble is not only a strong bubble, but also a common expected bubble. The first result partially answers what properties a bubble cannot have in a rational world, and the second result tells more about what a bubble might look like, given the results in AMP (1993) and Conlon (2004).

One direction for future work will be to extend the concept of strong bubbles to higher orders for any finite number of agents, following the work done in Conlon (2004) in which an example of higher order bubbles is constructed for the two agents case. Another direction will be to introduce some irrational agents into the model and to see whether a common strong bubble can exist in such a setting. Since the bubbles modeled in this paper are not robust to perturbations of agents' beliefs in a general sense, introducing noise into the model might be another good direction.

Appendix

Appendix 1:

Proof to Proposition 2:

Write $\Omega = \{\omega_n | n = 1, 2, \cdots, 3I+2\}$. Let $\Omega_D \equiv \{\omega_n \in \Omega | n = 3i-2, i = 1, 2, \cdots, I\}$, $\Omega_{2W} \equiv \{\omega_n \in \Omega | n = 3i-1, i = 1, 2, \cdots, I\}$, $\Omega_i \equiv \{\omega_{3i-2}, \omega_{3i-1}, \omega_{3i}\}$, $\Omega_i^- \equiv \Omega_i \backslash \{\omega_{3i}\} = \{\omega_{3i-2}, \omega_{3i-1}\}$, $i = 1, 2, \cdots I$.

Each share of the risky asset will pay a dividend of amount 4 at the end of period 3 if the state $\omega \in \Omega_D$ and will pay nothing otherwise. Each agent is endowed with I units of money and 1 share of the risk asset at the beginning of period 1.

The specific structures of S_{it}'s are given below.

$$\begin{aligned}
S_{11} &= \{\Omega \backslash \{\omega_{3I}, \omega_{3I+1}\}, \{\omega_{3I}, \omega_{3I+1}\}\} \\
S_{i1} &= \{\Omega \backslash \{\omega_{3i-3}, \omega_{3I+1}\}, \{\omega_{3i-3}, \omega_{3I+1}\}\} \ \forall i = 2, \cdots, I \\
S_{12} &= \left\{\Omega_1, \Omega_2, \cdots, \Omega_{I-1}, \Omega_I^-, \{\omega_{3I}\}, \{\omega_{3I+1}\}, \{\omega_{3I+2}\}\right\} \\
S_{i2} &= \left\{\Omega_1, \cdots, \Omega_{i-2}, \Omega_i, \cdots, \Omega_I, \Omega_{i-1}^-, \{\omega_{3i-3}\}, \{\omega_{3I+1}\}, \{\omega_{3I+2}\}\right\} \ \forall i = 2, \cdots, I \\
S_{i3} &= S^F \ \forall i = 1, 2, \cdots, I
\end{aligned}$$

The agents' beliefs about the states are given by the following functions.

$$\pi_i(\omega_n) = \left\{ \begin{array}{ll} 2W \text{ if } n = 3i-2 \text{ or } \omega_n \in \Omega_{2W} \backslash \{\omega_{3i-1}\} & \\ (4I-1)W & \text{if } n = 3I+2 \\ W & \text{otherwise} \end{array} \right\} \forall i = 1, 2, \cdots, I, W = \frac{1}{8I}$$

To see that the belief of agent i is well defined, note that the number of elements in Ω_{2W} is I, hence there are I states which are put with probability $2W$. Since there is only one state with probability $(3I+2)W$, the number of the states with probability W is $3I+2-I-1 = 2I+1$. Thus, $\sum_{\omega \in \Omega} \pi_i(\omega) = I \times 2W + 1 \times (4I-1)W + (2I+1) \times W = 8IW = 1$.

The equilibrium with the prices and trades given below is what we look for - the one in which there is a strong and common expected bubble at period 1 in state ω_{3I+1}.

$$\begin{aligned}
P_1(\omega) &= 1 \forall \omega \in \Omega \\
P_2(\omega_n) &= \left\{ \begin{array}{ll} 0 & \text{if } n = 3I+1 \text{ or } n = 3I+2 \\ 2 & \text{otherwise} \end{array} \right\} \\
P_3(\omega_n) &= \left\{ \begin{array}{ll} 4 & \text{if } n \in \Omega_D \\ 0 & \text{otherwise} \end{array} \right\}
\end{aligned}$$

$$\forall i = 1, 2, \cdots, I, \forall \omega \in \Omega, x_{i1}(\omega) = x_{i3}(\omega) = 0$$

$$x_{i2}(\omega_n) = \left\{ \begin{array}{cc} I-1 & \text{if } \omega_n \in \Omega_i \\ 0 \text{ if } n = 3I+1 \text{ or } n = 3I+2 & \\ -1 & \text{otherwise} \end{array} \right\} \forall i = 1, 2, \cdots, I$$

Observe that neither the prices nor the trades reveal any addition information with the settings above.

It can be similarly checked following the procedures described in the two-agent example that the above prices and trades constitute a rational expectations equilibrium. And since at period 1 in state ω_{3I+1}, each agent knows that he will receive nothing at the end of period 3, given the positive price of 1 at period 1, there exists a strong bubble in this equilibrium.

Note that $m_1^{PX}(\omega_{3I+1}) = \Omega$. To see that this bubble is robust to common knowledge in the expected sense, we need to check that $\forall i = 1, 2, \cdots, I, \forall \omega \in \Omega, 1 > \frac{1}{\sum_{\omega' \in s_{i1}^{PX}(\omega)} \pi_i(\omega')} \sum_{\omega' \in s_{i1}^{PX}(\omega)} \pi_i(\omega') d(\omega')$. Note that for agent 1 (or agent i, $i \geq 2$), either he will observe $\{\omega_{3I}, \omega_{3I+1}\}$ (or $\{\omega_{3i-3}, \omega_{3I+1}\}$), or he will observe $\Omega \backslash \{\omega_{3I}, \omega_{3I+1}\}$ (or $\Omega \backslash \{\omega_{3i-3}, \omega_{3I+1}\}$). If it is the first case, his expected dividend will be $\frac{1}{2}0 + \frac{1}{2}0 = 0$; If it is the second case, his expected dividend will be $\frac{I+1}{8I-2}4 + \frac{7I-3}{8I-2}0 = \frac{2I+2}{4I-1}$. In either case, the expected dividend is less than the price. Therefore, the bubble at period 1 in state ω_{3I+1} is a common expected bubble.

However it should noted in the structure above, $\forall \omega_n \in \Omega \backslash \{\omega_{3I+1}, \omega_{3I+2}\}$, at period 2 in state ω_n there are always $(I-1)$ agents who observes the same event $\Omega_i = \{\omega_{3i-2}, \omega_{3i-1}, \omega_{3i}\}$[19] where i is determined such that $\omega_n \in \Omega_i$. Obviously this violates Assumption 2. In order to ensure that agents' information differs from each other everywhere when this is no perfect information, the number of the states has to be great enough to guarantee the existence of bubbles.

Appendix 2:

Proof to Proposition 3:

The case of 2 agents has already been shown in section 3. Here it suffices to consider the case when $I \geq 3$.

Write $\Omega = \{\omega_n | n = 1, 2, \cdots, I^2 + 2\}$. Let $\Omega_D \equiv \{\omega_n \in \Omega | n = I(i-1) + 1, i = 1, 2, \cdots, I\}$, $\Omega_{(I-1)W} \equiv \{\omega_n \in \Omega | n = I(i-1) + 2, i = 1, 2, \cdots, I\}$, $\Omega_j \equiv \{\omega_n \in \Omega | I(j-1) + 1 \leq n \leq Ij\}$, $\Omega_j^{-k} \equiv \Omega_j \backslash \{\omega_{I(j-1)+k}\}$, $j, k = 1, 2, \cdots I$.

Again, each share of the risky asset will pay a dividend of amount 4 at the end of period 3 if the state $\omega \in \Omega_D$ and will pay nothing otherwise. Each agent is endowed with I units of money and 1 share of the risk asset at the beginning of period 1.

[19] Though there is one agent observing $\{\omega_n\}$ or $\Omega_i \backslash \{\omega_n\}$, Ω_i is common knowledge in this case. And this feature holds also for the constructed example under proposition.

Let a_{ij} be the ith row and jth column element of the following $I \times I$ matrix. Hence $\omega_{I(j-1)+a_{ij}}$ is the a_{ij}th element in Ω_j.

$$\begin{bmatrix} & 2 & 3 & \cdots & I-1 & I \\ I & & 2 & \cdots & I-2 & I-1 \\ I-1 & I & & 2 & \cdots & I-2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 3 & 4 & \cdots & I & & 2 \\ 2 & 3 & \cdots & I-1 & I & \end{bmatrix}$$

The specific structures of S_{it}'s are given below.

$$\begin{aligned} S_{i1} &= \left\{\Omega \backslash \left\{\omega_{Ik_i}, \omega_{I^2+1}\right\}, \left\{\omega_{Ik_i}, \omega_{I^2+1}\right\}\right\} \text{ where } k_i \text{ is determined by } a_{ik_i} = I \\ S_{i2} &= \left\{\left\{\omega_{I(j-1)+a_{ij}}\right\} : 1 \le j \le I, j \ne i\right\} \cup \left\{\Omega_j^{-a_{ij}} : 1 \le j \le I, j \ne i\right\} \cup \left\{\Omega_i, \left\{\omega_{I^2+1}\right\}, \left\{\omega_{I^2+2}\right\}\right\} \\ S_{i3} &= S^F \ \forall i = 1, 2, \cdots, I \end{aligned}$$

The agents' beliefs about the states are given by the following functions. $\forall i = 1, 2, \cdots, I$,

$$\pi_i(\omega_n) = \left\{ \begin{array}{cc} (I-1)W \text{ if } n = I(i-1)+1 \text{ or } \omega_n \in \Omega_{(I-1)W} \backslash \left\{\omega_{I(i-1)+2}\right\} & \\ (2I(I-1)-1)W & \text{if } n = I^2+2 \\ W & \text{otherwise} \end{array} \right\}, W = \frac{1}{4I(I-1)}$$

To see that the belief of agent i is well defined, note that the number of elements in $\Omega_{(I-1)W}$ is I, hence there are I states which are put with probability $(I-1)W$. Since there is only one state with probability $(2I(I-1)-1)W$, the number of the states with probability W is $I^2 + 2 - I - 1 = I(I-1)+1$. Thus, $\sum_{\omega \in \Omega} \pi_i(\omega) = I \times (I-1)W + 1 \times (2I(I-1)-1)W + (I(I-1)+1) \times W = 4I(I-1)W = 1$.

The equilibrium with the prices and trades given below is what we look for - the one in which there is a strong and common expected bubble at period 1 in state ω_{I^2+1}.

$$\begin{aligned} P_1(\omega) &= 1 \forall \omega \in \Omega \\ P_2(\omega_n) &= \left\{ \begin{array}{cc} 0 & \text{if } n = I^2+1 \text{ or } n = I^2+2 \\ 2 & \text{otherwise} \end{array} \right\} \\ P_3(\omega_n) &= \left\{ \begin{array}{cc} 4 & \text{if } n \in \Omega_D \\ 0 & \text{otherwise} \end{array} \right\} \end{aligned}$$

$$\forall i = 1, 2, \cdots, I, \forall \omega \in \Omega, x_{i1}(\omega) = x_{i3}(\omega) = 0$$

$$x_{i2}(\omega_n) = \left\{ \begin{array}{cc} I-1 & \text{if } \omega_n \in \Omega_i \\ 0 \text{ if } n = I^2+1 \text{ or } n = I^2+2 & \\ -1 & \text{otherwise} \end{array} \right\} \forall i = 1, 2, \cdots, I$$

Observe that neither the prices nor the trades reveal any addition information with the settings above.

It can be similarly checked following the procedures described in the two-agent example that the above prices and trades constitute a rational expectations equilibrium. And since at period 1 in state ω_{I^2+1}, each agent knows that he will receive nothing at the end of period 3, given the positive price of 1 at period 1, there exists a strong bubble in this equilibrium.

Note that $m_1^{PX}(\omega_{I^2+1}) = \Omega$. To see that this bubble is robust to common knowledge in the expected sense, we need to check that $\forall i = 1, 2, \cdots, I, \forall \omega \in \Omega, 1 > \frac{1}{\sum_{\omega' \in s_{i1}^{PX}(\omega)} \pi_i(\omega')} \sum_{\omega' \in s_{i1}^{PX}(\omega)} \pi_i(\omega') d(\omega')$. Note that for agent 1, either he will observe $\{\omega_{Ik_i}, \omega_{I^2+1}\}$, or he will observe $\Omega \backslash \{\omega_{Ik_i}, \omega_{I^2+1}\}$. If it is the first case, his expected dividend will be $\frac{1}{2}0 + \frac{1}{2}0 = 0$; If it is the second case, his expected dividend will be $\frac{2(I-1)}{4I(I-1)-2}4 + \frac{4I(I-1)-2-2(I-1)}{4I(I-1)-2}0 = \frac{4}{2I - \frac{1}{I-1}}$. In either case, the expected dividend is less than the price. Therefore, the bubble at period 1 in state ω_{I^2+1} is a common expected bubble.

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