MATCHING, ASPIRATION AND LONG-TERM RELATIONSHIP

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Abstract. This paper studies a simple matching model with a single population where players, who are identical *ex ante* but not necessarily so *ex post*, are randomly matched and decide whether or not to form a long-term relationship. When two agents are matched, they observe a relation-specific productivity as well as their respective shares and decide whether or not to agree to form a long-term relationship. The pair forms a long-term relation if and only if both the agents in the pair agree to do so; otherwise, they join the pool of singles, who continuously search for partners. A long-term relationship is subject to a small probability of break-up, which results in both the players going back to the pool of singles. Each agent agrees to form a long-term relationship if and only if his share exceeds his own past average payoff. It is shown that almost all the agents in the society are engaged in a long-term relationship and play the symmetric efficient outcome.

Keywords: Matching, Long-Term Relationship, Aspiration, Pure Symmetric Efficient State, Stochastic Approximation, Lyapunov Function

1. Introduction

Many social interactions occur among the same group of agents who voluntarily participate in a long-term relationship. Examples abound. Most business relations are repeated not by force, but by choice. A typical relationship between a worker and an employer is built upon a long-term relationship which starts with a matching between the two parties (e.g., Mortensen and Pissarides (1994)).

Extensive investigations of repeated games under various institutional and informational assumptions have offered us a useful insight on the outcomes of a long-term relationship. However, the existing literature on repeated games remains silent about how and why a

Date: February 20, 2009.

We are grateful to Drew Fudenberg, William Sandholm and participants of NSF/NBER/CEME Conference on General Equilibrium and Mathematical Economics at Brown University, the Pan-Pacific Conference on Game Theory at University of Auckland, and the seminars at University of Wisconsin and Korea University. Financial supports from the National Science Foundation (ECS-0523620, SES-0720592) and the Scientific Research Program (Creative) of the Japan Society for the Promotion of Science (19GS0101) are gratefully acknowledged. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation or the Japan Society for the Promotion of Science.

long-term relationship is formed among players, and which outcome becomes a pervasive social norm. This paper provides a social dynamic foundation for a long-term relationship by formulating and analyzing a decentralized matching model, where the players choose to, rather than are assumed to, initiate the long-term relationship. Our goal is to understand the conditions under which a long-term relationship can arise from decentralized social interactions to become a dominant form of social institution.

We shall examine a simple canonical matching model with a single population where players are randomly matched and decide whether or not to form a long-term relationship. When two agents are matched, they observe a relation-specific productivity as well as their respective shares and decide whether or not to agree to form a long-term relationship. The pair forms a long-term relation if and only if both the agents in the pair agree to do so; otherwise, they join the pool of singles, who continuously search for partners. A long-term relationship is subject to a small probability of break-up, which results in both the players going back to the pool of singles. The interaction between the matching process and the long-term relationship is the focus of our analysis.

One might pursue an equilibrium analysis by formulating the long-term relationship as a repeated game, embedded in a random matching process. We need to endow the agents with an exceedingly powerful computational capability to calculate one's best responses against a profile of other agents' strategies and a physical environment. However, as we are interested in the case where the long-term relationship is subject to a small probability of break-up, we have a large set of equilibria from the repeated game, and the equilibrium analysis provides little guidance to select a particular equilibrium. Unless the agent knows the expected gain from the long-term relationship through a commonly known equilibrium selection rule, it would be impossible to figure out the way in which the players decide to form a long-term relationship after on observing the relation specific outcome.

Instead, we opt for a model with bounded rationality, in which the players have only a limited computational capability and may not know the structure of the game, the distribution of strategies, or the future course of evolution. Each agent in our model uses a simple rule of thumb that uses the average payoff of the past as the threshold. If the current match gives a higher payoff than the past average, then the agent is willing to initiate the long-term relationship. On the other hand, if it gives a lower payoff than the past average, then the agent chooses to terminate the partnership.

This principle of decision making is more closely related to the satisficing theory of Simon (1987)–later elaborated by Rubinstein (1993), Gilboa and Schmeidler (1995), Cho (1995), Karandikar, Mookherjee, Ray, and Vega-Redondo (1998), Bendor, Mookherjee, and Ray (2007) and Cho and Matsui (2005)–than to the standard optimizing theory in the sense that each agent compares what is obtained today with what has been obtained in the past while making a decision on whether or not to continue the current relationship.

The above decision problem is simple, not only because each agent uses a threshold strategy as in the standard search theory, but because he uses the past average as a threshold–which may be called an *aspiration level*–by ignoring the context under which the payoff was realized. Being a simple average of his payoffs from the past plays, the aspiration level coincides with the steady state payoff, if one exists, which ensures an asymptotic consistency with the stationary distribution of the outcomes of our model. As we shall prove the existence of a unique steady state, our approach will offer a criterion for selecting a particular equilibrium outcome from a long-term relationship.

Even though we simplify the decision rule significantly, the model remains highly complex. We have to keep track of the state of a society populated with many players who can either be single or paired with someone else. As a result, a component of the state is the cross sectional description of the social interaction, identifying who is in the pool and who is paired with whom. In a large society, the number of all the possible configurations of social interactions is very large, and the number of states in our model is even larger. In addition to the large state space, we have to handle the non-stationarity of the aspiration level and resulting non-stationarity of the decision to initiate the long run relationship, both of which arise from the interaction between the aspiration level and the repeated game outcome. Our task is to characterize the asymptotic properties of a non-stationary stochastic process of aspiration levels in a high dimensional Euclidean space.

By invoking the stochastic approximation technique (Kushner and Yin (1997)), we can approximate the sample path of the stochastic process in terms of a trajectory induced by an ordinary differential equation (ODE).[1] Although this approximation dramatically simplifies the analysis, the resulting deterministic dynamics is still too complex to handle because the evolution of the individual aspiration level is affected by the entire profile of the aspiration vector. Inspired by the model reduction technique developed for the analysis of complex stochastic network (Meyn (2007)), we characterize the convergence properties of the aspiration vector by analyzing a lower dimensional vector of aspiration levels. That is, instead of examining the entire profile of the aspiration vector in the ODE, we focus on the dynamics of the largest and smallest element of the aspiration vector in the ODE to infer the asymptotic properties of the entire vector. The model reduction technique drastically simplifies the analysis, but also renders the characterization result very robust as the key asymptotic properties depend upon the two extreme elements instead of the entire profile of aspiration levels.

We demonstrate that if the probability of exogenous break down is small and the economy has a sufficiently large number of agents, then the proportion of agents who remain in a partnership with the same agent is close to one. Despite the absence of the centralized information processing or matching mechanism, the aspiration level of every agent converges to the the Pareto efficient symmetric outcome of the component game, which is a natural focal point outcome (Schelling (1960)). In this sense, the Nash bargaining solution is derived–as opposed to assumed, as is the case in many existing models–from the underlying matching process followed by a long-term relationship.

Any attempt to obtain more than the payoff in the symmetric efficient state triggers the termination of the partnership, dumping both agents into the pool of singles. Despite the absence of a centralized authority, every agent in the society appears to behave according to the rule, "Don't be greedy if you want to have a long-term relationship with your partner." The same rule applies to any agent regardless of his history of outcomes, which can be considered as a social norm.

[1] The same technique has been widely used in macro and microeconomic models, including Marcet and Sargent (1989) and Fudenberg and Kreps (1993).

The long-term relationship with an option to drop out was studied by Ghosh and Ray (1996), Carmichael and McLeod (1997) and recently, by Fujiwara-Greve and Okuno-Fujiwara (2009). Ghosh and Ray (1996) constructed a random matching model with an option of continuing the relationship upon a mutual agreement. They used a concept that allows the party to make a joint deviation and showed that the only efficient outcome emerges as a stable outcome. Their equilibrium concept has a flavor of strong equilibrium (Aumann (1959)) and selects the Pareto efficient outcomes from the model under a certain conditions. Fujiwara-Greve and Okuno-Fujiwara (2009) applied the standard equilibrium approach to a similar environment. They focused on a particular type of efficient stable state; otherwise, they would have gotten a result similar to the folk theorem. Carmichael and McLeod (1997) constructed a random matching model wherein agents are forced to form a long-term relation once matched with an option of cheap-talk and gift-giving. They then applied a weaker version of evolutionarily stable strategy to show that the symmetric efficient outcome is attained in the repeated prisoners' dilemma.

The present paper is closely related to Karandikar, Mookherjee, Ray, and Vega-Redondo (1998) and Cho and Matsui (2005) in terms of both the behavior rule and technique. Karandikar, Mookherjee, Ray, and Vega-Redondo (1998) and Cho and Matsui (2005) considered the standard repeated game, but following Simon (1987), they assumed that the agents are not utility maximizers, but that their actions are guided by the satisficing behavior. Bendor, Mookherjee, and Ray (2007) extended the analysis of Karandikar, Mookherjee, Ray, and Vega-Redondo (1998) from the class of repeated 2×2 games to a general class of games, implicitly assuming the existence of a centralized information processing unit that aggregates the aspiration levels and disseminates the summary statistics to the individual agents.

In section 2, we formally describe the model where some agents are successfully matched, while other agents are actively searching for a partner under the assumption that the strategy space of each agent is continuous. A natural state would be the profile of coalitions that describes the set of successfully matched pairs along with the actions they are playing and the group of agent who are yet to be matched. The number of states is significantly larger than the number of agents, which poses considerable challenge. In addition, the evolution of the aspiration vectors contains feedback features as the decision of each agent is affected by the aspiration, which in turn affects the evolution of aspiration vectors. In section 3, we address these two challenges. First, we identify the deterministic dynamics by characterizing the dynamics of the mean of the stochastic process of aspiration vectors in terms of an ordinary differential equation (ODE). Second, we prove the convergence property of the ODE by analyzing a lower dimensional dynamics. That is, instead of the dynamics of the entire profile, we shall focus on the dynamics of the two extreme values– the highest and lowest aspirations–and show that these two values must converge to a Pareto efficient pure state. Section 4 concludes the paper.

2. Model

2.1. **Environment.** Consider an economy populated with a finite number of infinitely lived agents. Time is discrete: $t = 1, 2, \ldots$. Let $I = \{1, 2, \ldots, n\}$ be the set of agents where n is an even number greater than or equal to four. Each agent searches for a partner. This search process is completely random among those who are not matched. When two agents meet, they work together in that period to realize how much output the two of them can produce and what the share of each agent will be. They then decide whether or not to form a partnership. The partnership is formed only if both the agents agree to do so. If one of them refuses to form a partnership, then they go back to the pool of search without producing anything in that period, which induces zero payoff, and wait for another match in the next period. On the other hand, if they agree to form a partnership, the partnership continues until it breaks up with an exogenous shock, which occurs with probability $1 - \delta \in (0, 1)$ at the end of each period. We call $\delta < 1$ the *continuation probability.*

The set of agents in the (random matching) pool at the beginning of period t is denoted by U_t. The other set $U_t^c = I \setminus U_t$ contains the agents who form a partnership in the beginning of the period. We suppress the time subscript from U_t and U_t^c whenever the meaning is clear and write U and U^c instead. We write the matched pair of agents i and j as $\{i, j\}$. U^c is partitioned into a finite number of pairs. Let us write $\mathcal{U}^c$ as a partition of U^c into pairs, i.e.,

$$\mathcal{U}^c = \{\{i_1, j_1\}, \ldots, \{i_k, j_k\}\},$$

where $i_s \neq i_{s'}$, $j_s \neq j_{s'}$ for $s \neq s'$ and $i_s \neq j_{s'}$ hold.

When two agents meet, the total value of their productivity as well as their respective shares is determined exogenously. Suppose that agents i and j meet. Their prescribed payoffs are then given by

$$\begin{aligned} u_i &= \theta_i u \\ u_j &= \theta_j u, \end{aligned}$$

where u is a relation-specific output from the partnership, and θ_i is player i's share; $0 \leq \theta_i \leq 1$ and $\theta_i + \theta_j = 1$, which reflects their bargaining power. Both θ_i and u are stochastically determined. Let $\nu_{u,\theta}$ be the probability distribution over (u, θ_1, θ_2). Let $V \subset \mathbb{R}^2$ be the set of all feasible (u_1, u_2) from the partnership. The measure $\nu_{u,\theta}$ induces a probability distribution over V, which is denoted by ν.

Assumption 2.1. *The measure ν has a full support over V, which is a compact convex set with a non-empty interior. The measure ν has no atom and has a density function f_ν, which is symmetric with respect to the 45 degree line, i.e., $f_\nu(x_i, x_j) = f_\nu(x_j, x_i)$ for all $(x_i, x_j) \in V$, continuously differentiable and concave over V.*

All conditions–except for the concavity of f_ν–are fairly standard regularity conditions. The symmetry around the 45 degree line is natural, as we treat the bargaining power of the two partners equally in the ex ante sense. The heterogeneity among players arises only from their different past experiences.

The concavity of f_ν is imposed only to address some technical issues in the analysis. Note that these assumptions imply that $f_\nu(x + z, x - z)$ is (weakly) decreasing in $|z|$.

The uniform distribution over V satisfies these assumptions, which is common in the labor market search models (e.g., Mortensen and Pissarides (1994)). In a certain sense, the uniform distribution poses the worst possible scenario toward achieving any form of coordination between the two partners, not to mention among n players in a society, because any pair of utilities is equally likely.

Let

$$u^* = \max_{(u_1,u_2)\in V} \frac{u_1 + u_2}{2},$$

then (u^*, u^*) corresponds to the Nash bargaining solution with the origin as a threat point. Roughly speaking, our task is to show that the social dynamics let (u_1, u_2) converge to the Nash bargaining solution, even if (u_1, u_2) is uniformly distributed over V initially.

If both i and j decide to keep the pair in period t, then $\{i, j\} \in \mathcal{U}^c$ at the beginning of period $t+1$ with probability $\delta \in (0, 1)$ and $i, j \in U$ with probability $1 - \delta$.

The timing of matches and decisions is illustrated in Figure 1.

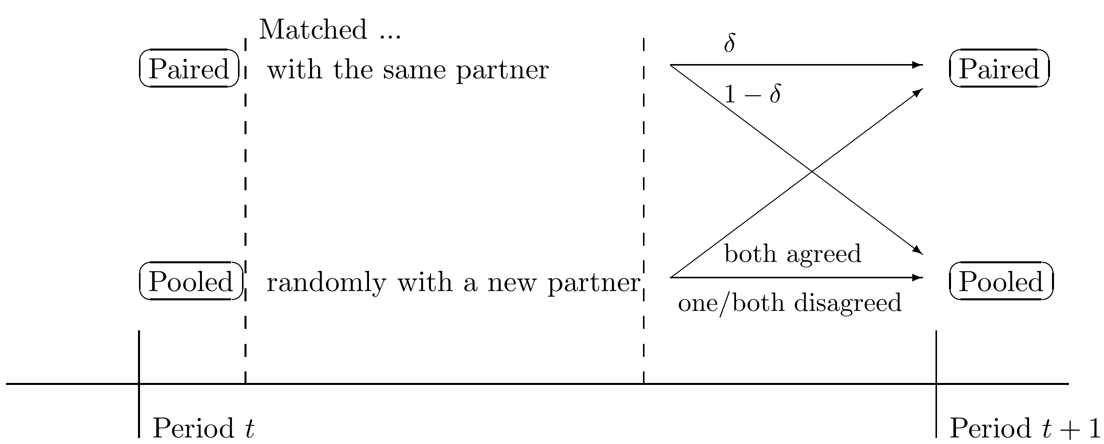

Figure 1: Timing of Matches and Decisions

2.2. **Behavior Rule.** To complete the formal description of the model, we have to specify how player i decides to form a partnership with player j after observing u_i. It is certainly an option to model the long-term relationship as a repeated game and to endow each player with a full set of repeated game strategies to invoke an equilibrium analysis. However, as we are interested in the case with continuation probability $\delta < 1$ being close to 1, a model with fully rational players leads to a large set of equilibria without any guidance to select a particular outcome.

Instead, we opt for a boundedly rational player who has a very limited computational capability. The agents of the economy do not know the distribution of outputs and their respective bargaining powers, nor do they know the strategies of other agents. Instead of solving the dynamic programming problem as a rational player in a repeated game would have done, our players use a simple rule of thumb to decide whether or not initiate a long-term relationship as envisioned by Simon (1987).

In period t with $t > 1$, whether or not agent i agrees to form a partnership, say, $\{i, j\}$, depends upon the payoff u_i and his past average payoff,

$$a_{it} = \frac{1}{t} \sum_{k=0}^{t-1} u_{ik}. \tag{2.1}$$

Following Simon (1987) and Karandikar, Mookherjee, Ray, and Vega-Redondo (1998), we call a_{it} *aspiration* of player i at t, or simply, aspiration. Agent i agrees to form a partnership if $u_{it} > a_{it}$, or chooses to go back to the pool otherwise.

The formation of a partnership requires mutual agreement. Therefore, players i and j form a partnership only if both $u_{it} > a_{it}$ and $u_{jt} > a_{jt}$.

Let v^{o} be the one-shot expected payoff of agent $i \in U$, which is calculated as

$$v^{\mathrm{o}} = \int_{(u',u'') \in V} u' \; d\nu(u', u'').$$

Note that $v^{\mathrm{o}} < u^*$.

Note that we assume very little computational capability of the players in their search for their partners and in deciding whether or not to initiate a long-term relationship. The agents of the economy do not know the distribution of outputs and their respective bargaining powers, nor do they know the strategies of other agents. Hence, the agents do not use dynamic programming to calculate the threshold above which they agree to form a partnership. Instead, they use a simple rule of thumb. It is instructive to understand the social outcome that arises under the minimal computational capability of the players before examining a more elaborate behavior.

3. Analysis

Let $a_t = (a_{it})_{i \in I}$ be the profile of average payoffs in period t. Our focus of interest is to understand the asymptotic properties of a_t, which is a random vector in $\mathbb{R}^I$. The dynamics of a_t poses a numbers of challenges. First, since agent i has generally been matched with many different agents in the past, a_{it} is affected by this history in a complex manner. Second, since the decision of agent i is affected by a_{it}, the feedback from the decision to the history makes the stochastic process a_t non-stationary. Third, the evolution of a_{it} is affected by the entire vector $a_t = (a_{1,t}, \ldots, a_{n,t})$. As we analyze an economy with a large finite number of agents, we have to keep track of a random vector in a high dimensional Euclidean space. Indeed, even with the simple decision rule, a state is described as

$$((a_i)_{i \in I}, (u_i)_{i \in I}, (U, \mathcal{U}^c)),$$

which is too large to handle as it is.

We first use the stochastic approximation technique, analyzing the dynamics of the mean of the average payoffs of individual agents instead of the sample path induced by the stochastic process. Since the average payoffs evolve much more slowly than the present payoffs and the matching patterns, we shall first investigate the evolution of the mean from a fixed profile of the average payoffs. Once we characterize the local evolution of the average payoffs, we can approximate the sample path of the average payoffs in terms of the trajectory induced by an associated ordinary differential equation (ODE).

We still have to handle different ODE's for different agents to understand the asymptotic properties. Instead, we use the model reduction technique (Meyn (2007)) by focusing on the dynamics of a couple of key variables in order to infer the relevant properties of the dynamics of a_t. We construct a simple Lyapunov function to summarize the key features of the dynamics into a function defined over a lower dimensional Euclidean space to pin down the stability and the convergence properties of a_t.

3.1. **Preliminaries.** In order to make this paper self-contained, we introduce notation and define concepts needed for later analysis. We draw our material from Kushner and Yin (1997), which provides a comprehensive introduction to the stochastic approximation technique.

We write (2.1) in a recursive form

$$a_{it} = a_{i,t-1} + \frac{1}{t}\left(u_{i,t-1} - a_{i,t-1}\right) \tag{3.2}$$

where $u_{i,t-1}$ is player i's utility realized in period $t-1$. Let $a_t = (a_{1,t}, \ldots, a_{n,t})$. The following conditions are needed.

A.1 $\sup_{t,i} \mathsf{E}|u_{i,t}|^2 < \infty$

A.2 There are measurable functions $\Psi_i(a_t)$ and $\beta_{i,t}$ such that

$$\mathsf{E}_t u_{i,t} = \Psi_i(a_t) + \beta_{i,t}$$

where E_t is the expectation conditioned on the information available at the beginning of period t.

A.3 Ψ_i is continuous.

A.4 $\sum_{t=0}^{\infty} \frac{|\beta_{i,t}|}{t} < \infty$ with probability 1.

Define $\tau_0 = 0$, and $\tau_K = \sum_{t=1}^{K} 1/t$ for $K \geq 1$. For $\tau > 0$, define $m(\tau)$ as the unique K such that $\tau_K \leq \tau < \tau_{K+1}$. Given a discrete time stochastic process $\{a_t\}_{t=1}^{\infty}$, we construct a continuous time stochastic process $a^0(\tau)$ through a continuous time interpolation:

$$a^0(\tau) = a_K \quad \text{if} \quad \tau_K \leq \tau < \tau_{K+1}.$$

Define the (left) shifted process $a^K(\tau)$ as

$$a^K(\tau) = a^0(\tau_K + \tau).$$

The classic results of the stochastic approximation consist of two parts: the approximation and the convergence of $\{a_t\}$. Let us state a basic theorem (Theorem 2.1 in Kushner and Yin (1997)) adapted for our model. A complete proof for much more general environments can be found in Kushner and Yin (1997).

Theorem 3.1. *Consider an ordinary differential equation (ODE)*

$$\dot{a}_i = \Psi_i(a) \qquad \forall i$$

or more compactly,

$$\dot{a} = \Psi(a) \tag{3.3}$$

where $\Psi = (\Psi_1, \ldots, \Psi_n)$.

Suppose that A.1–A.4 hold for (3.2)*. Then, the sample path of the discrete time stochastic process* $\{a_t\}_{t=1}^{\infty}$ *is approximated by a trajectory of* (3.3) *for a large* t*:*

$$\lim_{K\to\infty}\lim_{\tau\to\infty}\left[a_i^K(\tau)-a_i^K(0)-\int_0^\tau \Psi_i(a(s))ds\right]=0 \qquad \forall i \tag{3.4}$$

with probability 1 where $a(0)=a^K(0)$*.*

For a large t, the interpolated sample path of a_t remains close to the trajectory induced by (3.3). Since (3.3) approximates the stochastic process $\{a_t\}$, we call (3.3) the *associated ODE* or simply the ODE.

Among the four assumptions, A.1 and A.4 immediately follow the fact that the underlying game has the compact set V of feasible payoff vectors. In Section 3.2 we compute Ψ_i and check its continuity. In Section 3.3, we characterize the associated ODE and identify the neighborhood of the symmetric efficient outcome as the region within which any absorbing set is confined. Then, we obtain the main convergence result (Theorem 3.8) by invoking Theorem 3.1.

3.2. **Markov Process under a Fixed Aspiration Profile.** As the first step to invoke the stochastic approximation, let us for a moment fix a profile of average payoffs, $a=(a_i)_{i\in I}$, with

$$a_i \in (\min_{(u_i,u_j)\in V} u_i, \max_{(u_i,u_j)\in V} u_i) \qquad \forall i.$$

Although a evolves over time, its speed of change becomes significantly slower than the speed at which the agents' decisions to form a partnership as t increases. Our initial task is to describe the dynamics as a Markov process by considering for a moment that profile a is fixed.

By fixing a, we essentially eliminate the impact of payoffs in the long-term relationship. Thus, we are not only able to suppress a but also u from the description of the state. Hence, the state in the beginning of each period consists of the following two elements:

- the set of agents in the pool, and
- the pairs of agents continued from the previous period.

A state is, therefore, described as a pair $(U;\mathcal{U}^c)$ where U is a subset of I with an even number of elements and $\mathcal{U}^c$ is a partition of U^c into pairs, i.e.,

$$q = (U; \{\{i_1, j_1\}, \ldots, \{i_K, j_K\}\}), \quad i_k \neq i_\ell,\ j_k \neq j_\ell,\ i_k \neq i_\ell,$$

where $\{i_k, j_k\}$ $(k = 1, \ldots, K)$ is a pair continued from the previous period. Note that $\mathcal{U}^c = \emptyset$ holds if no pair is in the satisfied match from the previous play, and that $U = \emptyset$ holds if everyone is matched and satisfied. Let Q be the collection of all the states.

Given a state q, and given $i, j \in U$, let $x(a_i, a_j)$ be the probability that agents i and j form a long-term relation, i.e.,

$$\begin{aligned} x(a_i, a_j) &= \mathsf{P}\left(u_i > a_i \quad u_j > a_j | i, j \in U,\ i \text{ and } j \text{ meet}\right) \\ &= \int_{(u_i,u_j)\in V} \mathbb{I}(u_i - a_i > 0)\mathbb{I}(u_j - a_j > 0)d\nu(u_i, u_j), \end{aligned}$$

where $\mathbb{I}(\cdot)$ is an indicator function:

$$\mathbb{I}(u_i - a_i > 0) = \begin{cases} 1 & \text{if } \ u_i - a_i > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that x is symmetric, i.e., $x(a_i, a_j) = x(a_j, a_i)$ and monotonically nonincreasing in both the arguments.

Given an average payoff profile a, let

$$Q(a) = \{q = (U; \mathcal{U}^c) \in Q | x(a_i, a_j) > 0 \text{ for all } \{i, j\} \in \mathcal{U}^c\}.$$

Then the Markov process restricted to $Q(a)$ is ergodic, while all states in $Q \setminus Q(a)$ are transient. Let $\mu_t(q)$ be the probability of state q. There is a unique stationary distribution $\mu^*(a)$,

$$\lim_{t\to\infty} \mu_t = \mu^*(a)$$

satisfying $\mu^*(a)(q) = 0$ for all $q \in Q \setminus Q(a)$.[2] Note that $\mu^*(a)$ is continuous in a since ν has no mass point. This leads to A.3 of Subsection 3.1.

This stationary distribution $\mu^*(a)$ determines the unconditional probability $z_i(a)$ that agent i remains in the pool of singles:

$$z_i(a) = \sum_{i\in U} \mu^*(U; \mathcal{U}^c) \tag{3.5}$$

[2]This is because every state, including $(I; \emptyset)$, can be reached from $(I; \emptyset)$ in one step, and vice versa.

and $z_{i,t}(a)$ for the unconditional probability that agent i remains in the pool of singles at t. We can define the probability of his forming a (new) long-term relation with some agent conditioned on $i \in U$ at time t, which is denoted by $p_{i,t}(a)$:

$$p_{i,t}(a) = \frac{\displaystyle\sum_{j\neq i}\sum_{\{i,j\}\subset U_q} \frac{\mu_t(q)}{\#U_q - 1} x(a_i, a_j)}{\displaystyle\sum_{j\neq i}\sum_{\{i,j\}\subset U_q} \frac{\mu_t(q)}{\#U_q - 1}} \tag{3.6}$$

where $q = (U_q; \mathcal{U}_q^c)$ and $\#U_q$ is the number of elements in U_q. We also denote by $p_i(a)$ the corresponding probability under the stationary distribution μ^*. Notice that $p_{i,t}(a)$ is a positive linear combination of $x(a_i, a_j)$'s ($j \neq i$) and so is $p_t(a)$. Clearly,

$$\lim_{t\to\infty} z_{i,t}(a) = z_i(a)$$

and

$$\lim_{t\to\infty} p_{i,t}(a) = p_i(a).$$

Furthermore, since the probability of joining the pool is $1-\delta$ after both agree to continue the relationship, we have

$$z_{i,t+1}(a) = (1 - p_{i,t})z_{i,t} + (1-\delta)(1 - z_{i,t}(a)).$$

Letting t go to infinity, we have

$$z_i(a) = \frac{1-\delta}{1-\delta+p_i(a)}. \tag{3.7}$$

Intuitively, if a player has a low aspiration level, then his chance of forming a successful partnership would be higher than a player with a higher aspiration level. As a result, the probability that a player stays in the pool, $z_i(a)$, would be a decreasing function of the aspiration level of player i. The next lemma partially confirms this simple, but critical, observation.

To state the lemma, we need some preparation. Given $a = (a_1, \ldots, a_n)$, define

$$a_{(1)} = \min\{a_1, \ldots, a_n\}$$

and given $a_{(1)}, \ldots, a_{(i)}$, define

$$a_{(i+1)} = \min\left(\{a_1, \ldots, a_n\} \setminus \{a_{(1)}, \ldots, a_{(i)}\}\right).$$

By (i), we mean the player with the i-th lowest aspiration among $\{a_1, \ldots, a_n\}$. In particular,

$$a_{(n)} = \max\{a_1, \ldots, a_n\}.$$

Lemma 3.2. *For a given* $a = (a_1, \ldots, a_n)$,

$$z_{(1)}(a) \leq z_{(n)}(a).$$

Proof. The Markov process over Q converges to the unique stationary distribution regardless of the initial distribution μ_0 over Q. Thus, it suffices to prove

$$\lim_{t\to\infty} z_{(1),t}(a) \leq \lim_{t\to\infty} z_{(n),t}(a)$$

for an arbitrary initial distribution μ_0.

Recall that $x(a_i, a_j)$ is symmetric:

$$x(a_i, a_j) = z(a_j, a_i)$$

and a decreasing function of the aspiration levels:

$$x(a_{(i)}, a_{(k)}) \leq x(a_{(i')}, a_{(j')}) \qquad \forall i \geq i',\ \forall j \geq j', i \neq j.$$

Thus,

$$x(a_{(1)}, a_j) \geq x(a_k, a_{(n)}) = x(a_{(n)}, a_k)$$

for all $j \neq 1$ and $k \neq n$. From (3.6), we have

$$p_{(1),t}(a) \geq p_{(n),t}(a)$$

for all t. Thus,

$$p_{(1)}(a) \geq p_{(n)}(a)$$

holds. Then from (3.7), we have

$$z_{(1)}(a) \leq z_{(n)}(a).$$

□

Given a_i and a_j, let v_{ij} be the conditional expected payoff of player i when players i and j agree to form a long-term relation. Note that in this event, $u_i < a_i$ does not occur.

Then we have

$$x(a_i, a_j)v_{ij} = \int_{(u_i,u_j)} \mathbb{I}(u_i - a_i)\mathbb{I}(u_j - a_j)u_i d\nu(u_i, u_j) \geq x(a_i, a_j)a_i.$$

Let $v_i(a)$ be the conditional expected payoff when agent i stays in the same pair from the previous period. Using v_{ij}'s, we have

$$v_i(a) = \sum_{j \neq i} \frac{\mu^*(a)(\{i,j\} \in \mathcal{U}^c)}{\mu^*(a)(i \notin U)} v_{ij}.$$

Note that $v_i(a) > a_i$ always holds.

Let us define a state contingent expected utility function of player i for a given a:

$$\mathsf{u}_i : Q \to \mathbb{R}$$

suppressing its dependency on a to simplify notation. Given $q \in Q$, if $i \in U$, then

$$\mathsf{u}_i(q) = v^{\mathrm{o}}.$$

If $q = (U, \mathcal{U}^c)$ satisfies $\{i,j\} \in \mathcal{U}^c$, then $\mathsf{u}_i(q)$ represents the expected utility conditioned on forming a successful partnership, i.e.,

$$\mathsf{u}_i(q) = v_{ij}.$$

Thus, the expected payoff in this stationary state is written as

$$\begin{aligned} \mathsf{E}_{\mu^*(a)}[\mathsf{u}_i] &= \mu^*(a)(i \in U)v^{\mathrm{o}} + \sum_{j \neq i} \mu^*(a)(\{i,j\} \in \mathcal{U}^c)v_{ij} \\ &= z_i(a)v^{\mathrm{o}} + (1 - z_i(a))v_i(a). \end{aligned} \tag{3.8}$$

We state a series of preliminary results that show how the expected utility gain $\mathsf{E}_{\mu^*(a)}[\mathsf{u}_i - a_i]$ is affected by the profile of aspirations, a.

Lemma 3.3. *Given a, we have*

$$\mathsf{E}_{\mu^*(a)}[\mathsf{u}_i - a_i | \ \{i,k\} \in \mathcal{U}^c] > \mathsf{E}_{\mu^*(a)}[\mathsf{u}_j - a_j | \ \{j,k\} \in \mathcal{U}^c]$$

if $a_i < a_j$ holds.

Proof. Note that agents i (resp. j) and k form a long-term relation if and only if $u_i > a_i$ (resp. $u_j > a_j$) and $u_k > a_k$.

$$\mathsf{E}_{\mu^*(a)}[\mathsf{u}_i - a_i|\ \{i,k\} \in \mathcal{U}^c] - \mathsf{E}_{\mu^*(a)}[\mathsf{u}_j - a_j|\ \{j,k\} \in \mathcal{U}^c]$$

$$= \frac{\displaystyle\int_{\substack{u'>a_i\\u''>a_k}} (u' - a_i)d\nu(u', u'')}{\mathsf{P}[u' > a_i \text{ and } u'' > a_k]} - \frac{\displaystyle\int_{\substack{u'>a_j\\u''>a_k}} (u' - a_j)d\nu(u', u'')}{\mathsf{P}[u' > a_j \text{ and } u'' > a_k]}. \tag{3.9}$$

We claim that (3.9) is positive if $a_j > a_i$, using the assumptions on the distribution of (u', u''). To begin with, note that (3.9) is positive if and only if

$$\frac{\displaystyle\int_{\substack{u'>a_i\\u''>a_k}} (u' - a_i)d\nu(u', u'')}{\displaystyle\int_{\substack{u'>a_i\\u''>a_k}} d\nu(u', u'')}$$

is decreasing in a_i. Therefore, it suffices to show that its derivative with respect to a_i is negative:

$$\begin{aligned}
&- \left[\int_{u'>a_i} \int_{u''>a_k} f_\nu(u', u'')du''du'\right]^2 \\
&+ \left[\int_{u'>a_i} \int_{u''>a_k} (u' - a_i)f_\nu(u', u'')du''du'\right] \cdot \left[\int_{u''>a_k} f_\nu(a_i, u'')du''\right] < 0.
\end{aligned} \tag{3.10}$$

Define

$$F_{a_k}(u') = \int_{u''>a_k} f_\nu(u', u'')du'',$$

and

$$a_{\max} = \sup\{u' | F_{a_k}(u') > 0\}.$$

Then (3.10) is equivalent to

$$- \left[\int_{a_i}^{a_{\max}} F_{a_k}(u')du'\right]^2 + \left[\int_{a_i}^{a_{\max}} (u' - a_i)F_{a_k}(u')du'\right] \cdot F_{a_k}(a_i) < 0. \tag{3.11}$$

Noticing that $F_{a_k}(u')$ is concave, we consider the following three cases.

Case (i) $F_{a_k}(u')$ *is decreasing in* u' *on* $[a_i, a_{\max}]$. We have

$$\int_{a_i}^{a_{\max}} (u' - a_i)F_{a_k}(u')du' < \int_{a_i}^{a_{\max}} \frac{a_{\max} - a_i}{2} F_{a_k}(u')du'.$$

Then it suffices to show

$$- \int_{a_i}^{a_{\max}} F_{a_k}(u')du' + \frac{a_{\max} - a_i}{2} F_{a_k}(a_i) < 0,$$

which holds due to the concavity of F_{a_k}.

CASE (II) *$F_{a_k}(u')$ is increasing in u' on $[a_i, a_{\max}]$.* We have

$$\int_{a_i}^{a_{\max}} (u' - a_i) F_{a_k}(u') du' < \int_{a_i}^{a_{\max}} (a_{\max} - a_i) F_{a_k}(u') du'.$$

Then it suffices to show

$$-\int_{a_i}^{a_{\max}} F_{a_k}(u') du' \ + \ (a_{\max} - a_i) F_{a_k}(a_i) < 0,$$

which holds since $F_{a_k}(a_i) < F_{a_k}(u')$ holds for all $u' \in [a_i, a_{\max}]$.

CASE (III) *$F_{a_k}(u')$ is increasing and then decreasing on $[a_i, a_{\max}]$.* Fix a' that satisfies

$$\frac{\partial F_{a_k}(a')}{\partial u'} = 0.$$

Then we decompose the equation into two parts to invoke the logic from CASES (I) and (II) to obtain

$$\int_{a_i}^{a_{\max}} F_{a_k}(u') du' = \int_{a_i}^{a'} F_{a_k}(u') du' + \int_{a'}^{a_{\max}} F_{a_k}(u') du'.$$

Then we apply exactly the same arguments as in CASES (I) and (II) to the first and second term of the right hand side of the equation to prove the lemma. □

The following lemma is an easy corollary whose proof is omitted since it is essentially the same as that of Lemma 3.3.

Lemma 3.4. *Given the average payoff profile a, we have*

$$\mathsf{E}_{\mu^*(a)}[\mathsf{u}_i - a_i | \ \{i, k\} \in \mathcal{U}^c] > \mathsf{E}_{\mu^*(a)}[\mathsf{u}_i - a_i | \ \{i, \ell\} \in \mathcal{U}^c]$$

if $a_k < a_\ell$ holds.

The next lemma uses the assumption on the distribution of (u', u''). Its proof is a matter of tedious calculation, using the fact that

$$f_\nu(a_i + x + z, a_j + x - z) \geq f_\nu(a_i + x - z, a_j + x + z)$$

holds for all $x \geq 0$ and $0 \leq z \leq x$, which is due to the symmetry around the 45 degree line and the concavity of f_ν.

Lemma 3.5. *Given the average payoff profile a, we have*

$$\mathsf{E}_{\mu^*(a)}[\mathsf{u}_i - a_i | \ \{i, j\} \in \mathcal{U}^c] \geq \mathsf{E}_{\mu^*(a)}[\mathsf{u}_j - a_j | \ \{i, j\} \in \mathcal{U}^c]$$

if $a_i < a_j$ holds.

Note that since v^{o} is the average payoff in the pool and the agents form a long-term relationship only if their payoffs are higher than the past average payoffs, $a_{(n)} \geq v^{\mathrm{o}}$ must hold.

Lemmata 3.4 and 3.5 together imply

$$\min_{k \neq (1)} \mathsf{E}_{\mu^*(a)}[\mathsf{u}_{(1)} - a_{(1)} | \ \{(1), k\} \in \mathcal{U}^c] > \max_{\ell \neq (n)} \mathsf{E}_{\mu^*(a)}[\mathsf{u}_{(n)} - a_{(n)} | \ \{(n), \ell\} \in \mathcal{U}^c]$$

for all k, ℓ if $a_{(1)} < a_{(n)}$. Indeed, if $a_i < a_j$ holds, then for all $a_k, a_\ell \in [a_i, a_j]$, we have

$$\begin{aligned} \mathsf{E}_{\mu^*(a)}[\mathsf{u}_i - a_i | \ \{i,k\} \in \mathcal{U}^c] &\geq \mathsf{E}_{\mu^*(a)}[\mathsf{u}_i - a_i | \ \{i,j\} \in \mathcal{U}^c] \\ &\geq \mathsf{E}_{\mu^*(a)}[\mathsf{u}_j - a_j | \ \{i,j\} \in \mathcal{U}^c] \geq \mathsf{E}_{\mu^*(a)}[\mathsf{u}_j - a_j | \ \{j,\ell\} \in \mathcal{U}^c] \end{aligned}$$

where the first and the third inequalities come from Lemma 3.4, and the second inequality comes from 3.5. This together with the definition of $v_i(a)$ implies

$$v_{(1)}(a) - a_{(1)} \geq v_{(n)}(a) - a_{(n)}. \tag{3.12}$$

Now, we are in a position to state the following key result.

Proposition 3.6. *If $a_{(1)} < a_{(n)}$ holds, then we have*

$$\mathsf{E}_{\mu^*(a)}[\mathsf{u}_{(1)} - a_{(1)}] > \mathsf{E}_{\mu^*(a)}[\mathsf{u}_{(n)} - a_{(n)}]. \tag{3.13}$$

Proof. Suppose that $a_{(1)} < a_{(n)}$ holds. Using (3.8), we can write

$$\begin{aligned} \mathsf{E}_{\mu^*(a)}[\mathsf{u}_{(1)} - a_{(1)}] &= z_{(1)}(a)(v^{\mathrm{o}} - a_{(1)}) + (1 - z_{(1)})(v_{(1)}(a) - a_{(1)}) \\ &> z_{(1)}(a)(v^{\mathrm{o}} - a_{(n)}) + (1 - z_{(1)})(v_{(n)}(a) - a_{(n)}) \quad (3.14) \\ &\geq z_{(n)}(a)(v^{\mathrm{o}} - a_{(n)} + (1 - z_{(n)})(v_{(n)}(a) - a_{(n)}) \quad (3.15) \\ &= \mathsf{E}_{\mu^*(a)}[\mathsf{u}_{(n)} - a_{(n)}]. \end{aligned}$$

(3.12) together with the assumption of the proposition implies the strict inequality in (3.14), and (3.15) uses the fact that

$$z_{(1)}(a) \leq z_{(n)}(a) \quad \text{and} \quad v^{\mathrm{o}} < a_{(n)} < v_{(n)}(a).$$

□

3.3. **Evolution of Aspiration.** The key step toward analyzing the asymptotic properties of the distribution of average payoffs is identifying the associated ODE of the individual average payoff. The evolution of the mean of the average payoff around a small neighborhood of a_i is dictated by the difference between a_i and the expected payoff associated with the invariant distribution of the transition matrix (Kushner and Yin (1997)),

$$\dot{a}_i = \mathsf{E}_{\mu^*(a)}\left[\mathsf{u}_i\right] - a_i = \mathsf{E}_{\mu^*(a)}\left[\mathsf{u}_i - a_i\right]. \tag{3.16}$$

Note also that we eventually have $a_i \geq v^{\mathrm{o}}$ for all i with probability one since the expected value of a_i is a convex combination of v^{o} and $\mathsf{E}[u' > a_i]$, which is greater than or equal to v^{o}. This fact implies (3.13), and therefore, (3.16) implies the following proposition.

Proposition 3.7.

$$\dot{a}_{(n)} - \dot{a}_{(1)} < 0. \tag{3.17}$$

Suppose now that $a_{(n)} = a_{(1)} = \hat{a}$. Then $\hat{a}$ is no more than u^*. Fix $\zeta > 0$, a sufficiently small number, as given. Suppose that $\hat{a} < u^* - \zeta/2$. Let

$$\hat{p} = \int_{\substack{u' > \hat{a} \\ u'' > \hat{a}}} d\nu(u', u''),$$

and

$$\hat{v} = \int_{\substack{u' > \hat{a} \\ u'' > \hat{a}}} u' d\nu(u', u'')/\hat{p}.$$

Note that $\hat{v} > \hat{a}$ by definition. Moreover, using the assumption on ν, we can verify that $\hat{v} \geq (u^* + \hat{a})/2$.

Note that $Q(a) = Q$ in this case. This implies that $\mu^*(a)$ is continuous in a in the neighborhood of $(\hat{a}, \ldots, \hat{a})$. Also, since every agent has the same average payoff $\hat{a}$, the dynamics is relatively easy. Indeed, since the transition probability of each agent from the pool to the matched state is $\hat{p}$, while that from the matched state to the pool is given by $(1-\delta)$, each agent is in the pool and in the match with the probabilities of $(1-\delta)/(1-\delta+\hat{p})$ and $\hat{p}/(1-\delta+\hat{p})$, respectively. Hence, the expected payoff is given by

$$\mathsf{E}_{\mu^*(a)}[\mathsf{u}_i] = \frac{1-\delta}{1-\delta+\hat{p}} v^{\mathrm{o}} + \frac{\hat{p}}{1-\delta+\hat{p}} \hat{v}.$$

Since $\hat{p}$ is independent of δ, we have

$$\mathsf{E}_{\mu^*(a)}[\mathsf{u}_i] \to \hat{v} \geq \frac{u^* + \hat{a}}{2}, \text{ as } \delta \to 1.$$

In particular, if we let $\delta \in (0,1)$ satisfy

$$\frac{\zeta^2}{2(1-\delta)(u^*)^2+\zeta^2}\frac{u^*+\hat{a}}{2} > u^* - \zeta,$$

then $\hat{a} < u^* - \zeta$ implies

$$\frac{\hat{p}}{1-\delta+\hat{p}}\frac{u^*+\hat{a}}{2} > \hat{a}.$$

This in turn implies

$$\dot{a}_{it} = \mathsf{E}_{\mu^*(a)}[\mathsf{u}_i] - \hat{a} > 0,$$

for all $i \in I$.

As a result, we have the following theorem.

Theorem 3.8. $\forall \zeta > 0 \ \ \exists \underline{\delta} < 1 \ \forall \delta \in (\underline{\delta}, 1)$

$$u^* - \zeta \leq \liminf_{t\to\infty} a_{(1),t} \leq \limsup_{t\to\infty} a_{(n),t} \leq u^* + \zeta$$

with probability 1.

4. Concluding Remarks

In a society populated with players who have very limited computational and forecasting capabilities, the decentralized matching process can generate a uniform standard to initiate a long-term relationship. For example, if the underlying game has the structure of the prisoners' dilemma game, then Theorem 3.8 implies that a long-term relationship cannot be formed if one party is greedy in the sense that he receives more than the cooperative outcome.

In our model, the players are ex post heterogeneous because each player can go through a different experience, which is then summarized into a different level of aspiration. Yet, through a decentralized matching process, every player comes to have the same criterion for starting a long-term relationship.

The intuition is simple and robust. Whenever a player is trying to be greedy, he will be dumped into the pool of singles where he has to go through a matching process. If his aspiration is substantially higher than the cooperative outcome, then he is more likely to fail to form a successful long-term relationship. As he goes through a long period of failed attempts, his aspiration level goes down. With a lower aspiration level, he can then form a successful long-term relationship.

The robustness of the intuition in turn makes our result very much robust against a small perturbation of the basic model. For example, if we allow each player to deviate from the agreed action with a small probability in each period, then the same result holds. In the basic model, the aspiration is a simple average of the past payoffs. If we replace the simple average with the discounted average, then the same result is obtained as the rate at which the previous period's payoff is discounted converges to 1.

There are many tasks to be undertaken in the future. First, if we would like to use this analysis for labor search, we should consider two population. The second point, which is related to the first, is that in order to show its real usefulness, we should study economies with heterogeneous agents. The present model assumes ex ante identical agents, who are different only in terms of experiences. Differences in potential productivity may lead to different long-term payoffs. If we solve such a model, then we may be able to study income inequality based on differences in ability in the search theoretic context. Third, if we are interested in repeated games, we may need to consider more sophisticated strategies than the threshold ones. How we can extend the strategy space beyond the set of threshold strategies is a topic for future research.

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