

# Existence of Equilibria in Discontinuous and Nonconvex Games

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November, 2008/ Revised: May, 2009

## Abstract

This paper investigates the existence of pure strategy, dominant strategy, and mixed strategy Nash equilibria in discontinuous and/or nonconvex games. We introduce a new notion of very weak continuity, called *weak transfer quasi-continuity*, which is weaker than the most known weak notions of continuity, including diagonal transfer continuity in Baye *et al.* [1993] and better-reply security in Reny [1999], and holds in a large class of discontinuous games. We show that weak transfer quasi-continuity, together with the compactness of strategy space and the quasiconcavity or (strong/weak) diagonal transfer quasiconcavity of payoffs, permits the existence of a pure strategy Nash equilibrium. We provide sufficient conditions for weak transfer quasi-continuity by introducing notions of weak transfer continuity, weak transfer upper continuity, and weak transfer lower continuity. Moreover, an analogous analysis is applied to show the existence of dominant strategy and mixed strategy Nash equilibria in discontinuous games.

*Keywords:* Nash equilibrium, dominant strategy equilibrium, discontinuity, non-quasiconcavity, nonconvexity, and mixed strategy.

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†We thank Paulo Borelli for comments and suggestions. Financial support from the National Natural Science Foundation of China (NSFC-70773073) and the Program to Enhance Scholarly and Creative Activities at Texas A&M University as well as from Cheung Kong Scholars Program at the Ministry of Education of China is gratefully acknowledged. E-mail address: gtian@tamu.edu

# 1 Introduction

The concept of Nash equilibrium in Nash [1950, 1951] is probably the most important solution in game theory. It is immune from unilateral deviations, that is, each player has no incentive to deviate from his/her strategy given that other players do not deviate from theirs. Nash [1951] proved that a finite game has a Nash equilibrium in mixed strategies. Debreu [1952] then showed that games possess a pure strategy Nash equilibrium if (1) the strategy spaces are nonempty, convex and compact, and (2) players have continuous and quasiconcave payoff functions. Game theory has then been successfully applied in many areas in economics including oligopoly theory, social choice theory, and incentive mechanism design theory. These applications lead researchers from different fields to investigate the possibility of weakening equilibrium existence conditions to further enlarge its domain of applicability.

The uniqueness of pure strategy Nash equilibrium is established in Rosen [1965]. Nishimura and Friedman [1981] and Yao [1992] considered the existence of Nash equilibria in games where the payoff functions are not quasi-concave (but satisfying a *strong condition*) and  $\gamma$ -diagonally quasiconcave, respectively. Dasgupta and Maskin [1986] established the existence of pure and mixed strategy Nash equilibria in games where the strategy sets are convex and compact, and payoff functions are quasiconcave, upper semicontinuous and graph continuous by using an approximation technique. Simon [1987] and Simon and Zame [1990] used a similar approach to consider the existence of mixed strategy Nash equilibria in discontinuous games. Simon and Zame [1990] showed that if one is willing to modify the vector of payoffs at points of discontinuity so that they correspond to points in the convex hull of limits of nearby payoffs, then one can ensure a mixed strategy equilibrium of such a suitably modified game. Vives [1990] established the existence of Nash equilibria in games where payoffs are upper semicontinuous and satisfy certain monotonicity properties.

Baye *et al.* [1993] provided necessary and sufficient conditions for the existence of pure strategy Nash equilibria and dominant strategy equilibria in noncooperative games which may have discontinuous and/or non-quasiconcave payoffs. It is shown that diagonal transfer quasiconcavity is necessary, and further, under diagonal transfer continuity and compactness, sufficient for the existence of pure strategy Nash equilibrium. Both transfer quasiconcavity and diagonal transfer continuity are very weak notions of quasiconcavity and continuity and use a basic idea of transferring nonequilibrium strategies to a “securing” profile of strategies.

Reny [1999] established the existence of Nash equilibria in compact and quasiconcave games where the game is better-reply secure, which is a weak notion of continuity. Reny [1999] showed that better-reply security can be imposed separately as reciprocal upper semicontinuity introduced by Simon [1987] and payoff security. Bagh and Jofre [2006] further weakened reciprocal upper semicontinuity to weak reciprocal upper semicontinuity and showed that it, together with payoff

security, implies better-reply security. As one shall see, both better-reply security and payoff security use the same idea of transferring a (nonequilibrium) strategy to a “securing” profile of strategies, and they are actually also in the forms of transfer continuity.

Recently, Tian [2008] fully characterizes the existence of equilibria in games with general strategy spaces and payoffs. He establishes a single condition, called *recursive diagonal transfer continuity*, which is both necessary and sufficient for the existence of equilibria in games with arbitrary compact strategy spaces and payoffs. Like other existing characterization results, this is mainly for the purpose of providing a way of understanding equilibrium and identifying whether or not a game has an equilibrium, but not whether it is easy to check. In general, the weaker a condition in an existence theorem is, the harder it is to verify whether the conditions are satisfied in a particular game.

Most recently, Barelli and Soza [2009] further weaken the continuity and quasiconcavity conditions by using “majorized” approach. The conditions they developed allow more than one “securing strategy profiles” which depend on players and strategy profiles in a neighborhood of a non-equilibrium strategy profile under consideration. They unify and generalize most existing results, establishing existence of a pure strategy Nash equilibrium in the literature on discontinuous quasiconcave games and qualitative convex games. However, like a characterization result, the conditions are complicated and hard to check.

This paper investigates the existence of pure strategy, dominant strategy, and mixed strategy Nash equilibria in discontinuous and nonconvex games by providing a set of very weak sufficient conditions. We introduce a new notion of very weak continuity, called *weak transfer quasi-continuity*, which is weaker than the most known weak notions of continuity, including diagonal transfer continuity in Baye *et al.* [1993] and better-reply security in Reny [1999], and holds in a large class of discontinuous games and is relatively easy to check. Roughly speaking, a game is weakly transfer quasi-continuous if for every nonequilibrium strategy  $x^*$ , there exists a neighborhood and a securing strategy profile such that for every strategy profile in the neighborhood there is player  $i$  who is strictly better off by using his securing strategy. As a result, if  $x$  is not nonequilibrium, there is a neighborhood  $\mathcal{V}(x)$  of  $x$  that does not contain any equilibrium. Weak transfer quasi-continuity holds in many economic games and is easy to check. Besides those known sufficient conditions such as diagonal transfer continuity and better-reply security, we give four additional sets of sufficient conditions, each of which implies weak transfer quasi-continuity: (1) transfer continuity, (2) weak transfer continuity, (3) weak transfer upper continuity and payoff security,<sup>1</sup> and (4) upper semicontinuity and weak transfer lower continuity. These conditions are satisfied in many economic games and often quite simple to check.

We provide three main results on the existence of pure strategy Nash equilibria in games

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<sup>1</sup>It is worth pointing out that, while reciprocal upper semicontinuity combined with payoff security implies better-reply security, here weak transfer upper semicontinuity combined with payoff security implies weak transfer continuity.

with possibly discontinuous payoffs, which strictly generalize the most known existence results such as those in Baye *et al.* [1993] and Reny [1999]. Our Theorem 3.1 shows that weak transfer quasi-continuity, together with the boundedness and compactness of strategy space and the quasiconcavity of payoffs, permits the existence of a pure strategy Nash equilibrium. By relaxing the boundedness and quasiconcavity conditions, our Theorem 3.2 shows that under weak transfer quasi-continuity of payoffs and the compactness of strategy spaces, a game possesses a pure strategy Nash equilibrium if and only if it is strongly diagonal transfer quasiconcave. Furthermore, by introducing the notion of weak diagonal transfer quasiconcavity that is weaker than (strong) diagonal transfer quasiconcavity, our Theorem 3.3 shows that under weak transfer quasi-continuity of payoffs and the boundedness and compactness of strategy spaces, a game possesses a pure strategy Nash equilibrium if and only if it is weakly diagonal transfer quasiconcave.

Strong diagonal transfer quasiconcavity, diagonal transfer quasiconcavity, weakly diagonal transfer quasiconcavity are all very weak notions of quasiconcavity. A game is (strongly/weakly) diagonal transfer quasiconcave provided it has a pure strategy Nash equilibrium. We also give a results, Theorem 3.4, which shows the existence of pure strategy Nash equilibrium by further relaxing the compactness of strategy spaces. Moreover, by introducing the notion of weak dominant transfer upper continuity, an analogous analysis is applied to show the existence of dominant strategy and mixed strategy Nash equilibria in discontinuous games.

The remainder of the paper is organized as follows. Section 2 describes the notation, and provides a number of preliminary definitions. Section 3 introduces the notions of weak transfer continuity/weak transfer quasi-continuity and weak/strong diagonal transfer quasiconcavity, and provides the main results on the existence of pure strategy Nash equilibrium. We then generalize the results and those in Baye *et al.* [1993] and Reny [1999] without assuming any form of quasi-concavity of payoff functions or convexity of strategy spaces. Examples and applications illustrating the theorem are also given. Section 4 considers the existence of dominant strategy equilibrium by introducing a similar condition, weak dominant transfer continuity. We provide a main existence result of dominant strategy Nash equilibrium. Section 5 considers the existence of mixed strategy Nash equilibrium by applying the main result obtained in Section 3 on the existence of pure strategy Nash equilibrium. It is shown there that the mixed strategy theorems of Nash [1950], Glicksberg [1952], Dasgupta and Maskin [1986], Robson [1994], Simon [1987] and Reny [1999] imply our main result presented in section 5. Concluding remarks are offered in Section 6. The proofs of the theorem and propositions are presented in Appendix.

## 2 Preliminaries

Consider the following noncooperative game in a normal form:

$$G = (X_i, u_i)_{i \in I} \tag{2.1}$$

where  $I = \{1, \dots, n\}$  is a finite set of players,  $X_i$  is player  $i$ 's strategy space which is a nonempty subset of a topological space  $E_i$ , and  $u_i : X \rightarrow \mathbb{R}$  is the payoff function of player  $i$ . Denote by  $X = \prod_{i \in I} X_i$  the set of strategy profiles of the game. For each player  $i \in I$ , denote by  $-i$  all players rather than player  $i$ . Also denote by  $X_{-i} = \prod_{j \neq i} X_j$  the set of strategies of the players in coalition  $-i$ .

We say that a game  $G = (X_i, u_i)_{i \in I}$  is compact, convex, bounded, and semi-continuous, respectively if, for all  $i \in I$ ,  $X_i$  is compact and convex, and  $u_i$  is bounded and semi-continuous on  $X$ , respectively. We say that a game  $G = (X_i, u_i)_{i \in I}$  is quasiconcave if, for every  $i \in I$ ,  $X_i$  is convex and the function  $u_i$  is quasiconcave in  $x_i$ .

We say that a strategy profile  $x^* \in X$  is a *pure strategy Nash equilibrium* of game  $G$  if,

$$u_i(y_i, x_{-i}^*) \leq u_i(x^*) \quad \forall i \in I, \forall y_i \in X_i.$$

We say that a strategy profile  $x^* \in X$  is a *pure dominant strategy equilibrium* of a game  $G$  if,

$$\forall (y_i, y_{-i}) \in X, u_i(y_i, y_{-i}) \leq u_i(x_i^*, y_{-i}) \quad \forall i \in I.$$

Define a function  $U : X \times X \rightarrow \mathbb{R}$  by

$$U(x, y) = \sum_{i=1}^n u_i(y_i, x_{-i}), \quad \forall (x, y) \in X \times X. \quad (2.2)$$

Baye *et al.* [1993] study the existence of pure strategy Nash equilibria in games with possibly discontinuous and nonquasiconcave payoffs by introducing the concepts of diagonal transfer continuity and diagonal transfer quasiconcavity of  $U$ .

**DEFINITION 2.1** A game  $G = (X_i, u_i)_{i \in I}$  is diagonally transfer continuous if  $x$  is not an equilibrium, there exist a strategy profile  $y \in X$  and a neighborhood  $\mathcal{V}(x) \subset X$  of  $x$  such that  $U(z, y) > U(z, z)$  for all  $z \in \mathcal{V}(x)$ .

**REMARK 2.1** The point  $y$  in the above definition can be termed as a securing profile of strategies since whenever a strategy profile  $x$  is not an equilibrium, it secures a strictly higher utility for all strategy profiles in some neighborhood of  $x$ . It is clear that continuity of  $U$  implies diagonal transfer continuity of  $U$ .

**DEFINITION 2.2** A game  $G = (X_i, u_i)_{i \in I}$  is diagonally transfer quasiconcave if, for any finite subset  $Y^m = \{y^1, \dots, y^m\} \subset X$ , there exists a corresponding finite subset  $X^m = \{x^1, \dots, x^m\} \subset X$  such that for any subset  $\{x^{k^1}, x^{k^2}, \dots, x^{k^s}\} \subset X^m$ ,  $1 \leq s \leq m$ , and any  $x \in \text{co}\{x^{k^1}, x^{k^2}, \dots, x^{k^s}\}$  we have  $\min_{1 \leq l \leq s} U(x, y^{k^l}) \leq U(x, x)$ .

Theorem 1 in Baye *et al.* [1993] shows that, a game that is compact, convex, diagonally transfer continuous, and diagonally transfer quasiconcave must possess a pure strategy Nash equilibrium.

Reny [1999] studies the existence of pure strategy Nash equilibria in discontinuous games by introducing the concepts of payoff security and better-reply security.

The graph of the game is  $\Gamma = \{(x, u) \in X \times \mathbb{R}^n : u_i(x) = u_i, \forall i \in I\}$ . The closure of  $\Gamma$  in  $X \times \mathbb{R}^n$  is denoted by  $\text{cl } \Gamma$ . The frontier of  $\Gamma$ , which is the set of points that are in  $\text{cl } \Gamma$  but not in interior of  $\Gamma$ , is denoted by  $\text{Fr}(\Gamma)$ .

**DEFINITION 2.3** A game  $G = (X_i, u_i)_{i \in I}$  is payoff secure if for every  $x \in X$ , every  $\epsilon > 0$ , and every player  $i$ , there exists  $\bar{x}_i \in X_i$ , such that  $u_i(\bar{x}_i, y_{-i}) \geq u_i(x) - \epsilon$  for all  $y_{-i}$  in some open neighborhood of  $x_{-i}$ .

**DEFINITION 2.4** A game  $G = (X_i, u_i)_{i \in I}$  is better-reply secure if  $(x^*, u^*) \in \text{cl } \Gamma$  and  $x^*$  is not an equilibrium, there is a player  $i$  and a strategy  $\bar{x}_i \in X_i$  such that  $u_i(\bar{x}_i, y_{-i}) > u_i^*$  for all  $y_{-i}$  in some open neighborhood of  $x_{-i}$ .

**DEFINITION 2.5** A game  $G = (X_i, u_i)_{i \in I}$  is reciprocally upper semicontinuous if, whenever  $(x, u) \in \text{cl } \Gamma$  and  $u_i(x) \leq u_i$  for every player  $i$ , then  $u_i(x) = u_i$  for every player  $i$ .

The following notions are introduced by Bagh and Jofre [2006] and Morgan and Scalzo [2007], respectively.

**DEFINITION 2.6** A game  $G = (X_i, u_i)_{i \in I}$  is weakly reciprocal upper semicontinuous, if for any  $(x, u) \in \text{Fr}(\Gamma)$ , there is a player  $i$  and  $\hat{x}_i \in X_i$  such that  $u_i(\hat{x}_i, x_{-i}) > u_i$ .

**DEFINITION 2.7** Let  $Z$  be a topological space and  $f$  be an extended real valued function defined on  $Z$ .  $f$  is upper pseudocontinuous at  $z_0$  if for all  $z \in Z$  such that  $f(z_0) < f(z)$ , we have  $\limsup_{y \rightarrow z_0} f(y) < f(z)$ .  $f$  is said to be lower pseudocontinuous at  $z_0$  if  $-f$  is upper pseudocontinuous at  $z_0$ .  $f$  is said to be pseudocontinuous if it is both upper and lower pseudocontinuous.

Theorem 3.1 in Reny [1999] shows that a  $G = (X_i, u_i)_{i \in I}$  possesses a Nash equilibrium if it is compact, bounded, quasiconcave and better-reply secure. Reny [1999] and Bagh and Jofre [2006] provided sufficient conditions for a game to be better-reply secure. Reny [1999] showed that a game  $G = (X_i, u_i)_{i \in I}$  is better-reply secure if it is payoff secure and reciprocal upper semicontinuous. Bagh and Jofre [2006] further showed that  $G = (X_i, u_i)_{i \in I}$  is better-reply secure if it is payoff secure and weakly reciprocal upper semicontinuous. Morgan and Scalzo [2007] showed that  $G = (X_i, u_i)_{i \in I}$  is better-reply secure if  $u_i$  is pseudocontinuous,  $\forall i \in I$ .

**REMARK 2.2** Since payoff security requires taking an open neighborhood in the upper contour set of a given level of payoff, it is a weak notion of lower semicontinuity. Also, since better-reply

security requires the limit payoff resulted from strategies to approach a nonequilibrium point, it is a weak notion of continuity (which displays a certain form of both lower semicontinuity and upper semicontinuity). In addition, both notions use the same idea of transferring nonequilibrium strategy profile to a securing strategy profile that results in a strictly better-off payoff, and thus they actually fall in the forms of transfer continuity.

### 3 Existence of Nash Equilibria

In this section we investigate the existence of pure strategy Nash equilibria in games that may be discontinuous or nonquasiconcave. We first provide three main results on the existence of pure strategy Nash equilibria in discontinuous games, which strictly generalize the existence results of Baye *et al.* [1993], Reny [1999] and Bagh and Jofre [2006]. We also characterize the existence of pure strategy Nash equilibrium without assuming the convexity of strategy spaces or any form of quasiconcavity of payoff functions. We then show how our main existence results are applied to some important economic games.

#### 3.1 Nash Equilibria in Discontinuous Games

We start by introducing some weak forms of continuities.

**DEFINITION 3.1** A game  $G = (X_i, u_i)_{i \in I}$  is said to be *transfer continuous* if for all player  $i$ ,  $u_i$  is transfer continuous in  $x$  with respect to  $X_i$ , *i.e.*, if  $u_i(z_i, x_{-i}) > u_i(x)$  for  $z_i \in X_i$  and  $x \in X$ , then there is some neighborhood  $\mathcal{V}(x)$  of  $x$  and  $y_i \in X_i$  such that  $u_i(y_i, x'_{-i}) > u_i(x')$  for all  $x' \in \mathcal{V}(x)$ .

**DEFINITION 3.2** A game  $G = (X_i, u_i)_{i \in I}$  is said to be *weakly transfer continuous* if  $x \in X$  is not an equilibrium, then there exist player  $i$ ,  $y_i \in X_i$  and a neighborhood  $\mathcal{V}(x)$  of  $x$  such that  $u_i(y_i, x'_{-i}) > u_i(x')$  for all  $x' \in \mathcal{V}(x)$ .

Weak transfer continuity means that whenever  $x$  is not an equilibrium, some player  $i$  has a “securing” strategy  $y_i$  yielding a strictly better off payoff even if all players deviate slightly from  $x^*$ . Note that the notion of weak transfer continuity only requires some player, but not all players, who can have a “securing” strategy resulting in a strictly better off payoff even if all players deviate slightly from a non-Nash equilibrium. Games with continuous payoff functions are clearly (weakly) transfer continuous. It is also clear that a game  $G$  is weakly transfer continuous if it is transfer continuous, but the reverse may not be true. We will give such an example to show this in the next subsection.

An even weaker form of continuity is the notion of weak transfer quasi-continuity.

**DEFINITION 3.3** A game  $G = (X_i, u_i)_{i \in I}$  is said to be *weakly transfer quasi-continuous* if  $x \in X$  is not an equilibrium, then there exist a strategy profile  $y \in X$  and a neighborhood  $\mathcal{V}(x)$  of  $x$  so that for every  $x' \in \mathcal{V}(x)$ , there exists a player  $i$  such as  $u_i(y_i, x'_{-i}) > u_i(x')$ .

Roughly speaking, weak transfer quasi-continuity means whenever a strategy profile is not an equilibrium, there exist some of its neighborhood and a securing strategy profile such that for every strategy profile in the neighborhood, some player will be strictly better off by using his securing strategy. Thus, weak transfer quasi-continuity implies that if  $x$  is not nonequilibrium, some of its neighborhood  $\mathcal{V}(x)$  of  $x$  does not contain any equilibrium. This is a very weak form of continuity that covers almost all the known weak forms of continuity, and here we have the following proposition.

**PROPOSITION 3.1** *If a game  $G = (X_i, u_i)_{i \in I}$  is weakly transfer continuous, diagonal transfer continuous or better reply secure, then it is weakly transfer quasi-continuous.*

Now we are ready to state our first main result, which strictly generalizes Theorem 3.1 of Reny [1999] by weakening better-reply security.

**THEOREM 3.1** *Suppose  $G = (X_i, u_i)_{i \in I}$  is convex, compact, bounded, quasiconcave, and weakly transfer quasi-continuous. Then  $G$  possesses a pure strategy Nash equilibrium.*

By Proposition 3.1, Theorem 3.1 also extends Theorem 1 of Baye *et al.* [1993] by replacing diagonal transfer quasiconcavity with the conventional quasiconcavity that is much easier to verify. We then have the following corollary.

**COROLLARY 3.1** *Suppose  $G = (X_i, u_i)_{i \in I}$  is convex, compact, bounded, quasiconcave, and the function  $U(x, y)$  defined in (2.2) is diagonal transfer continuous in  $x$ . Then  $G$  possesses a pure strategy Nash equilibrium.*

**EXAMPLE 3.1** (Tullock: Baye *et al.* [1993]): Consider an  $n$ -person game played on the unit square  $X_i = [0, 1]$  and the payoffs:

$$u_i(p_i, p_{-i}) = \begin{cases} 1/n, & \text{if } p_h = 0, h = 1, \dots, n \\ \frac{p_i^\alpha}{\sum_{j=1, \dots, n} p_j^\alpha} - p_i, & \text{otherwise} \end{cases}$$

with  $\alpha \in (0, 1)$ .

This game is convex, compact, quasiconcave and the aggregate function  $U(x, y)$  is diagonally transfer continuous in  $y$ . Then, by Corollary 3.1, the game has a pure strategy Nash equilibrium.

To relax the boundedness and/or quasiconcavity conditions in Theorem 3.1 above and the result of Reny [1999], we now introduce the notion of strong diagonal transfer quasiconcavity.



**DEFINITION 3.4** A game  $G = (X_i, u_i)_{i \in I}$  is said to be *strongly diagonal transfer quasiconcave* if for any finite subset  $\{y^1, \dots, y^m\} \subset X$ , there exists a corresponding finite subset  $\{x^1, \dots, x^m\} \subset X$  such that for any subset  $\{x^{k^1}, x^{k^2}, \dots, x^{k^s}\} \subset X^m$ ,  $1 \leq s \leq m$ , and any  $x \in \text{co}\{x^{k^1}, x^{k^2}, \dots, x^{k^s}\}$ , there exists  $y^h \in \{y^{k^1}, \dots, y^{k^s}\}$  so that

$$u_i(y_i^{k^h}, x_{-i}) \leq u_i(x) \quad \forall i \in I. \quad (3.1)$$

Strong diagonal transfer quasiconcavity roughly says that given any finite subset  $Y^m = \{y^1, \dots, y^m\}$  of deviation profiles, there exists a corresponding finite subset  $X^m = \{x^1, \dots, x^m\}$  of candidate profiles such that for any subset  $\{x^{k^1}, x^{k^2}, \dots, x^{k^s}\} \subset X^m$ ,  $1 \leq s \leq m$ , its convex combinations are not upset by all deviation profiles in  $X^{\tilde{m}}$  for all players simultaneously. In the definition, *diagonal* refers to the fact that the convex combination is taken along the diagonal (i.e., it is taken not only for player  $i$  but also for all other players). It is clear that a game is diagonally transfer quasiconcave if it is strongly diagonal transfer quasiconcave. Indeed, summing up (3.1), we have  $\min_{1 \leq l \leq s} U(x, y^{k^l}) \leq U(x, x)$ .

**REMARK 3.1** Let the correspondence  $F : X \rightarrow 2^X$  be defined by  $F(y) = \{x \in X : u_i(y_i, x_{-i}) \leq u_i(x), \forall i \in I\}$ . Then it is transfer FS-convex if and only if the game is strongly diagonal transfer quasiconcave.<sup>2</sup>

We then have our second main theorem, which extends Theorem 3.1 above and Theorem 3.1 in Reny [1999] by relaxing the boundedness and quasiconcavity conditions.

**THEOREM 3.2** Suppose  $G = (X_i, u_i)_{i \in I}$  is convex, compact and weakly transfer quasiconcave. Then, the game  $G$  possesses a Nash equilibrium if and only if it is strongly diagonal transfer quasiconcave.

With strong diagonal transfer quasiconcavity, the proof of Theorem 3.2 is much simpler than the proof of Theorem 3.1. Strong diagonal transfer quasiconcavity can be further weakened if one is willing to impose the boundedness of payoffs. Indeed, we can do so by introducing weak diagonal transfer quasiconcavity.

Let  $m \in \mathbb{N}^*$ <sup>3</sup> and let the following special simplex:<sup>4</sup>

$$\Delta(n, m) = \{\lambda = (\lambda_{i,j})_{\substack{i=1, \dots, n \\ j=1, \dots, m}} \in \mathcal{M}_{\mathbb{R}}(n, m) : \lambda_{i,j} \geq 0 \text{ and } \sum_{i,j} \lambda_{i,j} = 1\}.$$

<sup>2</sup>A correspondence  $H : X \rightarrow 2^X$  is transfer FS-convex if for any finite subset  $\{y^1, \dots, y^m\} \subset X$ , there exists a corresponding finite subset  $\{x^1, \dots, x^m\} \subset X$  such that for each  $J \subset \{1, \dots, m\}$ , we have  $\text{co}\{x^j, j \in J\} \subset \bigcup_{j \in J} H(y^j)$ .

<sup>3</sup> $\mathbb{N}^*$  is the set of strictly positive integer numbers.

<sup>4</sup> $\mathcal{M}_{\mathbb{R}}(n, m)$  is the matrix space with  $n$  lines,  $m$  columns and scalars in  $\mathbb{R}$

**DEFINITION 3.5** A game  $G = (X_i, u_i)_{i \in I}$  is said to be *weakly diagonal transfer quasiconcave* if for any finite subset  $\{y^1, \dots, y^m\} \subset X$ , there exists a corresponding finite subset  $\{x^1, \dots, x^m\} \subset X$  such that for each  $\tilde{x} = \sum_{i,j} \lambda_{i,j} x^j \in \text{co}\{x^h, h = 1, \dots, m\}$ , we have

$$\min_{(i,j) \in J} [u_i(y_i^j, \tilde{x}_{-i}) - u_i(\tilde{x})] \leq 0, \text{ with } J = \{(i, j) : \lambda_{i,j} > 0\}. \quad (3.2)$$

**REMARK 3.2** The Definition 3.5 is equivalent to the following definition: A game  $G = (X_i, u_i)_{i \in I}$  is weakly diagonal transfer quasiconcave if and only if for any finite subset  $\{y^1, \dots, y^m\} \subset X$ , there exists a corresponding finite subset  $\{x^1, \dots, x^m\} \subset X$  such that for each  $\lambda \in \Delta(n, m)$ , there exists a player  $i \in I$  such that

$$\min_{j \in J(i)} u_i(y_i^j, \tilde{x}_{-i}) \leq u_i(\tilde{x}),$$

where  $J(i) = \{j = 1, \dots, m : \lambda_{i,j} > 0\}$  and  $\tilde{x} = \sum_{i,j} \lambda_{i,j} x^j$ .

Weak diagonal transfer quasiconcavity roughly says that given any finite subset  $Y^m = \{y^1, \dots, y^m\}$  of deviation profiles, there exists a corresponding finite subset  $X^m = \{x^1, \dots, x^m\}$  of candidate profiles such that for any subset  $\{x^{k^1}, x^{k^2}, \dots, x^{k^s}\} \subset X^m, 1 \leq s \leq m$ , there exists some player  $i$  so that its convex combinations are not upset by those deviation profiles in  $X^{\tilde{m}}$  which have nonzero weights.

Weak diagonal transfer quasiconcavity is also weaker than diagonal transfer quasiconcavity as shown in the following proposition.

**PROPOSITION 3.2** *If the aggregate function defined by (2.2) is diagonally transfer quasiconcave, then the game  $G$  is weakly diagonally transfer quasiconcave.*

Now we have our third main result on the existence of pure strategy Nash equilibrium, which generalizes Theorem 1 in (Baye *et al.* [1993]) by weakening both transfer diagonal continuity and transfer quasiconcavity, and extends Theorem 3.1 in Reny [1999] by weakening better-reply security and relaxing quasiconcavity condition.

**THEOREM 3.3** *Suppose  $G = (X_i, u_i)_{i \in I}$  is convex, compact, bounded and weakly transfer quasi-continuous, then  $G$  possesses a pure strategy Nash equilibrium if and only if it is weakly diagonal transfer quasiconcave.*

By Theorem 3.3, we have the following corollaries.

**COROLLARY 3.2** *Suppose  $G = (X_i, u_i)_{i \in I}$  is compact, bounded, and diagonally transfer continuous. Then, the game  $G$  possesses a Nash equilibrium if and only if it is weakly diagonal transfer quasiconcave.*

**COROLLARY 3.3** *Suppose that  $G = (X_i, u_i)_{i \in I}$  is compact, bounded, and better-reply secure. Then, the game  $G$  possesses a Nash equilibrium if and only if it is weakly diagonal transfer quasiconcave.*

**REMARK 3.3** Strong diagonal transfer quasiconcavity, diagonal transfer quasiconcavity as well as weak diagonal transfer quasiconcavity are all very weak notions of quasiconcavity, and in fact, one can see from the proof of necessity of Theorems 3.2 and 3.3 that a game must be (strongly/weakly) diagonal transfer quasiconcave as long as it possesses a pure strategy Nash equilibrium.<sup>5</sup>

### 3.2 Discussion and Examples

Various weak notions of continuity having appeared in our results, such as transfer continuity, weak transfer continuity, weak transfer quasi-continuity, diagonal transfer continuity, better-reply security, etc., are quite weak, which hold in a large class of discontinuous games. In this subsection we illustrate the relationships of these weak notions of continuity and show the usefulness of our main results with examples.

It is clear that a game  $G$  is weakly transfer continuous if it is transfer continuous. However, the following example shows the reverse may not be true.

**EXAMPLE 3.2** Consider a two-player game with  $X_1 = X_2 = [0, 1]$  and

$$u_1(x_1, x_2) = \begin{cases} 2 + x_1 + x_2, & \text{if } x_1 = x_2, \\ x_1 + x_2, & \text{otherwise,} \end{cases} \quad \text{and } u_2(x) = x_1 + x_2.$$

To show it is not transfer continuous, consider the nonequilibrium  $x = (1, 0)$ . Then, for any  $y_1 \in [0, 1]$  and any neighborhood  $\mathcal{V}(x) \subset X$  of  $x$ , choosing  $z \in \mathcal{V}(x)$  with  $z_1 = 1$  and  $1 \neq z_2 \neq y_1$ , we then have  $u_1(y_1, z_2) = y_1 + z_2 \leq 1 + z_2 = u_1(1, z_2)$ . Thus,  $u_1$  is not transfer continuous.

However, this game is weakly transfer continuous. Indeed, since the unique Nash equilibrium is given by  $x_1 = x_2 = 1$ , any nonequilibrium strategy profile  $(x_1, x_2)$  contains a component that is not equal to one.

If  $x_2 < 1$ , let  $y_2 = 1$ . Then, for any neighborhood  $\mathcal{V}(x)$  of  $x$  where  $\mathcal{V}(x) \subset [0, 1] \times [0, 1]$  such that for all  $z \in \mathcal{V}(x)$ , we have  $u_2(z_1, y_2) = 1 + z_1 > u_2(z_1, z_2) = z_1 + z_2$ .

If  $x_2 = 1$ , then  $x_1 < 1$ . Letting  $y_1 = 1$ , then for any neighborhood  $\mathcal{V}(x)$  of  $x$  such that  $\mathcal{V}(x) \subset [0, 1] \times [0, 1]$  and for all  $z \in \mathcal{V}(x)$ ,  $z_1 < z_2$ , we have  $u_1(y_1, z_2) = 3 + z_2$  if  $z_2 = 1$  and  $u_1(y_1, z_2) = 1 + z_2$  otherwise. Thus  $u_1(y_1, z_2) > z_1 + z_2 = u_1(z_1, z_2)$  for all  $z_2$ .

Hence, the game is weakly transfer continuous.

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<sup>5</sup>Strong diagonal transfer quasiconcavity, diagonal transfer quasiconcavity, and weak diagonal transfer quasiconcavity all become the same for one-player games.

Also, weak transfer quasi-continuity is strictly weaker than weak transfer continuity and better-reply security. To see this, consider the following example.

**EXAMPLE 3.3** Consider the two-player game with the following payoff functions defined on  $X = [0, 1] \times [0, 1]$ .

$$u_1(x_1, x_2) = \begin{cases} 0 & \text{if } x_1 \in (0, 1) \\ 1 & \text{if } x_1 = 0 \text{ and } x_2 \in \mathbb{Q} \\ 1 & \text{if } x_1 = 1 \text{ and } x_2 \notin \mathbb{Q} \\ 0 & \text{otherwise.} \end{cases}, \text{ and } u_2(x_1, x_2) = x_1 - x_2$$

where  $\mathbb{Q} = \{x \in [0, 1] : x \text{ is a rational number}\}$ .

The payoff function of player 1 is taken from Barelli and Soza [2009]. This game is neither weakly transfer continuous, better-reply secure, nor diagonally transfer continuous, but is weakly transfer quasi-continuous.

To show the game is not weakly transfer continuous, consider the nonequilibrium  $x = (1, 0)$ . Then, for any  $y_1 \in [0, 1]$  and any neighborhood  $\mathcal{V}(x) \subset X$  of  $x$ , choosing  $z \in \mathcal{V}(x)$  with  $z_1 = 1$  and  $z_2 \notin \mathbb{Q}$ , we have  $u_1(y_1, z_2) \leq u_1(z_1, z_2) = 1$ . Also, for any  $y_2 \in [0, 1]$  and any neighborhood  $\mathcal{V}(x) \subset X$  of  $x$ , choosing  $z \in \mathcal{V}(x)$  with  $z_2 = 0$ , we have  $u_2(z_1, y_2) = z_1 - y_2 \leq z_1 = u_2(z_1, z_2)$ . So it is not weakly transfer continuous.

To show the game is not better-reply secure either, consider  $x = (1, 0)$  and  $u = (0, 1)$ . Clearly  $(x, u)$  is in the closure of the graph of its vector function, and  $x$  is not a Nash equilibrium. We show that player 1 cannot obtain a payoff strictly above  $u_1 = 0$ . Indeed, for all  $y_1 \in [0, 1]$  and any neighborhood  $\mathcal{V}(x_2) \subset [0, 1]$  of  $x_2$ , choosing  $z_2 \in \mathcal{V}(x_2) \setminus \mathbb{Q}$  if  $y_1 = 0$ , or  $z_2 = 0$  otherwise, we then have  $u_1(y_1, z_2) = 0 \leq u_1 = 0$ . Player 2 cannot obtain a payoff strictly above  $u_2 = 1$  either. To see this, for all  $y_2 \in [0, 1]$  and any neighborhood  $\mathcal{V}(x_1) \subset [0, 1]$  of  $x_1$ , we have  $u_2(z_1, y_2) = z_1 - y_2 \leq z_1 \leq 1 = u_2$  for  $z_1 \in \mathcal{V}(x_1)$ . Thus, this game is not better-reply secure, so Theorem 3.1 of Reny [1999] can not be applied.

Now we show the game is not diagonally transfer continuous. Let  $x = (1, 0)$  and  $y = (0, 0)$ . Then,  $U(x, y) = u_1(y_1, x_2) + u_2(x_1, y_2) = u_1(0, 0) + u_2(1, 0) = 2 > U(x, x) = u_1(1, 0) + u_2(1, 0) = 1$ . However, for all  $y' \in [0, 1] \times [0, 1]$  and any neighborhood  $\mathcal{V}(x) \subset X$  of  $x$ , choosing  $z \in \mathcal{V}(x)$  with  $z_1 = 1$  and  $z_2 \notin \mathbb{Q}$  if  $y'_1 < 1$ , or  $z_2 = 0$  otherwise, we then have: (1) If  $y'_1 < 1$ , then  $z_2 \notin \mathbb{Q}$ . Thus,  $u_1(y'_1, z_2) = 0$ ,  $u_1(z_1, z_2) = 1$ ,  $u_2(z_1, y'_2) = 1 - y'_2$  and  $u_2(z_1, z_2) = 1 - z_2$ . Therefore  $U(z, y') = 1 - y'_2 \leq 2 - z_2 = U(z, z)$ . (2) If  $y'_1 = 1$ , then  $z_2 = 0$ . Thus,  $u_1(y'_1, z_2) = 0$ ,  $u_1(z_1, z_2) \leq 1$ ,  $u_2(z_1, y'_2) = 1 - y'_2$  and  $u_2(z_1, z_2) = 1$ . Therefore  $U(z, y') = 1 - y'_2 \leq 1 + u_1(z_1, z_2) = U(z, z)$ . Thus, this game is not diagonally transfer continuous, so Theorem 1 of Baye *et al.* [1993] can not be applied.

However, it is weakly transfer quasi-continuous. Indeed, let  $(x_1, x_2)$  be a nonequilibrium strategy profile with at least one non-zero coordinate. There are two cases to be considered.

1.  $x_2 > 0$ . Letting  $y = (y_1, 0)$  and taking a neighborhood  $\mathcal{V}(x) \subset [0, 1] \times (0, 1]$  of  $x$ , then for each  $z \in \mathcal{V}(x)$  and player  $i = 2$ , we have  $u_2(z_1, z_2) = z_1 - z_2 < z_1 = u_2(z_1, y_2)$ .
2.  $x_2 = 0$  and  $x_1 > 0$ . Letting  $y = (0, 0)$  and taking a neighborhood  $\mathcal{V}(x) \subset (0, 1] \times [0, 1]$  of  $x$ , then for each  $z \in \mathcal{V}(x)$ , we have  $u_1(z_1, z_2) = 0 < 1 = u_1(y_1, z_2)$  for player 1 when  $z_2 \in \mathbb{Q}$  and  $u_2(z_1, z_2) = z_1 - z_2 < z_1 = u_2(z_1, y_2)$  for player 2 when  $z_2 \notin \mathbb{Q}$ .

Since the game is also convex, compact, bounded and quasiconcave, by Theorem 3.1, the game considered possesses a Nash equilibrium.

Although diagonal transfer continuity, better-reply security, weak transfer (quasi-)continuity are all transfer types of continuities that are satisfied by many discontinuous economic games, a main difference among them is that, while better-reply security takes an open neighborhood of strategy profiles only for opponents' strategies rather than those of deviation player  $i$ , diagonal transfer continuity and the weak transfer (quasi-)continuity take open neighborhoods of the strategy profile  $x$  for all players to the aggregate payoff function  $U$  and individual payoffs  $u_i$ , respectively.

Also, although weak transfer quasi-continuity is implied by better-reply security in Reny [1999] or diagonal transfer continuity in Baye *et al.* [1993], weak transfer continuity neither implies nor is implied by better-reply security in Reny [1999] or diagonal transfer continuity in Baye *et al.* [1993]. The following examples can show this.

**EXAMPLE 3.4** Consider the two-player game with the following payoff functions defined on  $[0, 1] \times [0, 1]$  studied by Carmona [2008].

$$u_i(x_1, x_2) = \begin{cases} \varphi_i(x_1, x_2), & \text{if } x_1 = x_2, \\ \psi_i(x_1, x_2), & \text{otherwise,} \end{cases}$$

where  $\varphi_i, \psi_i : [0, 1]^2 \rightarrow \mathbb{R}$  are continuous functions. In addition, assume that  $G$  is bounded and quasiconcave and satisfies the following conditions:

- (i) For each  $i \in I$ ,  $\epsilon > 0$  and  $y \in [0, 1]$ , there exist  $\bar{x} \in [0, 1]$  and a neighborhood  $\mathcal{V}(y) \subset [0, 1]$  of  $y$  with  $\bar{x} \notin \mathcal{V}(y)$  such that  $\psi_i(\bar{x}, z) \geq \varphi_i(y, y) - \epsilon$  for each  $z \in \mathcal{V}(y)$ .
- (ii) If for each  $x, y \in [0, 1]$  with  $x \neq y$  and for some  $i$ ,  $\psi_i(x, y) > \varphi_i(y, y)$ , then there exist a player  $j$ ,  $\bar{x} \neq y \in [0, 1]$  and a neighborhood  $\mathcal{V}(y, y) \subset [0, 1]^2$  of  $(y, y)$  such that  $\psi_j(\bar{x}, y) > u_j(z)$  for each  $z \in \mathcal{V}(y, y)$ .

Carmona [2008] shows that the functions  $\varphi_i$  and  $\psi_i$  can be chosen so as to violate diagonal transfer continuity and/or better-reply security.

However, under conditions (i)-(ii), we can show that it is weakly transfer continuous so that it has a Nash equilibrium by Theorem 3.1. Indeed, suppose  $x$  is not a Nash equilibrium. Then there exist a player  $i$  and a strategy  $y_i \in [0, 1]$  such that  $u_i(y_i, x_{-i}) > u_i(x)$ .

1. If  $x_1 = x_2 = x$ , then  $\psi_i(y_i, x) > \varphi_i(x, x)$ . By condition (ii), there exist a player  $j$ ,  $\bar{x}_j \neq x \in [0, 1]$ , and a neighborhood  $\mathcal{V}(x_1, x_1) \subset [0, 1]^2$  of  $(x_1, x_2)$  such that  $\psi_j(\bar{x}_j, x) > u_j(z)$  for each  $z \in \mathcal{V}(x_1, y_2)$ . Let  $\epsilon > 0$  such that  $\psi_j(\bar{x}_j, x) - \epsilon > \sup u_j(z)$ . Since  $\bar{x}_j \neq x$  and the function  $\psi_j(\bar{x}_j, \cdot)$  is continuous, then there exists a neighborhood  $\mathcal{V}(x) \subset [0, 1]$  such that  $\bar{x}_j \notin \mathcal{V}(x)$  and  $\psi_j(\bar{x}_j, x) - \epsilon \leq \psi_j(\bar{x}_j, z_{-j})$  for all  $z_{-j} \in \mathcal{V}(x)$ . Thus, there exist a player  $j$ , a neighborhood  $\mathcal{V}(x_1, x_1) \subset [0, 1]^2$ , and a strategy  $\bar{x}_j \in [0, 1]$  with  $\bar{x}_j \neq z_{-j}$  such that  $u_j(\bar{x}_j, z_{-j}) > u_j(z)$  for all  $z \in \mathcal{V}(x_1, x_1)$ .
2. If  $x_1 \neq x_2$ , then  $u_i(y_i, x_{-i}) - \epsilon > \psi_i(x_1, x_2)$  for some  $\epsilon > 0$ . If  $y_i \neq x_{-i}$ , then by continuity of  $\psi_i$ , there exists a  $\mathcal{V}(x_1, x_2)$  such that for all  $z \in \mathcal{V}(x_1, x_1)$ ,  $z_1 \neq z_2$  and  $u_i(y_i, z_{-i}) > u_i(z)$ .

If  $y_i = x_{-i}$ , then  $\varphi_i(y_i, y_i) - \epsilon > \psi_i(x_1, x_2)$ . By condition (i), there exist  $\bar{x}_i \in [0, 1]$  and a neighborhood  $\mathcal{V}(y_i) \subset [0, 1]$  of  $y_i$  with  $\bar{x}_i \notin \mathcal{V}(y_i)$  such that  $\psi_i(\bar{x}_i, z) \geq \varphi_i(y_i, y_i) - \frac{\epsilon}{2}$  for each  $z \in \mathcal{V}(y_i)$ . Since the function  $\psi_j(\cdot, \cdot)$  is continuous, then there exists a neighborhood  $\mathcal{V}(x_1, x_2) \subset [0, 1]^2$  such that for all  $z \in \mathcal{V}(x_1, x_2)$ ,  $z_1 \neq z_2$  and  $\psi_i(x_1, x_2) + \frac{\epsilon}{2} \geq \psi_i(z_j, z_{-j})$ . Thus, for each  $z \in \mathcal{V}(x_1, x_2)$ , we have  $\psi_i(z_j, z_{-j}) \leq \psi_i(x_1, x_2) + \frac{\epsilon}{2} < \varphi_i(y_i, y_i) - \frac{\epsilon}{2} \leq \psi_i(\bar{x}_i, z_{-i}) = u_i(\bar{x}_i, z_{-i})$ , i.e.  $u_i(\bar{x}_i, z_{-i}) > u_i(z)$ .

### 3.3 Sufficient Conditions for Weak Transfer (Quasi-)Continuity

In this subsection we provide some new sufficient conditions for weak transfer (quasi-)continuity. While it is simple to verify weak transfer continuity, it is sometimes even simpler to verify other conditions leading to it and consequently weak transfer quasi-continuity. In addition to the fact that diagonal transfer continuity, better-reply security, transfer continuity, and weak transfer continuity all imply weak transfer quasi-continuity, weak transfer upper continuity and weak transfer lower continuity introduced below, when they are combined respectively with payoff security and upper semicontinuity, they also imply weak transfer continuity, and consequently weak transfer quasi-continuity.

**DEFINITION 3.6** A game  $G = (X_i, u_i)_{i \in I}$  is said to be *weakly transfer upper continuous* if  $x \in X$  is not an equilibrium, then there exist player  $i$ ,  $\hat{x}_i \in X_i$  and a neighborhood  $\mathcal{V}(x)$  of  $x$  such that  $u_i(\hat{x}_i, x_{-i}) > u_i(x')$  for all  $x' \in \mathcal{V}(x)$ .

**REMARK 3.4** If a game  $G$  is upper semicontinuous, then  $G$  is weakly transfer upper continuous. Indeed, suppose  $x$  is not a Nash equilibrium, then there exist a player  $i$  and a strategy  $y_i$  such that  $u_i(y_i, x_{-i}) > u_i(x)$ . Choose  $\epsilon > 0$  such that  $u_i(y_i, x_{-i}) - \epsilon > u_i(x)$ . Since  $G$  is upper

semicontinuous, then there exists a neighborhood  $\mathcal{V}(x)$  of  $x$  such that  $u_i(y_i, x_{-i}) - \epsilon > u_i(x) \geq u_i(x') - \epsilon$ , for each  $x' \in \mathcal{V}(x)$ .

**DEFINITION 3.7** A game  $G = (X_i, u_i)_{i \in I}$  is said to be *weakly transfer lower continuous* if  $x$  is not a Nash equilibrium, which implies that there exist a player  $i$ ,  $y_i \in X_i$ , and a neighborhood of  $\mathcal{V}(x_{-i})$  of  $x_{-i}$  such that  $u_i(y_i, x'_{-i}) > u_i(x)$  for all  $x'_{-i} \in \mathcal{V}(x_{-i})$ .

**REMARK 3.5** If a game  $G$  is payoff secure, then  $G$  is weakly transfer lower continuous. To see this, suppose  $x \in X$  and  $x$  is not a Nash equilibrium, then there exists a player  $i$  that has a strategy  $\hat{x}_i$  such that  $u_i(\hat{x}_i, x_{-i}) > u_i(x)$ . Choose  $\epsilon > 0$  such that  $u_i(\hat{x}_i, x_{-i}) - \epsilon > u_i(x)$ . Since  $G$  is payoff secure, then there exist a strategy  $y_i$  and a neighborhood  $\mathcal{V}(x_{-i})$  of  $x_{-i}$  such that  $u_i(y_i, x'_{-i}) \geq u_i(\hat{x}_i, x_{-i}) - \epsilon > u_i(x)$ , for each  $x'_{-i} \in \mathcal{V}(x_{-i})$ .

We then have the following propositions that provide sufficient conditions for weak transfer (quasi-)continuity.

**PROPOSITION 3.3** *If a game  $G = (X_i, u_i)_{i \in I}$  is weakly transfer upper continuous and payoff secure, then it is weakly transfer continuous.*

**PROPOSITION 3.4** *If a game  $G = (X_i, u_i)_{i \in I}$  is weakly transfer lower continuous and upper semicontinuous, then it is weakly transfer continuous.*

Propositions 3.3-3.4, together with Theorem 3.1 or Theorem 3.3, immediately yield the following useful results.

**COROLLARY 3.4** *If a game  $G = (X_i, u_i)_{i \in I}$  is convex, compact, bounded, weakly transfer upper continuous, payoff secure, and quasiconcave or weakly diagonal transfer quasiconcave, then  $G$  possesses a pure strategy Nash equilibrium.*

**COROLLARY 3.5** *If a game  $G = (X_i, u_i)_{i \in I}$  is convex, compact, bounded, weakly transfer lower continuous, upper semicontinuous, and quasiconcave or weakly diagonal transfer quasiconcave, then  $G$  possesses a pure strategy Nash equilibrium.*

As an application of the above proposition, consider the following well-known noisy game.

**EXAMPLE 3.5** Consider the two-player, nonzero sum, noisy games with the following payoff functions defined from  $[0, 1] \times [0, 1]$ .

$$f_i(x_i, x_{-i}) = \begin{cases} l_i(x_i), & \text{if } x_i < x_{-i}, \\ \phi_i(x_i), & \text{if } x_i = x_{-i}, \\ m_i(x_{-i}), & \text{if } x_i > x_{-i}, \end{cases}$$

where  $l_i(\cdot)$ ,  $m_i(\cdot)$  and  $\phi_i(\cdot)$  are upper semicontinuous over  $[0, 1]$ ,  $l_i(\cdot)$  is strictly nondecreasing on  $[0, 1]$  and satisfies the following additional conditions:

- a)  $\forall x \in [0, 1], \forall \epsilon > 0$ , there exists a neighborhood  $\mathcal{V}(x)$  of  $x$  such that  $\phi_i(x) \geq \max(l_i(z), m_i(z)) - \epsilon$ , for every  $z \in \mathcal{V}(x)$ .
- b) if  $m_i(x) > \phi_i(x)$  with  $x < 1$ , then there exists a neighborhood  $\mathcal{V}(x) \subset [0, 1)$  of  $x$  such that  $m_i(z) > \phi_i(x)$ , for every  $z \in \mathcal{V}(x)$ .
- c) if  $\phi_i(x) > m_i(x)$  with  $x < 1$ , then there exists a neighborhood  $\mathcal{V}(x) \subset [0, 1)$  of  $x$  such that  $\phi_i(z) > m_i(x)$ , for every  $z \in \mathcal{V}(x)$ .

It is clear that this game  $G$  is compact and convex. Suppose that  $G$  is quasiconcave. The condition a) and the upper semicontinuity of  $l_i(\cdot)$ ,  $m_i(\cdot)$  and  $\phi_i(\cdot)$  over  $[0, 1]$ , imply that the noisy game is upper semicontinuous. The conditions b) and c) imply that the game is weakly transfer lower continuous. Then, the game considered is weakly transfer continuous by Proposition 3.4.<sup>6</sup>, and thus it has a Nash equilibrium by Theorem 3.1.

**REMARK 3.6** All the definitions of weak transfer quasi-continuity, weak transfer continuity, weak transfer upper continuity, weakly transfer lower continuity and upper semicontinuity can be easily extended to the quasi-symmetric game and to get the existence results on symmetric Nash equilibrium.

### 3.4 Nash Equilibria in Discontinuous and Nonconvex Games

In this subsection we characterize the existence of pure strategy Nash equilibria in games that may be discontinuous or nonconvex. We generalize the results above as well as the most known existence results, such as Baye *et al.* [1993], Reny [1999] and Bagh and Jofre [2006], without assuming the convexity of strategy spaces or any form of quasiconcavity of payoff functions.

The following theorem generalizes Theorems 3.1– 3.3 by relaxing the convexity of strategy spaces, (weak) diagonal transfer quasiconcavity and weak diagonal transfer quasiconcavity of payoff functions.

**THEOREM 3.4** *Suppose  $G = (X_i, u_i)_{i \in I}$  is compact and weakly transfer quasi-continuous. Then, the game  $G$  has a Nash equilibrium if and only if for all  $A \in \langle X \rangle$ , there exists  $x \in X$  such that for all  $i \in I$ ,  $u_i(y_i, x_{-i}) \leq u_i(x)$ ,  $\forall y \in A$ .*

Since the diagonal transfer continuity and the better-reply security imply the weak transfer quasi-continuity, then Theorem 3.4 generalizes Theorem 3.1–3.3, Theorem 1 in Baye *et al.* [1993], and Theorem 3.1 in Reny [1999] by relaxing the boundedness convexity of strategy spaces and (strong/weak) diagonal transfer quasiconcavity or quasiconcavity of payoffs, respectively.

**EXAMPLE 3.6** Consider a game with  $n = 2$ ,  $X_1 = X_2 = [1, 2] \cup [3, 4]$ , and

<sup>6</sup>As Reny [1999] showed, if  $\phi_i(x) \in \text{co}\{l_i(x), m_i(x)\}$  and  $l_i(x)$  is nondecreasing, then the game is quasiconcave.



$$\begin{aligned} u_1(x) &= x_2 x_1^2, \\ u_2(x) &= -x_1 x_2^2. \end{aligned}$$

Note that,  $X_i$  is not convex for  $i = 1, 2$ , and the function  $y_i \mapsto u_i(y_i, x_{-i})$  is not quasiconcave for  $i = 1$  so that the existing theorems on Nash equilibrium are not applicable.

However, we can show the existence of Nash equilibrium by applying Theorem 3.4. Indeed, for each  $x = (x_1, x_2)$  and  $y = (y_1, y_2)$ ,

$$U(x, y) = x_2 y_1^2 - x_1 y_2^2.$$

The function  $U$  is continuous on  $X \times X$ . For any subset  $\{(1y_1, 2y_2), \dots, (ky_1, ky_2)\}$  of  $X$ , let  $x = (x_1, x_2) \in X$  such that  $x_1 = \max_{h=1, \dots, k} iy_1$  and  $x_2 = \min_{h=1, \dots, k} iy_2$ . Then, we have

$$\begin{cases} iy_2^2 \geq x_2^2, & \forall i = 1, \dots, k, \\ iy_1^2 \leq x_1^2. \end{cases}$$

Thus,

$$\begin{cases} -x_1 iy_2^2 \leq -x_1 x_2^2, & \forall i = 1, \dots, k, \\ x_2 iy_1^2 \leq x_2 x_1^2. \end{cases}$$

Therefore,  $U(x, iy) \leq U(x, x)$ ,  $\forall i = 1, \dots, k$ . According to Theorem 3.4, this game has a Nash equilibrium.

## 3.5 Applications

In this subsection we show how our main existence results are applied to some important economic games. We provide two applications: one is in the shared resource games that is intensively studied by Rothstein [2007], and the other is in the classic Bertrand price competition games.

### 3.5.1 The Shared Resource Games

Rothstein [2007] studies a class of shared resource games with discontinuous payoffs, which includes a wide class of games such as the canonical game of fiscal competition for mobile capital. In these games, players compete for a share of a resource that is in fixed total supply, except perhaps at certain joint strategies. Each player's payoff depends on her opponents' strategies only through the effect those strategies have on the amount of the shared resource that the player obtains.

Formally, for such a game  $G = (X_i, u_i)_{i \in I}$ , each player  $i$  has a convex and compact strategy space  $X_i \subset \mathbb{R}^l$  and a payoff function  $u_i$  that associates *sharing rule* defined by  $S_i : X \rightarrow [0, \bar{s}]$  with  $\bar{s} \in (0, +\infty)$ . That is to say, each player has a payoff function  $u_i : X \rightarrow \mathbb{R}$  with the form  $u_i(x_i, x_{-i}) = F_i[x_i, S_i(x_i, x_{-i})]$  where  $F_i : X_i \times [0, \bar{s}] \rightarrow \mathbb{R}$  and  $u_i$  is bounded.<sup>7</sup>

<sup>7</sup>For more details on this model, see Rothstein [2007]

Define  $D_i \subset X$  to be the set of joint strategies at which  $S_i$  is discontinuous and let the set  $\Delta = \bigcup_{i \in I} D_i$  be then all of the joint strategies at which one or more of the sharing rules is discontinuous. The set  $X \setminus \Delta$  is all of the joint strategies at which all of the sharing rules are continuous.

Rothstein [2007] shows a shared resource game possesses a pure strategy Nash equilibrium if the following conditions are satisfied:

- (1)  $X$  is compact and convex;
- (2)  $u_i$  is continuous on  $X$  and quasiconcave in  $x_i$ ,
- (3)  $S_i$  satisfies:

- (3.i) For all  $x \in X \setminus \Delta$ ,  $\sum_{i=1}^n S_i(x) = \bar{s}$ ;
- (3.ii) There exists  $\underline{s} \in [0, \bar{s}]$  such that for all  $x \in \Delta$ ,  $\sum_{i=1}^n S_i(x) = \underline{s}$ ;
- (3.iii) For all  $i$ , all  $(x_i, x_{-i}) \in D_i$  and every neighborhood of  $x_i$ , there exists  $x'_i \in X_i$  such that  $(x'_i, x_{-i}) \in X \setminus D_i$ ;
- (3.iv) For all  $i$ , there exists a constant  $\tilde{s}$  satisfying  $\bar{s} \geq \tilde{s} > \bar{s}/n$  such that for all  $(x_i, x_{-i}) \in \Delta$  and all  $(x_i, x_{-i}) \in X \setminus D_i$ ,  $S_i(x'_i, x_{-i}) \geq s_i \geq S_i(x_i, x_{-i})$ .

- (4) For all  $i$ ,  $F_i$  satisfies:

- (4.i)  $F_i$  is continuous;
- (4.ii) For all  $x_i \in X_i$ ,  $F_i(x_i, \cdot)$  is nondecreasing in  $s_i$ ;
- (4.iii) Given any  $s_i > \bar{s}/n$ ,  $\max_{x_i \in X_i} F_i(x'_i, s_i) > \max_{x_i \in X_i} F_i(x_i, \bar{s}/n)$ .

In the following, we will give an existence result with much simpler conditions:

**Assumption 1:** For each  $i \in I$ ,  $X_i$  is convex and compact and  $u_i(\cdot, x_{-i})$  is bounded and quasiconcave for each  $x_{-i} \in X_{-i}$ .

**Assumption 2:** If  $(y_i, x_{-i}) \in D_i$  and  $F_i(y_i, S_i(y_i, x_{-i})) > F_i(x_i, S_i(x))$  for player  $i$ , then there exist some player  $j \in I$  and  $y'_j$  such that  $(y'_j, x_{-j}) \in X \setminus D_j$  and  $F_j(y'_j, S_j(y'_j, x_{-j})) > F_j(x_j, S_j(x))$ .

**Assumption 3:** If  $(y_i, x_{-i}) \in X \setminus D_i$  and  $F_i(y_i, S_i(y_i, x_{-i})) > F_i(x_i, S_i(x))$  for player  $i$ , then there exist a deviation strategy profile  $y'$  and a neighborhood  $\mathcal{V}(x)$  of  $x$  such that for each  $z \in \mathcal{V}(x)$ , there exists a player  $j \in I$  such as  $F_j(y'_j, S_j(y'_j, z_{-j})) > F_j(z_j, S_j(z))$ .

Assumption 1 is standard. A well-known sufficient condition for a composite function  $u_i = F_i[x_i, S_i(x_i, x_{-i})]$  to be quasiconcave is that  $F_i$  is quasiconcave and nondecreasing in  $s_i$ , and  $S_i$  is concave. Assumption 2 means that if  $x$  is not an equilibrium and can be improved at a

discontinuous strategy profile  $(y_i, x_{-i})$  when player  $i$  uses the deviation strategy  $y_i$ , then there exists a player  $j$  such that it must also be improved by a continuous strategy profile  $(y'_i, x_{-i})$  when player  $j$  uses the deviation strategy  $y'_i$ . Assumption 3 means that if a strategy profile  $x$  is not an equilibrium and can be improved by a continuous strategy profile  $(y_i, x_{-i})$  when player  $i$  uses a deviation strategy  $y_i$ , then there exist a securing strategy profile  $y'$  and a neighborhood of  $x$  such that all points in the neighborhood cannot be equilibria. Note that, if  $F_i$  is continuous, then Assumption 3 is satisfied.

We then have the following result.

**PROPOSITION 3.5** *Each shared resource game possesses a pure strategy Nash equilibrium if it satisfies Assumptions 1-3.*

### 3.5.2 The Bertrand Price Competition Games

Bertrand competition is a normal form game in which each of  $n \geq 2$  firms,  $i = 1, 2, \dots, n$ , simultaneously sets a price  $p_i \in P_i = [0, \bar{p}]$ . Under the assumption of profit maximization, the payoff to each firm  $i$  is

$$\pi_i(p_i, p_{-i}) = p_i D_i(p_i, p_{-i}) - C_i(D_i(p_i, p_{-i})),$$

where  $p_{-i}$  denotes the vector of prices charged by all firms other than  $i$ ,  $D_i(p_i, p_{-i})$  represents the total demand for firm  $i$ 's product at prices  $(p_i, p_{-i})$ , and  $C_i(D_i(p_i, p_{-i}))$  is firm  $i$ 's total cost of producing the output  $D_i(p_i, p_{-i})$ . A Bertrand equilibrium is a Nash equilibrium of this game.

Let  $A_i \subset X = \prod_{i \in I} P_i$  be the set of joint strategies at which  $\pi_i$  is discontinuous,  $\Delta = \bigcup_{i \in I} A_i$  be the set of all of the joint strategies at which one or more of the payoff is discontinuous, and  $X \setminus \Delta$  be the set of all joint strategies at which all of payoffs are continuous.

We make the following assumptions:

**Assumption 1:** For each  $i \in I$ ,  $\pi_i(\cdot, p_{-i})$  is quasiconcave for each  $p_{-i} \in X_{-i}$ .

**Assumption 2:** If  $(q_i, p_{-i}) \in A_i$  and  $\pi_i(q_i, p_{-i}) > \pi_i(p_i, p_{-i})$  for  $i \in I$ , then there exist a firm  $j \in I$ , and  $q'_j$  such that  $(q'_j, p_{-j}) \in X \setminus A_j$  and  $\pi_j(q'_j, p_{-j}) > \pi_j(p_j, p_{-j})$ .

We then have the following result.

**PROPOSITION 3.6** *Each Bertrand price competition game has a pure strategy Nash equilibrium if it satisfies (A1)-(A2).*

**PROOF.** It is similar to the proof of Proposition 3.5. ■

**EXAMPLE 3.7** Consider a two-player Bertrand price competition game on the square  $[0, a] \times [0, a]$ , with  $a > 0$ . Assume that the demand function is discontinuous and is defined by

$$D_i(p_i, p_{-i}) = \begin{cases} \alpha_i f(p_i) & \text{if } p_i < p_{-i} \\ \beta_i f(p_i) & \text{if } p_i = p_{-i} \\ \gamma_i f(p_i) & \text{if } p_i > p_{-i} \end{cases}$$

where  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a continuous function,  $\alpha_i, \beta_i > 0, \gamma_i \geq 0$  and  $\alpha_i > \beta_i$ . Suppose that the total cost of production is zero for each firm. Then, the payoff of each firm  $i$  becomes

$$\pi_i(p_i, p_{-i}) = \begin{cases} \alpha_i p_i f(p_i) & \text{if } p_i < p_{-i} \\ \beta_i p_i f(p_i) & \text{if } p_i = p_{-i} \\ \gamma_i p_i f(p_i) & \text{if } p_i > p_{-i} \end{cases}$$

We show that Assumption 2 is satisfied. To see this, note that  $A_1 = A_2 = \{(p_1, p_2) : p_1 = p_2 \in [0, a]\}$ . Suppose  $(q_i, q_i) \in A_i$  and  $\pi_i(q_i, q_i) > \pi_i(p_i, q_i)$  for some  $p_i \in [0, a]$ . We then must have  $q_i = 0$ . Thus

$$\beta_i q_i f(q_i) > \pi_i(p_i, q_i), \quad (3.3)$$

and therefore  $f(q_i) > 0$ . Since the function  $xf(x)$  is continuous then for  $\epsilon = q_i f(q_i)/\theta > 0$  with  $\theta = \frac{\alpha_i}{\alpha_i - \beta_i}$ , there exists  $\delta > 0$  such as for all  $x$  with  $q_i - \delta < x < q_i + \delta$ ,  $q_i f(q_i) - \epsilon \leq xf(x) \leq q_i f(q_i) + \epsilon$ . Thus, there exists  $q'_i \in [0, a]$  such that

$$0 < q'_i < q_i \text{ and } \alpha_i q'_i f(q'_i) \geq \beta_i q_i f(q_i). \quad (3.4)$$

(3.3) and (3.4) imply that there exists  $q'_i \in [0, a]$  such that  $(q'_i, p_{-i}) \in X \setminus A_i$  and  $\pi_i(q'_i, p_{-i}) > \pi_i(p_i, p_{-i})$ . Then, by Proposition 3.6, the game has a pure strategy Nash equilibrium, if it is quasiconcave.

## 4 Existence of Dominant Strategy Equilibria

In this section we investigate the existence of dominant strategy equilibria in discontinuous and/or nonconvex games.

### 4.1 Dominant Strategy Equilibria in Discontinuous Games

We start by reviewing some of the basic definitions and results introduced and obtained in Baye *et al.* [1993].

**DEFINITION 4.1** A game  $G = (X_i, u_i)_{i \in I}$  is transfer upper semicontinuous if for each  $i \in I$ ,  $x_i \in X_i$  and  $y \in X$ ,  $u_i(y) > u_i(x_i, y_{-i})$  implies that there exist a point  $y' \in X$  and a neighborhood  $\mathcal{V}(x_i)$  of  $x_i$  such that  $u_i(y') > u_i(x'_i, y'_{-i})$ , for all  $x'_i \in \mathcal{V}(x_i)$ .

**DEFINITION 4.2** A game  $G = (X_i, u_i)_{i \in I}$  is uniformly transfer quasiconcave on  $X$  if, for each  $i \in I$  and any finite subset  $Y^m = \{y^1, \dots, y^m\} \subset X$ , there exists a corresponding finite subset  $\{x_i^1, \dots, x_i^m\} \subset X_i$  such that for any subset  $\{y_i^{k^1}, y_i^{k^2}, \dots, y_i^{k^s}\}$ ,  $1 \leq s \leq m$ , and any  $x_i \in \text{co}\{x_i^{k^1}, x_i^{k^2}, \dots, x_i^{k^s}\}$ , we have  $\min_{1 \leq l \leq s} \{u_i(y^{k^l}) - u_i(x_i, y_{-i}^{k^l})\} \leq 0$ .

Baye *et al.* [1993] showed that a game  $G = (X_i, u_i)_{i \in I}$  that is convex, compact and transfer upper continuous must possess a dominant strategy equilibrium if and only if it is uniformly transfer quasiconcave.

In the following, we provide a new result on the existence of dominant strategy equilibria in discontinuous games. We start by introducing the notion of weak dominant transfer upper continuity.

**DEFINITION 4.3** A game  $G = (X_i, u_i)_{i \in I}$  is said to be *weakly dominant transfer upper continuous* if  $\bar{x} \in X$  is not a dominant strategy equilibrium, then there exist a player  $i$ , a strategy  $y \in X$  and a neighborhood  $\mathcal{V}(\bar{x}_i)$  of  $\bar{x}_i$  such that  $u_i(y) > u_i(z_i, y_{-i})$ , for each  $z_i \in \mathcal{V}(\bar{x}_i)$ .

A game is weakly dominant transfer upper continuous if for every non dominant strategy equilibrium  $x^*$ , some player  $i$  has a strategy  $y_i$  which dominates all other strategy  $z_i$  in a neighborhood of  $x_i^*$  when other players play  $y_{-i}$ .

An even weaker form of dominant transfer continuity is the following.

**DEFINITION 4.4** A game  $G = (X_i, u_i)_{i \in I}$  is said to be *weakly dominant transfer upper quasi-continuous* if  $\bar{x}$  is not a dominant strategy equilibrium, then there exist a strategy  $y \in X$  and a neighborhood  $\mathcal{V}(\bar{x})$  of  $\bar{x}$  so that for each  $z \in \mathcal{V}(\bar{x})$  there exists a player  $i \in I$  such as  $u_i(y) > u_i(z_i, y_{-i})$ .

A game is weakly dominant transfer upper quasi-continuous if for every nondominant strategy equilibrium  $x$ , there is a neighborhood  $\mathcal{V}(x)$  of  $x$  that does not contain a dominant strategy equilibrium.

**REMARK 4.1** It is clear that if the game  $G$  is weakly dominant transfer upper continuous or transfer upper semicontinuous (See Definition 4.1), then it is weakly dominant transfer upper quasi-continuous.

**DEFINITION 4.5** A game  $G = (X_i, u_i)_{i \in I}$  is said to be *strongly uniformly transfer quasiconcave* if for any finite subset  $\{y^1, \dots, y^m\} \subset X$ , there exists a corresponding finite subset  $\{x^1, \dots, x^m\} \subset X$  such that for any subset  $\{x^{k^1}, x^{k^2}, \dots, x^{k^s}\} \subset X^m$ ,  $1 \leq s \leq m$ , and any  $x \in \text{co}\{x^{k^1}, x^{k^2}, \dots, x^{k^s}\}$ , there exists  $y^h \in \{y^{k^1}, \dots, y^{k^s}\}$  so that

$$u_i(y^{k^h}) \leq u_i(x_i, y_{-i}^{k^h}) \quad \forall i \in I. \quad (4.1)$$

Strong uniform transfer quasiconcavity roughly says that given any finite subset  $Y^m = \{y^1, \dots, y^m\}$  of deviation profiles, there exists a corresponding finite subset  $X^m = \{x^1, \dots, x^m\}$  of candidate profiles such that for any subset  $\{x^{k^1}, x^{k^2}, \dots, x^{k^s}\} \subset X^m$ ,  $1 \leq s \leq m$ , its convex combinations are not dominated simultaneously by all deviations in  $X^m$  for all players. We will see from Theorem 4.1 below that strong uniform transfer quasiconcavity is necessary for the existence of a dominant strategy equilibrium of a game when it is weakly dominant transfer upper quasi-continuous. It is clear that a game is uniformly transfer quasiconcave if it is strongly uniformly transfer quasiconcave. Indeed, by (4.1), we have  $\min_{1 \leq l \leq s} \{u_i(y^{k^l}) - u_i(x_i, y_{-i}^{k^l})\} \leq 0$ .

**REMARK 4.2** Let the correspondence  $F : X \rightarrow 2^X$  be defined by  $F(y) = \{x \in X : u_i(y) \leq u_i(x_i, y_{-i}), \forall i \in I\}$ . Then it is transfer FS-convex if and only if the game is strongly uniformly transfer quasiconcave.

The following theorem characterizes the existence of dominant strategy equilibrium if the game is weakly dominant transfer upper quasi-continuous and the strategy spaces of players are convex.

**THEOREM 4.1** Suppose  $G = (X_i, u_i)_{i \in I}$  is convex, compact and weakly dominant transfer upper quasi-continuous. Then, the game  $G$  possesses a dominant strategy equilibrium if and only if it is strongly uniformly transfer quasiconcave.

Let  $m \in \mathbb{N}^*$  and let the following special simplex:

$$\Delta(n, m) = \{\lambda = (\lambda_{i,j})_{\substack{i=1, \dots, n \\ j=1, \dots, m}} \in \mathcal{M}_{\mathbb{R}}(n, m) : \lambda_{i,j} \geq 0 \text{ and } \sum_{i,j} \lambda_{i,j} = 1\}.$$

**DEFINITION 4.6** A game  $G = (X_i, u_i)_{i \in I}$  is said to be *weakly uniformly transfer quasiconcave* if for any finite subset  $\{y^1, \dots, y^m\} \subset X$ , there exists a corresponding finite subset  $\{x^1, \dots, x^m\} \subset X$  such that for each  $\tilde{x} = \sum_{i,j} \lambda_{i,j} x^j \in \text{co}\{x^h, h = 1, \dots, m\}$ , we have

$$\min_{(i,j) \in J} [u_i(y^j) - u_i(\tilde{x}_i, y_{-i}^j)] \leq 0, \quad (4.2)$$

where  $J = \{(i, j) : \lambda_{i,j} > 0\}$ .

**REMARK 4.3** Definition 4.6 is equivalent to the following definition: A game  $G = (X_i, u_i)_{i \in I}$  is weakly transfer quasiconcave if and only if for any finite subset  $\{y^1, \dots, y^m\} \subset X$ , there exists a corresponding finite subset  $\{x^1, \dots, x^m\} \subset X$  such that for each  $\lambda \in \Delta(n, m)$ , there exist a player  $i \in I$  such that

$$\min_{j \in J(i)} [u_i(y^j) - u_i(\tilde{x}_i, y_{-i}^j)] \leq 0,$$

where  $J(i) = \{j = 1, \dots, m : \lambda_{i,j} > 0\}$  and  $\tilde{x} = \sum_{i,j} \lambda_{i,j} x^j$ .

Weak uniform transfer quasiconcavity roughly says that given any finite subset  $Y^m = \{y^1, \dots, y^m\}$  of deviation profiles, there exists a corresponding finite subset  $X^m = \{x^1, \dots, x^m\}$  of candidate profiles such that for any subset  $\{x^{k^1}, x^{k^2}, \dots, x^{k^s}\} \subset X^m$ ,  $1 \leq s \leq m$ , there exists some player  $i$  so that its convex combinations are not dominated by all deviations in  $X^{\tilde{m}}$  which have nonzero weights. We will see from Theorem 4.2 below that weak uniform transfer quasiconcavity is necessary for the existence of a dominant strategy equilibrium of a game when it is weakly dominant transfer upper continuous. If a game  $G$  is strongly uniformly transfer quasiconcave, then it is weakly uniformly transfer quasiconcave.

The following theorem characterizes the existence of dominant strategy equilibrium if the game is weakly dominant transfer upper continuous and the strategy spaces of players are convex.

**THEOREM 4.2** *Suppose  $G = (X_i, u_i)_{i \in I}$  is compact, bounded, convex and weakly dominant transfer upper continuous. Then, the game  $G$  has a dominant strategy equilibrium if and only if  $G$  is weakly uniformly transfer quasiconcave.*

The following proposition provides sufficient conditions for a game to be weakly dominant transfer upper continuous.

**PROPOSITION 4.1** *Any of the following conditions implies that the game  $G = (X_i, u_i)_{i \in I}$  is weakly dominant transfer upper continuous.*

- (a)  $u_i$  is continuous in  $x_i$ .
- (b)  $u_i$  is upper semi-continuous in  $x_i$ .
- (c)  $u_i$  is transfer upper continuous in  $x_i$ .

## 4.2 Dominant Strategy Equilibria in Discontinuous and Nonconvex Games

In this subsection we characterize the existence of dominant strategy equilibria in games that may be discontinuous or nonconvex. We generalize the results above as well as the existence results of Baye *et al.* [1993] without assuming the convexity of strategy spaces or any form of quasiconcavity of payoff functions.

The following theorem generalizes Theorems 4.1 and 4.2 by relaxing the convexity of strategy spaces and uniform transfer quasiconcavity of payoff functions.

**THEOREM 4.3** *Suppose  $G = (X_i, u_i)_{i \in I}$  is compact, and weakly dominant transfer upper quasi-continuous. Then, the game  $G$  has a dominant strategy equilibrium if and only if for all  $A \in \langle X \rangle$ , there exists  $x \in X$  such that  $u_i(y) \leq u_i(x_i, y_{-i})$ , for each  $y \in A$  and  $i \in I$ .*

Since weak dominant transfer upper continuity and transfer upper semicontinuity imply weak dominant transfer upper quasi-continuity, Theorem 4.3 generalizes Theorem 4.2 and Theorem 4

in Baye *et al.* [1993] by relaxing the convexity of strategy spaces and uniform transfer quasiconcavity.

**DEFINITION 4.7** Let  $X$  be a nonempty subset of a topological space and  $Y$  be a nonempty subset. A function  $f : X \times Y \rightarrow \mathbb{R}$  is said to be  $\alpha$ -transfer lower continuous in  $x$  with respect to  $Y$  if for  $(x, y) \in X \times Y$ ,  $f(x, y) > \alpha$  implies that there exist some point  $y' \in Y$  and some neighborhood  $\mathcal{V}(x) \subset X$  of  $x$  such that  $f(z, y') > \alpha$  for all  $z \in \mathcal{V}(x)$ .

Let  $\hat{X} = \prod_{i \in I} X = X^n$ . A generic element of  $\hat{X}$  is denoted by  $\hat{y} = (y^1, \dots, y^n)$ . Define a function  $\phi : X \times \hat{X} \rightarrow \mathbb{R}$  by

$$\phi(x, \hat{y}) = \sum_{i=1}^n \{u_i(y^i) - u_i(x_i, y_{-i}^i)\}, \quad \forall (x, \hat{y}) \in X \times \hat{X}.$$

Assume that for each  $i$ ,  $X_i$  is a nonempty and compact subset of a topological space  $E_i$  and  $u_i$  is continuous on  $X$ . Then, for all  $x \in X$ , the maximum of  $\phi(x, \cdot)$  over  $\hat{X}$  and  $\min_{x \in X} \max_{\hat{y} \in \hat{X}} \phi(x, \hat{y})$  exist.

Note that, by the definition of  $\phi$ , we have

$$\forall x \in X, \quad \max_{\hat{y} \in \hat{X}} \phi(x, \hat{y}) \geq 0. \quad (4.3)$$

The following lemma shows the relationship between the solution of  $\phi$  and dominant strategy equilibrium for  $G = (X_i, u_i)_{i \in I}$ .

**LEMMA 4.1** A strategy profile  $\bar{x} \in X$  is a dominant strategy equilibrium for  $G = (X_i, u_i)_{i \in I}$  if and only if  $\max_{\hat{y} \in \hat{X}} \phi(\bar{x}, \hat{y}) = 0$ .

By the inequality (4.3) and Lemma 4.1, we have the following proposition.

**PROPOSITION 4.2** Suppose that  $X$  is compact and  $u_i$  is continuous on  $X$ . Let

$$\alpha = \min_{x \in X} \max_{\hat{y} \in \hat{X}} \phi(x, \hat{y}). \quad (4.4)$$

Then, the game  $G = (X_i, u_i)_{i \in I}$  has at least one dominant strategy equilibrium if and only if  $\alpha = 0$ .

**DEFINITION 4.8**  $G = (X_i, u_i)_{i \in I}$  is 0-transfer lower continuous if  $\phi$  is 0-transfer lower continuous in  $x$  with respect to  $\hat{X}$ .

We then have the following result.

**THEOREM 4.4** Suppose  $G = (X_i, u_i)_{i \in I}$  is compact and 0-transfer lower continuous in  $x$  with respect to  $\hat{X}$ . Then,  $G = (X_i, u_i)_{i \in I}$  has a dominant strategy equilibrium if and only if for all  $A \in \langle X^n \rangle$ , there exists  $x \in X$  such that  $\phi(x, \tilde{y}) \leq 0$ , for each  $\tilde{y} \in A$ .



**COROLLARY 4.1** Suppose that the game (2.1) is partially separable<sup>8</sup>,  $X_i$  is a nonempty and compact subset of a topological space  $E_i$ , and  $h_i(x_i)$  is upper semicontinuous over  $X_i$ ,  $\forall i \in I$ . Then, the game  $G = (X_i, u_i)_{i \in I}$  has a dominant strategy equilibrium.

**EXAMPLE 4.1** Again consider Example 3.7.

$$\begin{aligned} u_1(x) &= x_2 x_1^2, \\ u_2(x) &= -x_1 x_2^2. \end{aligned}$$

Since  $X_i$  is not convex  $\forall i \in I$ , Theorem 4 in Baye *et al.* [1993] is not applicable.

For  $x = (x_1, x_2)$  and  $y = (y_1, y_2)$ , we have

$$\phi(x, (y, z)) = y_2 y_1^2 - x_1^2 y_2 - z_1 z_2^2 + z_1 x_2^2.$$

Note that  $\phi$  is continuous on  $X \times \hat{X}$ . For any subset  $\{((1y_1, 2y_2), (1z_1, 2z_2)), \dots, ((ky_1, ky_2), (kz_1, kz_2))\}$  of  $\hat{X}$ , let  $x = (x_1, x_2) \in X$  such that  $x_1 = \max_{h=1, \dots, k} iy_1$  and  $x_2 = \min_{h=1, \dots, k} iz_2$ . Then

$$\begin{cases} iz_2^2 \geq x_2^2, \quad \forall i = 1, \dots, k, \\ iy_1^2 \leq x_1^2. \end{cases}$$

Thus,

$$\begin{cases} iz_1 iz_2^2 \geq iz_1 x_2^2, \quad \forall i = 1, \dots, k, \\ iy_2 iy_1^2 \leq iy_2 x_1^2. \end{cases} \quad \text{and then } \begin{cases} -iz_1 iz_2^2 + iz_1 x_2^2 \leq 0, \quad \forall i = 1, \dots, k, \\ iy_2 iy_1^2 - iy_2 x_1^2 \leq 0. \end{cases}$$

Therefore,  $\phi(x, (iy, iz)) \leq 0, \forall i = 1, \dots, k$ . According to Theorem 4.3, this game has a dominant strategy equilibrium. Indeed,  $\bar{x} = (4, 1)$  is such a point.

## 5 Nash Equilibria in Mixed Strategies

In this section, we consider the existence of mixed strategy Nash equilibrium by applying the pure strategy existence results derived in the previous sections. Assume that each  $X_i$  is a compact *Hausdorff* space. Let  $u_i$  be bounded and measurable for all  $i \in I$  and let  $M_i$  be the regular, countably additive probability measures on the *Borel* subsets of  $X_i$ ,  $M_i$  is compact in the weak\* topology. Let us consider  $U_i$  be the extension of  $u_i$  to  $M = \prod_{i \in I} M_i$  by defining  $U_i(\mu) = \int_X u_i(x) d\mu(x)$  for all  $\mu \in M$  with  $d\mu(x) = d\mu_1(x_1) \times d\mu_2(x_2) \times \dots \times d\mu_n(x_n)$ , and let  $\bar{G} = (M_i, U_i)_{i \in I}$  denote the mixed extension of  $G$ .

<sup>8</sup>A game  $G = (I, (X)_{i \in I}, (f)_{i \in I})$  is partially separable if for each  $i \in I$  there exist two functions  $h_i : X_i \rightarrow \mathbb{R}$  and  $g_{-i} : X_{-i} \rightarrow \mathbb{R}$  such that  $u_i(x) = h_i(x_i) + g_{-i}(x_{-i})$  for all  $x \in X$ .

**DEFINITION 5.1** A mixed strategy Nash equilibrium of the game  $G$  is an  $n$ -tuple of probability measures  $(\mu_1^*, \dots, \mu_n^*) \in M$  such that for all  $i \in I$

$$U_i(\mu^*) = \int_X u_i(x) d\mu^*(x) \geq \max_{\mu_i \in M_i} \int_X u_i(x) d\mu_1^*(x_1) \times \dots \times d\mu_i(x_i) \times \dots \times d\mu_n^*(x_n).$$

The definitions of weak transfer continuity, weak transfer upper continuity, weak transfer lower continuity, upper semicontinuity, payoff security, etc. given in Subsection 3.1 apply in obvious ways to the mixed extension  $\bar{G}$  by replacing  $X_i$  with  $M_i$  in each definition. However, it may be noted that weak transfer continuity (resp., weak transfer upper continuity, weak transfer lower continuity, payoff security) of  $\bar{G}$  neither implies nor is implied by weak transfer continuity (resp., weak transfer upper continuity, weak transfer lower continuity, payoff security) of  $G$ .

**LEMMA 5.1** *If  $G$  is upper semicontinuous, then the mixed extension of  $G$  is also upper semicontinuous.*

**PROOF.** See the proof of Proposition 5.1 in Reny [1999] page 1052. ■

Nash [1950] and Glicksberg [1952] shows that a game that is compact, Hausdorff, and continuous possesses mixed strategy Nash equilibrium. Robson [1994] proves that in a compact game with metric strategy spaces, if each player's payoff is upper semicontinuous in all players' strategies, and continuous in other players' strategies, then the game possesses a mixed strategy Nash equilibrium.

The following theorem strictly generalizes the mixed strategy Nash equilibrium existence of Nash [1950], Glicksberg [1952], Dasgupta and Maskin [1986], Robson [1994], Simon [1987] and Reny [1999] by weakening continuity conditions.

**THEOREM 5.1** *Suppose that  $G = (X_i, u_i)_{i \in I}$  is a compact, Hausdorff game. Then  $G$  has a mixed strategy Nash equilibrium if its mixed extension  $\bar{G}$  is weakly transfer quasi-continuous. Moreover,  $\bar{G}$  is weakly transfer quasi-continuous if it is 1) weakly transfer continuous, 2) better reply secure, 3) weakly transfer upper continuous and payoff secure, or 4) weakly transfer lower continuous and upper semicontinuous.*

Monteiro and Page [2007] introduce the concept of uniform payoff security for games that are compact, Hausdorff, bounded and measurable. They show that if a game is compact and uniformly payoff secure, then its mixed extension  $\bar{G}$  is payoff secure, but the reverse may not be true, as shown by an example in Carmona [2005].

**DEFINITION 5.2** The game  $G$  is *uniformly payoff secure* if for each  $i \in I$ ,  $x_i \in X_i$ , and every  $\epsilon > 0$ , there is a strategy  $\bar{x}_i \in X_i$  such that for every  $y_{-i} \in X_{-i}$  there exists a neighborhood  $\mathcal{V}(y_{-i})$  of  $y_{-i}$  such that  $u_i(\bar{x}_i, z_{-i}) \geq u_i(x_i, y_{-i}) - \epsilon$ , for all  $z_{-i} \in \mathcal{V}(y_{-i})$ .

**DEFINITION 5.3** The game  $G$  is said to be *uniformly transfer continuous* if for each  $i \in I$ ,  $x_i \in X_i$ , and every  $\epsilon > 0$ , there is a strategy  $\bar{x}_i \in X_i$  such that for every  $y_{-i} \in X_{-i}$  there exists a neighborhood  $\mathcal{V}(x_i, y_{-i})$  of  $(x_i, y_{-i})$  such that

$$u_i(\bar{x}_i, z_{-i}) + \epsilon \geq u_i(x_i, y_{-i}) \geq u_i(z) - \epsilon, \text{ for all } z \in \mathcal{V}(x_i, y_{-i}).$$

Thus, a game  $G$  is uniformly transfer continuous if for any strategy  $x_i \in X_i$ , player  $i$  can choose a strategy  $\bar{x}_i \in X_i$  to secure a payoff of  $u_i(x_i, y_{-i}) - \epsilon$  against deviations by other players in some neighborhood of  $y_{-i} \in X_{-i}$ , and would be better off at  $(x_i, y_{-i})$  even if all players deviate slightly from  $(x_i, y_{-i})$  for all strategy profiles  $y_{-i} \in X_{-i}$ .

**PROPOSITION 5.1** *If a game  $G = (X_i, u_i)_{i \in I}$  is 1) uniformly payoff secure and upper semicontinuous or 2) uniformly transfer continuous, then the mixed extension  $\bar{G}$  is weakly transfer quasi-continuous.*

Proposition 5.1, together with THEOREM 5.1, immediately yields the following useful result.

**COROLLARY 5.1** *If a game  $G = (X_i, u_i)_{i \in I}$  is compact, bounded, Hausdorff, and 1) uniformly payoff secure and upper semicontinuous or 2) uniformly transfer continuous, then it possesses a mixed strategy Nash equilibrium.*

As an application of the above proposition, consider the following well-known concession game.

**EXAMPLE 5.1** Let us consider  $i = 1, 2$  and  $x_1, x_2 \in [0, 1]$  with:

$$u_i(x_i, x_{-i}) = \begin{cases} l_i(x_i), & \text{if } x_i < x_{-i}, \\ \phi_i(x_i), & \text{if } x_i = x_{-i}, \\ m_i(x_i), & \text{if } x_i > x_{-i}. \end{cases}$$

We make the following assumption on  $u_i$ :

**ASSUMPTION 5.1**

- a)  $\forall x \in [0, 1], \forall \epsilon > 0$ , there exists a neighborhood  $\mathcal{V}(x)$  of  $x$  such that  $\phi_i(x) \geq \max(m_i(z), l_i(z)) - \epsilon$ , for every  $z \in \mathcal{V}(x)$ .
- b)  $\forall x \in [0, 1], \forall \epsilon > 0$ , there exists  $y \in [0, 1]$  such that  $\min\{\phi_i(y), m_i(y), l_i(y)\} \geq \max\{\phi_i(x), m_i(x), l_i(x)\} - \epsilon$ .

Then we have the following result.

**PROPOSITION 5.2** *Suppose the concession game satisfies Assumption 5.1, and the functions  $l_i(\cdot)$ ,  $m_i(\cdot)$  and  $\phi_i(\cdot)$  are upper semicontinuous on  $[0, 1]$ . Then, the game has a mixed strategy Nash equilibrium.*

## 6 Conclusion

In this paper, we characterize the existence of equilibria in games with possibly nonconvex strategy spaces or non-quasiconcave payoffs. We first offer new Nash equilibrium existence results for a large class of discontinuous games, which rely on weak transfer (quasi-) continuities. We then characterize the existence of pure strategy, dominant strategy, and mixed strategy Nash equilibria in noncooperative games which may not have convex strategy spaces or non-quasiconcave payoff functions.

These results permit us to significantly weaken the key assumptions, such as continuity, convexity, and quasi-concavity on the existence of Nash equilibrium, and contains almost all the known results in the literature such as those in Baye *et al.* [1993] and Reny [1999]. We also provide examples and economic applications where our general results are applicable, but the existing theorems for pure strategy, dominant strategy, and mixed strategy Nash equilibria fail to hold. These new results help us understand the existence or non-existence of pure strategy, dominant strategy, and mixed strategy Nash equilibria in discontinuous and non-concave games.

## Appendix

**PROOF OF PROPOSITION 3.1.** It is clear that a game  $G$  is weakly transfer quasi-continuous if it is weakly transfer continuous. We only need to prove either of diagonal transfer continuity and better-reply security implies weak transfer quasi-continuity.

We first consider the case of diagonal transfer continuity. Suppose  $x^* \in X$  is not an equilibrium. Then, by diagonal transfer continuity of  $U$ , there exist a strategy  $\bar{y} \in X$  and a neighborhood  $\mathcal{V}(x^*)$  of  $x^*$  such that  $U(z, \bar{y}) > U(z, z)$  for each  $z \in \mathcal{V}(x^*)$ , i.e.  $\sum_{i \in I} [u_i(\bar{y}_i, z_{-i}) - u_i(z)] > 0$  for each  $z \in \mathcal{V}(x^*)$ . Thus, for each  $z \in \mathcal{V}(x^*)$ , there exists a player  $i \in I$  such as  $u_i(\bar{y}_i, z_{-i}) - u_i(z) > 0$ .

We now consider the case of better-reply security. Suppose, by way of contradiction, that the game is not weakly transfer quasi-continuous. Then, there exists a nonequilibrium  $x^* \in X$  such that, for every  $\bar{y} \in X$  and every neighborhood  $\mathcal{V}(x^*)$  of  $x^*$ , there exists  $z \in \mathcal{V}(x^*)$  with

$$u_i(\bar{y}_i, z_{-i}) \leq u_i(z) \quad \text{for all } i \in I. \quad (6.1)$$

Letting  $\bar{u}$  be the limit of the vector of payoffs corresponding to some sequence of strategies converging to  $x^*$ , and  $U^*$  the set of all such points, which is a compact set by the boundedness of payoffs, we have  $(x^*, \bar{u}) \in \text{cl } \Gamma$  for all  $\bar{u} \in U^*$ . Then, by better-reply security, for each  $(x^*, \bar{u}) \in \text{cl } \Gamma$  with  $\bar{u} \in U^*$ , there exist a player  $i$ , a strategy  $\bar{y}_i$ , and a neighborhood  $\bar{\mathcal{V}}(x_{-i}^*)$  of  $x_{-i}^*$  such that  $u_i(\bar{y}_i, z_{-i}) > \bar{u}_i$  for all  $z_{-i} \in \bar{\mathcal{V}}(x_{-i}^*)$ . Then, we have  $\varphi_i(\bar{y}_i, x_{-i}^*) > \bar{u}_i$ <sup>9</sup>. Choose  $\epsilon > 0$  with  $\varphi_i(\bar{y}_i, x_{-i}^*) > \bar{u}_i + \epsilon$ . Since  $\varphi_i(\bar{y}_i, \cdot)$  is lower semi-continuous (cf. Reny [1999]), then

$$u_i(\bar{y}_i, z_{-i}) > \bar{u}_i + \epsilon, \quad \text{for each } z_{-i} \in \mathcal{V}^*(x_{-i}^*).$$

Let  $U_i^*$  be the projection of  $U^*$  to coordinate  $i$  and  $u_i^* = \sup\{\bar{u}_i \in U_i^* : u_i(\bar{y}_i, z_{-i}) > \bar{u}_i + \epsilon \text{ for all } z_{-i} \in \bar{\mathcal{V}}(x_{-i}^*)\}$ . Then, for  $\epsilon/2 > 0$ , there is a  $y_i^*$  such that  $u_i(y_i^*, z_{-i}) > (u_i^* + \epsilon) - \epsilon/2 = u_i^* + \epsilon/2$  for all  $z_{-i} \in \mathcal{V}^*(x_{-i}^*)$ .

Now, since the game is not weakly transfer quasi-continuous, for such a securing strategy  $y_i^*$ , we can find a directed system of neighborhoods  $\{\mathcal{V}^\alpha(x^*)\}_{\alpha \in \Lambda}$  and a net  $z^\alpha \in \mathcal{V}^\alpha(x^*)$  such that  $z^\alpha \rightarrow_\alpha x^*$  and

$$u_i(y_i^*, z_{-i}^\alpha) \leq u_i(z^\alpha) \rightarrow_\alpha \bar{u}_i \leq u_i^*.$$

Thus, for  $\epsilon/2 > 0$ , there exists  $\alpha_1$  such that whenever  $\alpha > \alpha_1$ , we have

$$u_i(y_i^*, z_{-i}^\alpha) \leq u_i^* + \epsilon/2 < u_i(y_i^*, z_{-i}), \quad \text{for each } z_{-i} \in \mathcal{V}^*(x_{-i}^*). \quad (6.2)$$

<sup>9</sup>The function  $\varphi_i(y_i, x_{-i})$  is defined by  $\varphi_i(y_i, x_{-i}) = \sup_{\mathcal{V} \in \Omega(x_{-i})} \inf_{z_{-i} \in \mathcal{V}} u_i(y_i, z_{-i})$  (cf. Reny [1999]).

Since the net  $\{z^\alpha\}_{\alpha \in \Lambda}$  also converges to  $x^*$ , then for each  $\mathcal{V}(x^*)$  of  $x^*$  with  $Proj_{-i}(\mathcal{V}(x^*)) \subseteq \mathcal{V}(x_{-i}^*)$ ,<sup>10</sup> there exists  $\alpha_2$  such that  $z^\alpha \in \mathcal{V}(x^*)$  for  $\alpha > \alpha_2$ . Consequently,  $z_{-i}^\alpha \in Proj_{-i}(\mathcal{V}^*(x^*))$  with  $\alpha > \max(\alpha_1, \alpha_2)$ . Thus, by (6.2), we obtain  $u_i(y_i^*, z_{-i}^\alpha) \leq u_i^* + \epsilon < u_i(y_i^*, z_{-i}^\alpha)$ , a contradiction. Hence, the game must be weakly transfer quasi-continuous. ■

To prove Theorem 3.1, we need the following lemma.

A correspondence  $C : X \rightarrow 2^Y$  is *open inverse-image* or have *lower open sections* if the set  $\{x \in X : y \in C(x)\}$  is open in  $X$ , for all  $y \in Y$ .

**LEMMA 6.1** (See Theorem 3a, page 264 in Deguire and Lassonde [1995]) *Let  $X_i$  be a nonempty, compact and convex space,  $i \in I$  and  $\{C_i : X \rightarrow X_i, i \in I\}$  be a family of correspondences such that:*

- (1) *for all  $i \in I$ ,  $C_i(x)$  is convex for every  $x \in X$ ,*
- (2) *for all  $i \in I$ ,  $C_i$  is open inverse-image,*
- (3) *for each  $x \in X$ , there exists  $i \in I$  such that  $C_i(x) \neq \emptyset$ .*

*Then, there as  $\bar{x} \in X$  and  $i \in I$  such that  $\bar{x}_i \in C_i(\bar{x})$ .*

**PROOF OF THEOREM 3.1.** For each player  $i \in I$  and every  $(x_i, y) \in X_i \times X$ , let

$$\varphi_i(x_i, y) = \sup_{\mathcal{V} \in \Omega(y)} \inf_{z \in \mathcal{V}} [u_i(x_i, z_{-i}) - u_i(z)]$$

where  $\Omega(y)$  is the set of all open neighborhoods of  $y$ .

For each  $i$  and every  $x_i \in X_i$ , the function  $\varphi_i(x_i, \cdot)$  is real-valued by boundedness of payoff function. We show it is also lower semicontinuous over  $X$ . Indeed, for each  $i \in I$ , let  $x_i \in X_i$  and  $\mathcal{V}$  be an open neighborhood. Consider the following function

$$g_{\mathcal{V}}^i(x_i, y) = \begin{cases} \inf_{z \in \mathcal{V}} [u_i(x_i, z_{-i}) - u_i(z)], & \text{if } y \in \mathcal{V}, \\ -\infty, & \text{otherwise.} \end{cases}$$

We want to show that  $g_{\mathcal{V}}^i(x_i, \cdot)$  is lower semicontinuous on  $X$ , which is equivalent to show the set

$$A(x_i) = \{y \in X : g_{\mathcal{V}}^i(x_i, y) \leq \alpha\}, \quad \alpha \in \mathbb{R}.$$

is closed for all  $x_i \in X_i$ . Suppose that there exists a point  $y \in X$  such that  $y$  is in the closure of  $A(x_i)$ , but not in  $A(x_i)$ . Then, there exists a net  $\{y^p\}_{p \in \Lambda} \subset A(x_i)$  converging to  $y$ . Since  $y \notin A(x_i)$ ,  $\inf_{z \in \mathcal{V}} [u_i(x_i, z_{-i}) - u_i(z)] > \alpha$ . If  $y \notin \mathcal{V}$ , then  $-\infty > \alpha$ , which is impossible, and thus

<sup>10</sup> $Proj_i(A)$  is the projection of  $A$  on space  $X_i$

$y \in \mathcal{V}$  and  $g_{\mathcal{V}}^i(x_i, y) > \alpha$ . Thus, we have  $\{y^p\}_{p \in \Lambda} \subset A(x_i)$ , and then  $g_{\mathcal{V}}^i(x_i, y^p) \leq \alpha$  for every  $p \in \Lambda$ . If there exists  $\bar{p} \in \Lambda$  such that  $y^{\bar{p}} \in \mathcal{V}$ , then  $\inf_{z \in \mathcal{V}} [u_i(x_i, z_{-i}) - u_i(z)] \leq \alpha$ , which contradicts the fact that  $\inf_{z \in \mathcal{V}} [u_i(x_i, z_{-i}) - u_i(z)] > \alpha$ . Thus, for all  $p \in \Lambda$ ,  $y^p \notin \mathcal{V}$ . Since the net  $\{y^p\}_{p \in \Lambda}$  converges to  $y \in \mathcal{V}$ , there exists  $\eta \in \Lambda$  such that, for all  $p \geq \eta$ ,  $y^p \in \mathcal{V}$ , which contradicts the fact that  $y^p \notin \mathcal{V}$  for all  $p \in \Lambda$ . Thus,  $A(x_i)$  is closed, which means that the function  $g_{\mathcal{V}}^i(x_i, \cdot)$  is lower semicontinuous over  $X$ . Since the function  $\varphi_i(x_i, \cdot)$  is the pointwise supremum of a collection of lower semicontinuous functions on  $X$ , by Lemma 2.39, page 43 in Aliprantis and Border [1994],  $\varphi_i(x_i, \cdot)$  is lower semicontinuous on  $X$ .

Let us consider the following sets:

For each  $y \in X$ , let

$$F(y) = \{x \in X : u_i(y_i, x_{-i}) \leq u_i(x), \forall i \in I\},$$

$$G(y) = \{x \in X : \varphi_i(y_i, x) \leq 0, \forall i \in I\},$$

and for each  $x \in X$  and  $i \in I$ , let

$$C_i(x) = \{y_i \in X_i : \varphi_i(y_i, x) > 0\}.$$

We first prove that  $F$  is transfer closed valued.<sup>11</sup> Let  $x, y \in X$  with  $x \notin F(y)$ . Then  $x$  is not an equilibrium. By the weak transfer quasi-continuity of the game  $G$ , there exist a strategy  $y' \in X$  and a neighborhood of  $x$  so that for every  $z \in \mathcal{V}(x)$ , there exists a player  $i$  such as  $u_i(y'_i, z_{-i}) > u_i(z)$ . Therefore, for all  $z \in \mathcal{V}(x)$   $z \notin F(y')$ , i.e.  $x \notin \text{cl } F(y')$ . Consequently,

$$\bigcap_{y \in X} F(y) = \bigcap_{y \in X} \text{cl } F(y).$$

Since  $\varphi_i(x_i, \cdot)$  is lower semicontinuous on  $X$ , then  $G(y)$  is closed. Moreover,  $G(y) \subset \text{cl } F(y)$  for all  $y \in X$ . Indeed, let  $y \in X$ , and  $x \in G(y)$ . Then, for all players  $i \in I$ ,  $\varphi_i(y_i, x) \leq 0$ , i.e., each  $i \in I$  there exists a neighborhood  $\mathcal{V}^i(x)$  of  $x$  such that

$$\forall z \in \mathcal{V}^i(x), u_i(y_i, z_{-i}) \leq u_i(z). \quad (6.3)$$

If  $x \notin \text{cl } F(y)$ , then there exists a neighborhood  $\mathcal{V}(x)$  of  $x$  such that

$$\forall z \in \mathcal{V}(x), \exists i \in I, \text{ such as } u_i(y_i, z_{-i}) > u_i(z). \quad (6.4)$$

Let  $\tilde{z} \in \mathcal{V}(x) \cap (\bigcap_{i \in I} \mathcal{V}^i(x))$ . Then, inequality (6.4) implies that there exists  $i \in I$  such that  $u_i(y_i, \tilde{z}_{-i}) > u_i(\tilde{z})$ , a contradiction to (6.3). Thus, for all  $y \in X$ ,  $G(y) \subset \text{cl } F(y)$ . Therefore

$$\bigcap_{y \in X} G(y) \subset \bigcap_{y \in X} \text{cl } F(y) = \bigcap_{y \in X} F(y). \quad (6.5)$$

<sup>11</sup> $F$  is transfer closed valued if  $x \notin F(y)$  implies that there exists  $y' \in X$  such that  $x \notin \text{cl } F(y')$ .

We now show  $\{C_i\}_{i \in I}$  is convex and open inverse-image for all  $i \in I$ . Indeed, let  $i \in I$ ,  $x \in X$ ,  $\bar{y}_i, \underline{y}_i$  be two elements of  $C_i(x)$  and  $\theta \in [0, 1]$ . Since  $\bar{y}_i$  and  $\underline{y}_i$  are in  $C_i(x)$ ,  $\varphi_i(\bar{y}_i, x) > 0$  and  $\varphi_i(\underline{y}_i, x) > 0$ . Then, there exist  $\mathcal{V}^1(x)$  and  $\mathcal{V}^2(x)$  of  $x$  such that for all  $(z^1, z^2) \in \mathcal{V}^1(x) \times \mathcal{V}^2(x)$

$$\begin{cases} u_i(\bar{y}_i, z_{-i}^1) > u_i(z^1) \\ u_i(\underline{y}_i, z_{-i}^2) > u_i(z^2). \end{cases}$$

Thus, there exists a neighborhood  $\mathcal{V}(x) = \mathcal{V}^1(x) \cap \mathcal{V}^2(x)$  such that

$$\min\{u_i(\bar{y}_i, z_{-i}), u_i(\underline{y}_i, z_{-i})\} > u_i(z), \quad \forall z \in \mathcal{V}(x).$$

Since  $G$  is quasiconcave in  $x_i$ , then  $\min\{u_i(\bar{y}_i, z_{-i}), u_i(\underline{y}_i, z_{-i})\} \leq u_i(\theta\bar{y}_i + (1-\theta)\underline{y}_i, z_{-i})$ , for each  $z_{-i}$ . Therefore,  $u_i(\theta\bar{y}_i + (1-\theta)\underline{y}_i, z_{-i}) > u_i(z)$ ,  $\forall z \in \mathcal{V}(x)$ . Thus,  $\theta\bar{y}_i + (1-\theta)\underline{y}_i \in C_i(x)$ . Also, let  $i \in I$ . Since  $\varphi_i(\bar{y}_i, \cdot)$  is lower semicontinuous, the set  $\{x \in X : \varphi_i(\bar{y}_i, x) > 0\}$  is open in  $X$ , for each  $y_i \in X_i$ , which means  $C_i$  is open inverse-image.

Now suppose, by way of contradiction, that for each  $x \in X$ , there exists a player  $i \in I$  such that  $C_i(x) \neq \emptyset$ . Then, by Lemma 6.1, there exist a point  $\tilde{x} \in X$  and  $i \in I$  such that  $\tilde{x}_i \in C_i(\tilde{x})$ , i.e.,  $\varphi_i(\tilde{x}_i, \tilde{x}) > 0$ . Thus, by lower semicontinuity of  $\varphi_i(x_i, \cdot)$ , there exists a neighborhood  $\mathcal{V}(\tilde{x})$  of  $\tilde{x}$  such that  $u_i(\tilde{x}_i, z_{-i}) > u_i(z)$ , for each  $z \in \mathcal{V}(\tilde{x})$ . Letting  $z = \tilde{x}$  in the last inequality, we obtain  $u_i(\tilde{x}_i) > u_i(\tilde{x})$ , which is impossible. Thus, there exists  $\bar{x} \in X$  such that for each  $i \in I$ , we have  $C_i(\bar{x}) = \emptyset$ . Therefore, for each  $i \in I$  and for each  $y_i \in X_i$ ,  $\varphi_i(y_i, \bar{x}) \leq 0$ . Hence,  $\bar{x} \in \bigcap_{y \in X} G(y) \subset \bigcap_{y \in X} F(y)$ , i.e.  $\bar{x}$  is a Nash equilibrium. ■

**PROOF OF THEOREM 3.2. Necessity ( $\Rightarrow$ ):** Suppose the game  $G$  has a pure strategy Nash equilibrium  $\bar{x} \in X$ . We want to prove that  $G$  is strongly diagonal transfer quasiconcave. Indeed, for any finite subset  $\{y^1, \dots, y^m\} \subset X$ , let the corresponding finite subset  $\{x^1, \dots, x^m\} = \{\bar{x}\}$ . Thus, for any  $J \subset \{1, \dots, m\}$  and each  $x \in \text{co}\{x^h, h \in J\} = \{\bar{x}\}$ , we have  $\forall h \in J$ ,  $u_i(y_i^h, \bar{x}_{-i}) \leq u_i(\bar{x})$  for each player  $i$ .

*Sufficiency ( $\Leftarrow$ ).* For each  $y \in X$ , let

$$F(y) = \{x \in X : u_i(y_i, x_{-i}) \leq u_i(x), \quad \forall i \in I\}.$$

Since  $G$  is weakly transfer quasi-continuous, then  $F$  is transfer closed valued.

For  $y \in X$ , let  $\bar{F}(y) = \text{cl } F(y)$ . Then  $\bar{F}(y)$  is closed, and by the strong diagonal transfer quasiconcavity (Remark 3.4), it is also transfer FS-convex. From Lemma 1 in Tian [1993], we deduce  $\bigcap_{y \in X} F(y) = \bigcap_{y \in X} \bar{F}(y) \neq \emptyset$ . Thus, there exists a strategy profile  $\bar{x} \in X$  such that

$$u_i(y_i, \bar{x}_{-i}) \leq u_i(\bar{x}), \quad \text{for all } y \in X \text{ and } i \in I.$$

Thus  $\bar{x}$  is a pure strategy Nash equilibrium of game  $G$ . ■



**PROOF OF PROPOSITION 3.2.** Suppose that the aggregate function  $U(x, y) = \sum_{i=1}^n u_i(y_i, x_{-i})$  is diagonally transfer quasiconcave. Then, for any finite subset  $Y^m = \{y^1, \dots, y^m\} \subset X$ , there exists a corresponding finite subset  $X^m = \{x^1, \dots, x^m\} \subset X$  such that for each  $\tilde{x} = \sum_{i,j} \lambda_{i,j} x^j \in \text{co}\{x^h, h = 1, \dots, m\}$  we have  $\min_{s \in J_1} U(x, y^s) \leq U(x, x)$  where  $J_1 = \{j = 1, \dots, m : \sum_{i \in I} \lambda_{i,j} > 0\}$  and  $\lambda \in \Delta(n, m)$ . Thus,  $\min_{s \in J_1} \sum_{i=1}^n [u_i(y_i^s, x_{-i}) - u_i(x)] \leq 0$ . Therefore, there exists  $(i, j) \in J = \{(i, j) : \lambda_{i,j} > 0\}$  such that  $u_i(y_i^j, x_{-i}) - u_i(x) \leq 0$ . We conclude that  $\min_{(i,j) \in J} [u_i(y_i^j, \tilde{x}_{-i}) - u_i(\tilde{x})] \leq 0$  with  $J = \{(i, j) : \lambda_{i,j} > 0\}$ . ■

**PROOF OF THEOREM 3.3. Necessity ( $\Rightarrow$ ):** Suppose the game  $G$  has a pure strategy Nash equilibrium  $\bar{x} \in X$ . We want to prove that  $G$  is weakly diagonally transfer quasiconcave. Indeed, for any finite subset  $\{y^1, \dots, y^m\} \subset X$ , let the corresponding finite subset  $\{x^1, \dots, x^m\} = \{\bar{x}\}$ . Thus, for any  $\lambda \in \Delta(n, m)$   $x = \sum_{i,j} \lambda_{i,j} x^j = \bar{x}$ , we have for each  $i \in I$

$$\min_{j \in J(i)} u_i(y_i^j, \bar{x}_{-i}) \leq u_i(y_i^h, \bar{x}_{-i}) \leq u_i(\bar{x}),$$

for each  $h$  such that  $\lambda_{i,h} > 0$  where  $J(i) = \{j : \lambda_{i,j} > 0\}$ .

*Sufficiency ( $\Leftarrow$ ):* For each player  $i \in I$  and every  $(x_i, y) \in X_i \times X$ , let

$$\varphi_i(x_i, y) = \sup_{\mathcal{V} \in \Omega(y)} \inf_{z \in \mathcal{V}} [u_i(x_i, z_{-i}) - u_i(z)]$$

where  $\Omega(y)$  is the set of all open neighborhoods of  $y$ . For each  $i$  and every  $x_i \in X_i$ , the function  $\varphi_i(x_i, \cdot)$  is lower semicontinuous over  $X$  from the proof of Theorem 3.1.

Let us consider the following sets: for each  $y \in X$ , let

$$F(y) = \{x \in X : u_i(y_i, x_{-i}) \leq u_i(x), \forall i \in I\},$$

$$G(y) = \{x \in X : \varphi_i(y_i, x) \leq 0, \forall i \in I\}.$$

By proof of Theorem 3.1,  $F$  is also transfer closed valued, and thus  $G(y)$  is closed and

$$\bigcap_{y \in X} G(y) \subset \bigcap_{y \in X} \text{cl } F(y) = \bigcap_{y \in X} F(y). \quad (6.6)$$

Now, suppose, by way of contradiction, that  $\bigcap_{y \in X} G(y) = \emptyset$ . Then, we have

$$\forall x \in X, \text{ there exists } y \in X, i \in I \text{ such that } \varphi_i(y_i, x) > 0. \quad (6.7)$$

Thus,  $X$  can be covered by the following subsets

$$\theta_{i,y} = \{x \in X : \varphi_i(y_i, x) > 0\}, i \in I \text{ and } y \in X.$$

Since  $\varphi_i(y_i, \cdot)$  is lower semicontinuous on  $X$ , the subset  $\theta_{i,y}$  is open in  $X$ , for each  $i \in I$  and  $y \in X$ . Also, since  $X$  is compact, it can be covered by a finite number of subsets  $\{\theta_{i,y^j} : i = 1, \dots, n \text{ and } j = 1, \dots, m\}$ . Consider a continuous partition of unity  $\{\alpha_{i,j}\}_{i=1, \dots, n, j=1, \dots, m}$  associated to the finite covering  $\{\theta_{1,y^1}, \dots, \theta_{n,y^m}\}$ .

Since  $G$  is weakly diagonal transfer quasiconcave, there exists a corresponding finite subset  $\{x^1, \dots, x^m\} \subset X$  such that for each  $\tilde{x} = \sum_{i,j} \lambda_{i,j} x^j \in \text{co}\{x^h, h = 1, \dots, m\}$  and if  $J = \{(i, j) : \lambda_{i,j} > 0\}$ , then

$$\min_{(i,j) \in J} [u_i(y_i^j, \tilde{x}_{-i}) - u_i(\tilde{x})] \leq 0. \quad (6.8)$$

Let us now consider the following function defined on  $X$  into  $X$  by

$$f(x) = \sum_{i,j} \alpha_{i,j}(x) x^j.$$

Since the functions  $\alpha_{i,j}$  are continuous over the compact convex  $X$  into  $X$ , by Brouwer Fixed-Point Theorem, there exists  $\tilde{x} = f(\tilde{x}) = \sum_{i,j} \alpha_{i,j}(\tilde{x}) x^j$ . Let  $J(\tilde{x}) = \{(i, j) : \alpha_{i,j}(\tilde{x}) > 0\}$ .

If  $(i, j) \in J(\tilde{x})$ , then  $\tilde{x} \in \text{supp}(\alpha_{i,j}) \subset \theta_{i,y^j}$ . Thus,  $\varphi_i(y_i^j, \tilde{x}) > 0$  for each  $(i, j) \in J(\tilde{x})$ . Therefore,

$$\min_{(i,j) \in J(\tilde{x})} \varphi_i(y_i^j, \tilde{x}) > 0. \quad (6.9)$$

Since  $\varphi_i(y_i^j, \tilde{x}) \leq u_i(y_i^j, \tilde{x}_{-i}) - u_i(\tilde{x})$ , then inequalities (6.8) and (6.9) imply  $0 < \min_{(i,j) \in J(\tilde{x})} \varphi_i(y_i^j, \tilde{x}) \leq 0$ , which is impossible. Therefore,

$$\emptyset \neq \bigcap_{y \in X} G(y) \subset \bigcap_{y \in X} F(y).$$

Thus,  $\bar{x} \in X$  such that  $\bar{x} \in \bigcap_{y \in X} F(y)$  is a Nash equilibrium.

■

**PROOF OF PROPOSITION 3.3.** Suppose  $\bar{x} \in X$  is not a Nash equilibrium. Then, by weak transfer upper continuity, some player  $i$  has a strategy  $\hat{x}_i \in X_i$  and a neighborhood  $\mathcal{V}(\bar{x})$  of  $\bar{x}$  such that  $u_i(\hat{x}_i, \bar{x}_{-i}) > u_i(z)$  for all  $z \in \mathcal{V}(\bar{x})$ . Choose  $\epsilon > 0$  such that  $u_i(\hat{x}_i, \bar{x}_{-i}) - \epsilon > \sup_{z \in \mathcal{V}(\bar{x})} u_i(z)$ .

The payoff security of  $G$  implies that there exist a strategy  $y_i$  and a neighborhood  $\tilde{\mathcal{V}}(\bar{x}_{-i})$  of  $\bar{x}_{-i}$  such that  $u_i(y_i, z_{-i}) \geq u_i(\hat{x}_i, \bar{x}_{-i}) - \epsilon$  for all  $z_{-i} \in \tilde{\mathcal{V}}(\bar{x}_{-i})$ . Thus, there exist  $y_i \in X_i$  and a neighborhood of  $\hat{\mathcal{V}}(\bar{x})$  of  $\bar{x}$  such that  $u_i(y_i, z_{-i}) > u_i(z)$  for all  $z \in \hat{\mathcal{V}}(\bar{x})$ . ■

**PROOF OF PROPOSITION 3.4.** Suppose  $\bar{x} \in X$  is not a Nash equilibrium. Then, by weak transfer lower continuity, some player  $i$  has a strategy  $\hat{x}_i \in X_i$  and a neighborhood

$\mathcal{V}(\bar{x}_{-i})$  of  $\bar{x}_i$  such that  $u_i(\hat{x}_i, z_{-i}) > u_i(\bar{x})$  for all  $z_{-i} \in \mathcal{V}(\bar{x}_{-i})$ . Choose  $\epsilon > 0$  such that  $\inf_{z_{-i} \in \mathcal{V}(\bar{x}_{-i})} u_i(\hat{x}_i, z_{-i}) > u_i(\bar{x}) + \epsilon$ . The upper semicontinuity of  $G$  implies that there exists a neighborhood  $\tilde{\mathcal{V}}(\bar{x})$  of  $\bar{x}$  such that  $u_i(\bar{x}) + \epsilon \geq u_i(z)$  for all  $z \in \tilde{\mathcal{V}}(\bar{x})$ . Thus, there exist  $y_i \in X_i$  and a neighborhood of  $\hat{\mathcal{V}}(\bar{x})$  of  $\bar{x}$  such that  $u_i(y_i, z_{-i}) > u_i(z)$  for all  $z \in \hat{\mathcal{V}}(\bar{x})$ . ■

**PROOF OF THEOREM 3.4.** *Sufficiency* ( $\Leftarrow$ ). The proof of sufficiency is the same as that of Theorem 3.2 except the last paragraph. Note that

$$F(x) = \{y \in X : u_i(x_i, y_{-i}) \leq u_i(y), \forall i \in I\},$$

for each  $x \in X$ ,  $F$  is transfer closed valued. Then,  $\bigcap_{x \in X} F(x) = \bigcap_{x \in X} \bar{F}(x)$ . For  $x \in X$ , let  $\bar{F}(x) = \text{cl } F(x)$ . Then,  $\bar{F}(x)$  is closed in the compact set  $X$ . Therefore,  $F(x)$  is a compact subset for every  $x \in X$ . Thus, it suffices to show that the family  $\{F(x)\}_{x \in X}$  possesses the finite intersection property. Indeed, by assumption, for every  $A \in \langle X \rangle$ , there exists  $\hat{y} \in X$  such that  $u_i(x_i, \hat{y}_{-i}) \leq u_i(\hat{y}), \forall x \in A$  and  $\forall i \in I$ . Then, for every  $A \in \langle X \rangle$ , there exists  $\hat{y} \in X$  such that  $\hat{y} \in \bigcap_{x \in A} F(x)$ .

*Necessity* ( $\Rightarrow$ ): Let  $x^* \in X$  be a pure strategy Nash equilibrium of the game  $G$ . Then for all  $i \in I$ ,  $u_i(y_i, x_{-i}^*) \leq u_i(x^*)$  for all  $y \in X_i$ , and thus we have  $\max_{y \in A} u_i(y_i, x_{-i}^*) \leq u_i(x^*)$  for any subset  $A = \{y^1, \dots, y^m\} \in \langle Y \rangle$ . ■

**PROOF OF PROPOSITION 3.5.** Suppose  $x$  is not an equilibrium. Then some player  $i$  has a strategy  $y_i$  such that  $u_i(y_i, x_{-i}) > u_i(x)$ , i.e.,  $F_i(y_i, S_i(y_i, x_{-i})) > F_i(x_i, S_i(x))$ . If  $(y_i, x_{-i}) \in X \setminus D_i$ , then by Assumption 3, there exist a strategy profile  $y'$  and a neighborhood  $\mathcal{V}(x)$  of  $x$  so that for each  $z \in \mathcal{V}(x)$ , there exists a player  $j \in I$  such as  $F_j(y'_j, S_j(y'_j, z_{-j})) > F_j(z_j, S_j(z))$ , i.e.  $u_j(y'_j, z_{-j}) > u_j(z)$ . If  $(y_i, x_{-i}) \in D_i$ , then by Assumption 2, then there exist a player  $j \in I$  and  $y'_j$  such that  $(y'_j, x_{-j}) \in X \setminus D_j$  and  $F_j(y'_j, S_j(y'_j, x_{-j})) > F_j(x_j, S_j(x))$ . Thus, by Assumption 3, there exist a strategy profile  $\tilde{y}$  and a neighborhood  $\mathcal{V}(x)$  of  $x$  so that for each  $z \in \mathcal{V}(x)$ , there exists a player  $k \in I$  such that  $F_k(\tilde{y}_k, S_k(\tilde{y}_k, z_{-k})) > F_k(z_k, S_k(z))$ , i.e.  $u_k(\tilde{y}_k, z_{-k}) > u_k(z)$ . Thus, the game is weakly transfer quasi-continuous. It is also convex, compact, bounded and quasiconcave, then by Theorem 3.1 it has a pure strategy Nash equilibrium. ■

**PROOF OF THEOREM 4.1.** *Sufficiency* ( $\Leftarrow$ ): Suppose the game  $G$  has a dominant strategy equilibrium  $\bar{x} \in X$ . We want to prove that  $G$  is strongly uniformly transfer quasiconcave. Indeed, for any finite subset  $\{y^1, \dots, y^m\} \subset X$ , let the corresponding finite subset  $\{x^1, \dots, x^m\} = \{\bar{x}\}$ . Thus, for any  $J \subset \{1, \dots, m\}$  and each  $x \in \text{co}\{x^h, h \in J\} = \{\bar{x}\}$ , we have  $\forall h \in J$ ,  $u_i(y^h) \leq u_i(\bar{x}_i, y_{-i}^h)$  for each player  $i$ .

*Sufficiency* ( $\Leftarrow$ ). For each  $y \in X$ , let

$$F(y) = \{x \in X : u_i(y) \leq u_i(x_i, y_{-i}), \forall i \in I\}.$$

We first prove that  $F$  is transfer closed valued. Let  $x, y \in X$  with  $x \notin F(y)$ . Then  $x$  is not a dominant strategy equilibrium. By the weak dominant transfer upper quasi-continuity of the game  $G$ , there exist a strategy  $y' \in X$  and a neighborhood  $\mathcal{V}(x)$  of  $x$  so that for every  $z \in \mathcal{V}(x)$ , there exists a player  $i$  such as  $u_i(y') > u_i(z_i, y'_{-i})$ . Therefore, for all  $z \in \mathcal{V}(x)$   $z \notin F(y')$ , i.e.  $x \notin \text{cl } F(y')$ .

For  $y \in X$ , let  $\bar{F}(y) = \text{cl } F(y)$ . Then  $\bar{F}(y)$  is closed, and by the strong uniform transfer quasiconcavity (Remark 4.3), it is also transfer FS-convex. From Lemma 1 in Tian [1993], we deduce  $\bigcap_{y \in X} F(y) = \bigcap_{y \in X} \bar{F}(y) \neq \emptyset$ . Thus, there exists a strategy profile  $\bar{x} \in X$  such that

$$u_i(y) \leq u_i(\bar{x}_i, y_{-i}), \text{ for all } y \in X \text{ and } i \in I.$$

Thus  $\bar{x}$  is a dominant strategy equilibrium of game  $G$ . ■

**PROOF OF THEOREM 4.2.** *Sufficiency* ( $\Leftarrow$ ): For each player  $i \in I$  and every  $(y, x_i) \in X \times X_i$ , let

$$\pi_i(y, x_i) = \sup_{\mathcal{V} \in \Omega(x_i)} \inf_{z_i \in \mathcal{V}} [u_i(y) - u_i(z_i, y_{-i})]$$

where  $\Omega(x_i)$  is the set of all open neighborhoods of  $x_i$ .

For each  $i$  and every  $y \in X$ , the function  $\pi_i(y, \cdot)$  is both real-valued and lower semicontinuous over  $X_i$  (see the sufficiency proof of Theorem 3.3).

If there exists  $\bar{x} \in X$  such that for all  $i \in I$ ,

$$\sup_{y \in X} \pi_i(y, \bar{x}_i) \leq 0,$$

then  $\bar{x}$  is a dominant strategy equilibrium.

Now, suppose, by way of contradiction, that any strategy profile  $x \in X$ ,  $x$  is not a dominant strategy equilibrium. Then, by weakly dominant transfer upper continuous, there exist a player  $i$ , a strategy  $y \in X$  and a neighborhood  $\mathcal{V}(x_i)$  of  $x_i$  such that  $u_i(y) - u_i(z_i, y_{-i}) > 0$  for each  $z_i \in \mathcal{V}(x_i)$ . Thus,

$$\forall x \in X, \text{ there exists } y \in X, i \in I \text{ such that } \pi_i(y, x_i) > 0.$$

Thus,  $X$  can be covered by the following open subsets:

$$\theta_{i,y} = \{x_i \in X_i : \pi_i(y, x_i) > 0\} \times X_{-i}.$$

Since  $X$  is compact, then it can be covered by a finite number of subsets  $\{\theta_{i,y^j} : i \in I \text{ and } j = 1, \dots, m\}$ . Consider a continuous partition of unity  $\{\alpha_{i,j}\}_{\substack{i=1,\dots,n \\ j=1,\dots,m}}$  associated to the finite covering  $\{\theta_{1,y^1}, \dots, \theta_{n,y^m}\}$ .

Since  $G$  is strongly uniformly transfer quasiconcave, then there exists a corresponding finite subset  $\{x^1, \dots, x^m\} \subset X$  such that for each  $\tilde{x} = \sum_{i,j} \lambda_{i,j} x^j \in \text{co}\{x^h, h = 1, \dots, m\}$  and if  $J = \{(i, j) : \lambda_{i,j} > 0\}$ , then

$$\min_{(i,j) \in J} [u_i(y^j) - u_i(\tilde{x}_i, y_{-i}^j)] \leq 0. \quad (6.10)$$

Let us now consider the following function defined on  $X$  into  $X$  by

$$f(x) = \sum_{i,j} \alpha_{i,j}(x) x^j.$$

Since the functions  $\alpha_{i,j}$  are continuous over the compact convex  $X$  into  $X$ , then by Brouwer Fixed-Point Theorem, there exists  $\tilde{x} = f(\tilde{x}) = \sum_{i,j} \alpha_{i,j}(\tilde{x}) x^j$ . Let  $J(\tilde{x}) = \{(i, j) : \alpha_{i,j}(\tilde{x}) > 0\}$ .

If  $(i, j) \in J(\tilde{x})$ , then  $\tilde{x} \in \text{supp}(\alpha_{i,j}) \subset \theta_{i,y^j}$ . Thus,  $\pi_i(y^j, \tilde{x}_i) > 0$  for each  $(i, j) \in J(\tilde{x})$ . Therefore,

$$\min_{(i,j) \in J(\tilde{x})} \pi_i(y^j, \tilde{x}_i) > 0. \quad (6.11)$$

Since  $\pi_i(y^j, \tilde{x}_i) \leq u_i(y^j) - u_i(\tilde{x}_i, y_{-i}^j)$ , then inequalities (6.10) and (6.11) imply  $0 < \min_{(i,j) \in J(\tilde{x})} \pi_i(y^j, \tilde{x}_i) \leq 0$ , which is impossible.

*Necessity* ( $\Rightarrow$ ): It is the same as that of Theorem 3.3, so it is omitted here. ■

**PROOF OF THEOREM 4.3.** *Sufficiency* ( $\Leftarrow$ ). The proof of sufficiency is the same as that of Theorem 4.1 except the last paragraph. Note that

$$F(y) = \{x \in X : u_i(y) \leq u_i(x_i, y_{-i}), \forall i \in I\}.$$

$F$  is transfer closed valued. Then,  $\bigcap_{y \in X} F(y) = \bigcap_{y \in X} \bar{F}(y)$ . For  $y \in X$ , let  $\bar{F}(y) = \text{cl } F(y)$ . Then,  $\bar{F}(y)$  is closed in the compact set  $X$ . Therefore,  $F(y)$  is a compact subset for every  $y \in X$ . Thus, it suffices to show that the family  $\{F(y)\}_{y \in X}$  possesses the finite intersection property. Indeed, by assumption, for every  $A \in \langle X \rangle$ , there exists  $\hat{x} \in X$  such that  $u_i(y) \leq u_i(\hat{x}_i, y_{-i}), \forall y \in A$  and  $\forall i \in I$ . Then, for every  $A \in \langle X \rangle$ , there exists  $\hat{x} \in X$  such that  $\hat{x} \in \bigcap_{y \in A} F(y)$ .

*Necessity* ( $\Rightarrow$ ): Let  $x^* \in X$  be a dominant strategy equilibrium of the game  $G$ . Then for all  $i \in I$ ,  $u_i(y) \leq u_i(x_i^*, y_{-i})$  for all  $y \in X$ , and thus we have  $\max_{y \in A} [u_i(y) - u_i(x_i^*, y_{-i})] \leq 0$  for any

subset  $A = \{y^1, \dots, y^m\} \in \langle X \rangle$  and for all  $i \in I$ . ■

**PROOF OF LEMMA 4.1.** *Necessity* ( $\Rightarrow$ ): Let  $\bar{x} \in X$  be a dominant strategy equilibrium for  $G = (X_i, u_i)_{i \in I}$ . Then,  $u_i(\bar{x}_i, y_{-i}^i) \geq u_i(y^i), \forall y^i \in X, \forall i \in I$ . Hence,  $\phi(\bar{x}, \hat{y}) = \sum_{i=1}^n \{u_i(y^i) - u_i(\bar{x}_i, y_{-i}^i)\} \leq 0, \forall \hat{y} \in \hat{X}$ , i.e.,  $\max_{\hat{y} \in \hat{X}} \phi(\bar{x}, \hat{y}) \leq 0$ . Combining this inequality with inequality (4.3), we have  $\max_{\hat{y} \in \hat{X}} \phi(\bar{x}, \hat{y}) = 0$ .

*Sufficiency* ( $\Leftarrow$ ): Let  $\bar{x} \in X$  be a strategy profile such that  $\max_{\hat{y} \in \hat{X}} \phi(\bar{x}, \hat{y}) = 0$ . This equality implies  $\forall \hat{y} \in \hat{X}, \phi(\bar{x}, \hat{y}) = \sum_{i=1}^n \{u_i(y^i) - u_i(\bar{x}_i, y_{-i}^i)\} \leq 0$ . For each  $i \in I$ , we write  $\phi(\bar{x}, \hat{y}) = u_i(y^i) - u_i(\bar{x}_i, y_{-i}^i) + \sum_{j=1, j \neq i}^n \{u_j(y^j) - u_j(\bar{x}_j, y_{-j}^j)\} \leq 0 \forall \hat{y} \in \hat{X}$ . Letting  $\hat{y} = (\bar{x}, \dots, \bar{x}, y^i, \bar{x}, \dots, \bar{x}) \in \hat{X}$  with  $y^i$  arbitrarily chosen in  $X$ , we have  $\sum_{j=1, j \neq i}^n \{u_j(\bar{x}) - u_j(\bar{x}_j, \bar{x}_{-j})\} = 0$ , and thus  $u_i(y^i) \leq u_i(\bar{x}_i, y_{-i}^i) \forall y^i \in X, i = 1, \dots, I$ . Thus,  $\bar{x}$  is a dominant strategy equilibrium for the game  $G = (X_i, u_i)_{i \in I}$ . ■

**PROOF OF THEOREM 4.4.** It is similar to the proof of Theorem 4.3.

*Necessity* ( $\Rightarrow$ ): Let  $\bar{x} \in X$  be a dominant strategy equilibrium of the game  $G$ . According to Lemma 4.1,  $\phi(\bar{x}, \hat{y}) \leq 0$ , for each  $\hat{y} \in \hat{X}$ . Then, for each  $A \in \langle X^n \rangle$ ,  $\max_{\hat{y} \in A} \phi(\bar{x}, \hat{y}) \leq \max_{\hat{y} \in \hat{X}} \phi(\bar{x}, \hat{y}) = 0$ .

*Sufficiency* ( $\Leftarrow$ ). Let

$$G(\hat{y}) = \{x \in X : \phi(x, \hat{y}) \leq 0\}.$$

Since  $\phi$  is 0-transfer lower continuous in  $x$  with respect to  $\hat{X}$ , then  $G$  is transfer closed valued. Thus,  $\bigcap_{\hat{y} \in X^n} G(\hat{y}) = \bigcap_{\hat{y} \in X^n} \bar{G}(\hat{y})$ . For  $\hat{y} \in X^n$ , let  $\bar{G}(\hat{y}) = \text{cl } G(\hat{y})$ . Then,  $\bar{G}(\hat{y})$  is closed in the compact set  $X$ . Therefore,  $G(\hat{y})$  is a compact subset for every  $\hat{y} \in X^n$ . Thus, it suffices to show that the family  $\{G(\hat{y})\}_{\hat{y} \in X^n}$  possesses the finite intersection property. Indeed, by assumption, for every  $A \in \langle X^n \rangle$ , there exists  $x \in X$  such that  $\phi(x, \tilde{y}) \leq 0$ , for each  $\tilde{y} \in A$ . Then, for every  $A \in \langle X^n \rangle$ , there exists  $\hat{x} \in X$  such that  $\hat{x} \in \bigcap_{\tilde{y} \in A} G(\tilde{y})$ . ■

**PROOF OF PROPOSITION 5.1.** Suppose  $\bar{\mu} \in X$  is not a mixed strategy Nash equilibrium. Then, there exist a player  $i$ , a measure  $\mu_i^* \in M_i$  and a  $\epsilon > 0$  such that

$$U_i(\mu_i^*, \bar{\mu}_{-i}) - \epsilon = \int_X u_i(x) d\mu_i^*(x_i) d\bar{\mu}_{-i}(x_{-i}) - \epsilon > U_i(\bar{\mu}) = \int_X u_i(x) d\bar{\mu}(x). \quad (6.12)$$

Since the game  $G$  is uniformly transfer continuous, then the function  $u_i$  is upper semicontinuous over  $X$  and uniformly payoff secure. According to Proposition 5.1 of Reny [1999], the

function  $\int_X u_i(x) d\bar{\mu}(x)$  is upper semicontinuous in  $\mu$ . Thus, there exists  $\mathcal{V}_1(\bar{\mu})$  such that:

$$\int_X u_i(x) d\bar{\mu}(x) \geq \int_X u_i(x) d\mu(x) - \epsilon/2, \text{ for all } \mu \in \mathcal{V}_1(\bar{\mu}). \quad (6.13)$$

Also, according to the proof of Theorem 1 in Monteiro and Page [2007], there exist a measure  $\tilde{\mu}_i \in M_i$  and a neighborhood  $\mathcal{V}_2(\bar{\mu}_{-i})$  of  $\bar{\mu}_{-i}$  such that

$$\int_X u_i(x) d\tilde{\mu}_i(x_i) d\mu_{-i}(x_{-i}) \geq \int_X u_i(x) d\mu_i^*(x_i) d\bar{\mu}_{-i}(x_{-i}) - \epsilon/2, \text{ for all } \mu_{-i} \in \mathcal{V}_2(\bar{\mu}_{-i}). \quad (6.14)$$

Combining (6.12), (6.13) and (6.14), we conclude: there exist a measure  $\tilde{\mu}_i \in M_i$  and a neighborhood  $\mathcal{V}(\bar{\mu})$  of  $\bar{\mu}$  such that for all  $\mu \in \mathcal{V}(\bar{\mu})$ , we have

$$\begin{aligned} \int_X u_i(x) d\tilde{\mu}_i(x_i) d\mu_{-i}(x_{-i}) + \epsilon/2 &\geq \int_X u_i(x) d\mu_i^*(x_i) d\bar{\mu}_{-i}(x_{-i}) \\ &> \int_X u_i(x) d\bar{\mu}(x) + \epsilon \\ &\geq \int_X u_i(x) d\mu(x) + \epsilon/2 \end{aligned}$$

Thus, the mixed game  $\bar{G}$  is weakly transfer continuous. ■

**PROOF OF PROPOSITION 5.2.** Upper semicontinuity of  $l_i(\cdot)$ ,  $m_i(\cdot)$  and  $\phi_i(\cdot)$ , together with condition a) in Assumption 5.1, implies that the concession game is upper semicontinuous. Condition b) implies that for each  $x_i \in X_i$ , and  $\epsilon > 0$  there exists a strategy  $\bar{x}_i \in X_i$  such that for every  $y_i \in X_{-i}$  there exists a neighborhood  $\mathcal{V}(y_i)$  of  $y_i$  such that  $u_i(\bar{x}_i, z_i) \geq u_i(x_i, y_i) - \epsilon$ , for all  $z_i \in \mathcal{V}(y_i)$ . Then, it is uniformly transfer continuous. It is clear that this game  $G$  is compact, then by Corollary 5.1, we conclude that the game has a mixed strategy Nash equilibrium. ■

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