# The refined best-response correspondence and backward induction

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**Abstract**

Fixed points of the (most) refined best-reply correspondence, introduced in Balkenborg, Hofbauer, and Kuzmics (2009), in the agent normal form of extensive form games have a remarkable, one might call subgame consistency property. They automatically induce fixed points of the same correspondence in the agent normal form of every subgame. Furthermore, in a well-defined sense fixed points of this correspondence refine even trembling-hand perfect equilibria, while, on the other hand, reasonable equilibria that are not even weak perfect Bayesian are fixed points of this correspondence.

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# 1 Introduction

What constitutes a "reasonable" backward induction solution to an extensive form game is debatable. The first concept developed to eliminate some commonly agreed-on unreasonable Nash equilibria is Selten's (1965) subgame perfection. Yet, there are many subgame perfect equilibria that researchers agreed on are not reasonable. Selten (1975) introduced the concept of an extensive form trembling-hand perfect equilibrium, a trembling-hand perfect equilibrium of the agent normal form of the game to further eliminate unreasonable equilibria. This concept, while in many ways a fine one, was "recast and slightly weakened" by Kreps and Wilson (1982) in their sequential equilibrium. Kreps and Wilson (1982) also coined the terms "assessment" and "sequential rationality", now to be found in every textbook on game theory. Kreps and Wilson (1982) thus offer a reinterpretation of (a slightly weaker notion than) extensive form trembling-hand perfect equilibria, in terms of what players believe about what happened so far in the game (when it is their turn to move) and what they should choose given that and given the likely continuation of other players after them. One commonly agreed-on problem of the concept of sequential equilibrium is that, while it is based on these assessments, their justification is still derived from trembles. This is commonly considered theoretically clumsy.[1] Various attempts were made to get rid of these trembles, while maintaining the concept's other attractive properties. Fudenberg and Tirole (1991b) introduce a notion of *perfect Bayesian equilibrium* with this view in mind. Much like subgame perfection a variant of this notion, now often called weak perfect Bayesian equilibrium (see e.g. (Mas-Collel, Whinston, and Green 1995, Definition 9.C.3) or (Ritzberger 2002, Definition 6.2)) and defined for all extensive form games, is now commonly considered a minimal requirement for a reasonable solution to extensive form games. Battigalli (1996) demonstrates that Fudenberg and Tirole's (1991b) "generally reasonable extended assessment[s] ... may violate independence, full consistency, and invariance with respect to interchanging of essentially simultaneous moves.", where full consistency is the criterion an assessment has to justify in addition to the strategy profile being sequentially rational to obtain a sequential equilibrium.

Thus, efforts have been made to find conditions on assessments (not based on trembles) such that ultimately we obtain sequential equilibria, if not extensive form trembling-hand equilibria.

On the other hand, however, there are strategy profiles which are trembling-hand equilibria (or even only weak perfect Bayesian) that are not unreasonable. In all these solution concepts assessments generally need to be justified by the same strategy-profile for all players. It is not a priori clear why this is a reasonable requirement (see the game in Figure 4 in Section 4). Indeed (Fudenberg and Tirole 1991a, pp. 332-333), Bonanno (1995), and Battigalli (1996) argue that this is not a necessary requirement of a reasonable solution and allow for *heterogeneous* assessments. We even argue (see the game in Figure 5 in Section 4) that a player might randomize (at least in the mind of her opponents) over two or more pure strategies, where each pure strategy is independently justifiable by a fully consistent (or some such requirement) assessment, but the mixture itself is not.

We first prove that fixed points of the refined best-reply correspondence, introduced by Balkenborg, Hofbauer, and Kuzmics (2009), in the agent normal form of an extensive form game[2] have the remarkable conceptual consistency property that they automatically induce

[1] Another argument against these trembles underlying the justification of assessments but not of strategy choices, is that it is impractical. See (Battigalli 1996, p. 203).

[2] Such fixed points can be understood as the potential convergence points of a (most refined, while perhaps

fixed points of the same correspondence in the agent normal form of every subgame.

We then prove that fixed points of this refined best-reply correspondence, in the agent normal form, satisfy exactly the properties discussed above. Each pure strategy that is used by some player with positive probability must be justified by, in fact, a stronger requirement than full consistency (even stronger than trembling-hand perfection). However, there does not need to be one assessment justifying all these pure strategies.

The paper proceeds as follows. Section 2 provides the necessary definitions. The subgame consistency property as well as a full characterization of fixed points of the refined best-reply correspondence of the agent normal form are both stated and proven in Section 3. This section also contains the result that the unique subgame perfect equilibrium in generic extensive form games of perfect information is the unique rationalizable strategy profile under the refined best reply correspondence. This section, finally, also contains a brief discussion of the (ideological) relationship of fixed points of the refined best reply correspondence to Fudenberg and Levine's (1993) self-confirming equilibria. Section 4 provides examples demonstrating the degree of "reasonableness" of fixed points of the refined best-reply correspondence, while also documenting one flaw. Finally, we conclude without conclusion, but with two appendices. Appendix A demonstrates how, not surprisingly, none of our results extend to fixed points of the refined best-reply correspondence for the reduced normal form of an extensive form game. Appendix B demonstrates that rationalizability based on the refined best-reply correspondence in the agent normal form has no connection to forward induction reasoning and, thus, very little in common, with the notion of extensive form rationalizability of Pearce (1984) and Battigalli (1997).

# 2 Preliminaries

Let $\Gamma = (I, S, u)$ be a finite $n$-player normal form game, where $I = \{1, ..., n\}$ is the set of players, $S = \times_{i \in I} S_i$ is the set of pure strategy profiles, and $u : S \to \mathbb{R}^n$ the payoff function[3]. Let $\Theta_i = \Delta(S_i)$ denote the set of player $i$'s mixed strategies, and let $\Theta = \times_{i \in I} \Theta_i$ denote the set of all mixed strategy profiles. Let $\text{int}(\Theta) = \{x \in \Theta : \; x_{is} > 0 \; \forall s \in S_i \; \forall i \in I\}$ denote the set of all completely mixed strategy profiles.

For $x \in \Theta$ let $\mathcal{B}_i(x) \subset S_i$ denote the set of pure-strategy best-replies to $x$ for player $i$. Let $\mathcal{B}(x) = \times_{i \in I} \mathcal{B}_i(x)$. Let $\beta_i(x) = \Delta(\mathcal{B}_i(x)) \subset \Theta_i$ denote the set of mixed-strategy best-replies to $x$ for player $i$. Let $\beta(x) = \times_{i \in I} \beta_i(x)$.

As in Balkenborg, Hofbauer, and Kuzmics (2009) we shall restrict attention to games with a normal form in which the set of mixed-strategy profiles $\Psi = \{x \in \Theta | \; \mathcal{B}(x) \text{ is a singleton}\}$ is dense in $\Theta$. We denote this class by $\mathcal{G}^*$.

Not every normal form derived from even a generic extensive form game of perfect information (GEFGOPI) is in $\mathcal{G}^*$. Consider the 1-player extensive form game, given in Figure 1, in which at node 1 the player has two choices, $L$ and $R$, where $L$ terminates the game, while $R$ leads to a second node, where the player again faces two choices $l$ and $r$. The two pure strategies $Ll$ and $Lr$ are obviously equivalent. The semi-reduced normal form has been introduced to eliminate exactly this type of equivalences. The semi-reduced normal form of any GEFGOPI

not necessarily most plausible) learning dynamics. See Balkenborg, Hofbauer, and Kuzmics (2009). While this dynamics is defined for normal form games, there are no problems with extending it to extensive form games, as long as we look at convergence points from interior states only, thus always reaching all information sets. Note that these convergence points may well satisfy more stringent properties than the ones we characterize here.

[3]The function $u$ will also denote the expected utility function in the mixed extension of the game $\Gamma$.

is again in $\mathcal{G}^*$. In the appendix in Balkenborg, Hofbauer, and Kuzmics (2009) it is generally demonstrated that generic finite extensive form games have semi-reduced normal forms that are in $\mathcal{G}^*$. Also the agent normal form of such games is in $\mathcal{G}^*$ as long as no player has 2 or more equivalent actions at any of her information sets.

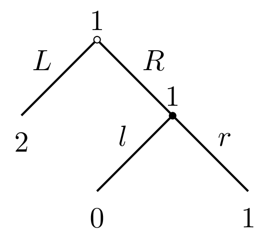


Figure 1: A 1-player extensive form game.

For games in $\mathcal{G}^*$ let $\sigma : \Theta \rightrightarrows \Theta$ be the *refined best-reply correspondence* as defined in Balkenborg, Hofbauer, and Kuzmics (2009) and as follows. For $x \in \Theta$ let

$$\mathcal{S}_i(x) = \{s_i \in S_i | \; \exists \{x_t\}_{t=1}^{\infty} \in \Psi : x_t \to x \wedge \mathcal{B}_i(x_t) = \{s_i\} \; \forall t\}.$$

Then $\sigma_i(x) = \Delta\left(\mathcal{S}_i(x)\right)$ and $\sigma(x) = \times_{i \in I} \sigma_i(x) \; \forall \; x \in \Theta$.

# 3 Results

Fixed points of $\sigma$ in the agent normal form have a surprising property. They induce fixed points of $\sigma$ in every subgame. This is much more than saying that fixed points of $\sigma$ are subgame perfect. One might call it a *conceptual consistency* property.

**Proposition 1** *Let $\Gamma \in \mathcal{G}^*$ be the agent normal form of a given extensive form game. Then if a strategy profile $x$ is a fixed point of $\sigma$ it is also a fixed point of $\sigma$ in the agent normal form of every subgame of this extensive form game.*

Proof: Let $\Gamma \in \mathcal{G}^*$ be the agent normal form of the given extensive form game. Let $\Theta$ denote the space of mixed strategies. Let $x \in \Theta$ be a fixed point of $\sigma$. Consider player (agent) $i \in I$, where $I$ is the set of all agents. Player $i$ only moves once, i.e., has only one information set. By the definition of $\sigma_i(x) = \Delta\left(\mathcal{S}_i(x)\right)$, player $i$, in $x_i \in \sigma_i(x)$ is exclusively randomizing over pure strategies, each of which are unique best replies to some $x' \in \Theta$ (possibly different for different $s_i \in \mathcal{S}_i$) in a neighborhood of $x$. In fact, for each $s_i \in \mathcal{S}_i(x)$, and, hence, for each $s_i \in C(x_i)$ we have that there is a sequence of $x^t \in \Theta$ such that $x^t \to x$ and $\mathcal{B}_i(x^t) = \{s_i\}$. Now consider any subgame in which player $i$ also moves. Let $\Gamma' = (I', S', u')$ denote its agent normal form. Obviously $I' \subset I$ and for all $i \in I'$ we have $S'_i = S_i$ and $u'$ is defined accordingly. Now for every $s_i \in C(x_i)$ consider the projection of $x^t \in \Theta$ onto the reduced game $\Gamma'$. Let it be denoted by $\hat{x}^t \in \Theta'$. Hence we simply have that $\hat{x}^t_i = x^t_i$ for all $i \in I'$. Now consider for the exact sequence of $x^t \in \Theta$ such that $x^t \to x$ and $\mathcal{B}_i(x^t) = \{s_i\}$ its projection. Obviously we have that $\hat{x}^t \to \hat{x}$ and also we must have that $\mathcal{B}'_i(\hat{x}^t) \subset \mathcal{B}_i(x^t)$. This is so because either in $x$ player $i$'s information is reached, in which case player $i$'s best responses cannot change in the subgame, or player $i$'s information set is not reached in $x$, and, hence, every strategy of player $i$ is a best response against $x$ in the full game. But now given $\mathcal{B}_i(x^t) = \{s_i\}$ we must also have $\mathcal{B}'_i(\hat{x}^t) = \{s_i\}$ for the whole sequence of $\hat{x}^t$. But this is nothing but saying that $s_i \in \mathcal{S}'_i(\hat{x})$ and, as this is true for all $s_i \in C(x_i)$ and all players $i \in I'$ we have that $\hat{x}$ is a fixed point of $\sigma$ in $\Gamma'$. QED

In fact a fixed point of $\sigma$ of the agent normal form of an extensive form game is both stronger in some respects and weaker in others than a sequential equilibrium. To clarify these issues we separate the equilibrium definitions of the various equilibrium concepts into parts.

For a given strategy profile $x \in \Theta$ in the agent normal form of an extensive form game $\Gamma$ let $\mu$ be a system of probability distributions, one for each information set (and, thus, one for each agent) of the extensive form game. This system $\mu$, which must be derived from $x$ using Bayes' rule whenever possible, is often called a **system of beliefs** and the pair $(x, \mu)$ an **assessment**. Recall that $C(x_i) \subset \Theta_i$ denotes the support of mixed strategy $x_i$ of player $i$. Given assessment $(x, \mu)$ we call $s_i \in C(x_i)$ **sequentially rational** if it maximizes the conditional expected payoff given $(x, \mu)$ at agent $i$'s (only) information set.

**Definition 1** *A strategy profile $x \in \Theta$, in the agent normal form of an extensive form game, is* **very weakly idio-justifiable** *if for every $s_i \in C(x_i)$ and every player $i \in I$ there is a system of beliefs $\mu = \mu(s_i)$ such that $s_i$ is sequentially rational given $(x, \mu)$. It is* **weakly idio-justifiable** *if for every $s_i \in C(x_i)$ and every player $i \in I$ there is sequence of assessments $(x^k, \mu^k)$ such that $x^k$ is in the strict interior of $\Theta$ and converges to $x$ and $\mu^k$ is the appropriate and unique system of beliefs derived from $x^k$ using Bayes' rule and converges to $\mu = \mu(s_i)$ such that $s_i$ is sequentially rational for player $i$ given $(x, \mu)$. It is* **strongly idio-justifiable** *if it is weakly idio-justifiable and each $s_i \in C(x_i)$ is also sequentially rational for every assessment along the sequence. It is* **very strongly idio-justifiable** *if for every $s_i \in C(x_i)$ and every player $i \in I$ there is an open set $U^x \subset int(\Theta)$, with closure containing $x$, such that $s_i$ is sequentially rational also for all assessments $(x', \mu')$, where $x' \in U^x$ and $\mu'$ derived from $x'$ using Bayes' rule. A strategy profile is* **very weakly, weakly, strongly, or very strongly pan-justifiable** *if it is very weakly, weakly, strongly, or very strongly idio-justifiable, respectively, and, in addition, the assessment, or the sequence or open set of assessments, that respectively justifies any one of the various pure strategies $s_i \in C(x_i)$ for the various players $i \in I$ is the same for all $s_i \in C_i(x_i)$ and all players $i \in I$.*

Note that when a possibly mixed strategy profile $x$ is idio-justifiable (of some degree) it means that **each of its parts**, i.e. each pure strategy in the support of some player's strategy part of $x$, is **idiosyncratically** justifiable through its **very own** assessment, or sequence of assessments, or open set of assessments (depending on which degree of idio-justifiability we speak of). Thus, even for a single player we might have two different assessments justifying two different pure strategies in the support of this player's mixed strategy. When a possibly mixed strategy profile $x$ is pan-justifiable (of some degree) it means that it is idio-justifiable (of the same degree) and, in addition, there is a single assessment, or sequence of assessments, or open set of assessment, that justifies **all** pure strategies in the support of **any** player's part of $x$.

Obviously very strong idio-justifiability implies strong idio-justifiability, which implies weak idio-justifiability, which, in turn, implies very weak idio-justifiability. The same is true for the respective four notions of pan-justifiability. Also any level of pan-justifiability implies the same level of idio-justifiability by definition. However there are no other relationships. I.e., it is not true that very strong idio-justifiability implies even very weak pan-consistency.

Using these definitions we can give characterizations of the various equilibrium notions in terms of these properties. A strategy profile $x$ is weak perfect Bayesian (see e.g., Definition 9.C.3

in Mas-Collel, Whinston, and Green (1995) or Definition 6.2 in Ritzberger (2002)) if and only if it is very weakly pan-justifiable. Note that, although there is a unique assessment that thus justifies a weak-perfect Bayesian equilibrium, this does not imply that the beliefs of any two agents, whose information sets are off the equilibrium path, have to be in any way consistent. It only requires that if one-player uses two or more pure strategies in this equilibrium, all these have to be justifiable by the same assessment. A strategy profile is then sequential (Kreps and Wilson (1982)), i.e. its assessment is fully consistent (Battigalli (1996)), if and only if it is weakly pan-justifiable. Note that weak pan-justifiability does require consistency of beliefs of players. A strategy profile is extensive form trembling hand perfect (Selten (1975)) if and only if it is strongly pan-justifiable. Note that they all share the pan prefix. A fixed point of $\sigma$ in the agent normal form of an extensive form game does not satisfy any of these pan-notions as the game in Figure 5 in Section 4 demonstrates. However, any fixed point of $\sigma$ in the agent normal form of an extensive form game is very strongly idio-justifiable.

**Proposition 2** *Let $\Gamma \in \mathcal{G}^*$ be the agent normal form of an extensive form game. A strategy profile $x$ is a fixed point of $\sigma$ if and only if it is very-strongly idio-justifiable.*

Proof: Let $s_i$ be in the support of player $i$'s part of $x$, i.e. $s_i \in C(x_i)$. Strategy profile $x$ is a fixed point of $\sigma$ if and only if, by the definition of $\sigma$, there is an open set $U^x \subset \text{int}(\Theta)$ with closure containing $x$, such that $s_i \in \mathcal{B}_i(y)$ for any $y \in U^x$. Note that this set $U^x$ in general depends on the player $i$ and on pure strategy $s_i$ and can well be different for different pure strategies and different players. For all these $y$ there is a unique system of beliefs $\mu^y$ derived from $y$ using Bayes' rule. Again, the thus derived beliefs $\mu^y$ can be different for different pure strategies $s_i' \in C(x_i)$ and for different players. Then $s_i \in \mathcal{B}_i(y)$ if and only if it is also sequentially rational given $(y, \mu^y)$. QED

Thus fixed points of $\sigma$ of the agent normal form of an extensive form game are more strongly justifiable than weak-perfect Bayesian, sequential, and extensive form trembling hand equilibria in terms of our notion of idio-justifiability. Thus, every single pure strategy used in a fixed point of $\sigma$ is justifiable in a very strong sense. The sense in which fixed points of $\sigma$ are weaker than even weak perfect Bayesian equilibria is that for fixed points of $\sigma$ we cannot necessarily guarantee that the system (or systems) of beliefs that justifies any single pure strategy in its support is the same that justifies other pure strategies in its support. This is, thus, reminiscent of Fudenberg and Levine's (1993) self-confirming equilibria, in which players can disagree about other players' strategies. Note, however, that for fixed points in $\sigma$ in the agent normal form, players can only disagree about how to interpret deviations from the prescribed strategy profile. They cannot disagree as to what a given player's originally intended strategy choice was. Thus, fixed points of $\sigma$ can remain to be Nash equilibria, while even the most stringent of self-confirming equilibria, which are the rationalizable self-confirming equilibria of Dekel, Fudenberg, and Levine (1999), may not be Nash equilibria. In some sense, thus, fixed points of $\sigma$ of the agent normal form are highly justifiable self-confirming equilibria within the bounds of being Nash equilibria as well.

**Corollary 1** *Let $\Gamma \in \mathcal{G}^*$ be the agent normal form of an extensive form game. Then if a pure strategy profile $s$ is a fixed point of $\sigma$ it induces a weak-perfect Bayesian equilibrium in every subgame.*

Proof: follows immediately from Propositions 1 and 2 and the realization that there is only one strategy in the support of each players $x_i$. QED

Still, Corollary 1 is really a gross understatement, as we know from Proposition 2. Another example that demonstrates that being a pure fixed points of $\sigma$ is a much stronger requirement than being a weak perfect Bayesian equilibrium is the game in Figure 2, which is taken from (Battigalli 1996, Fig. 1). In this game $(R_1, R_2, R_3)$ is weak perfect Bayesian for all $u \leq 1$. Note first that $R$ is strictly dominant for player 1. However, for all strategy profiles close to player 1 playing $R_1$, $R_3$ is only best on a thin set (where player 2 puts probability 0 on $L_2$). Thus, $R_3$ is not in $\mathcal{S}_3(x)$ for any $x \in \Theta$ close to $(R_1, R_2, R_3)$. Thus $(R_1, R_2, R_3)$ is not a fixed point of $\sigma$. Indeed, the only fixed point of $\sigma$ is the "reasonable" equilibrium $R_1, L_2, L_3$.

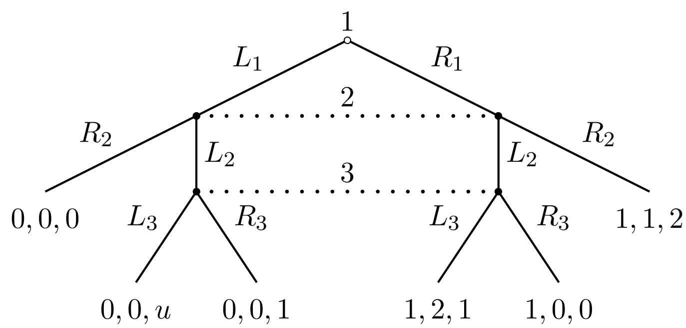


Figure 2: A game with a pure weakly perfect Bayesian equilibrium $(R, R, R)$ for all $u \leq 1$, which is not a fixed point of $\sigma$.

Define a notion of rationalizability (Bernheim (1984) and Pearce (1984)) based on $\sigma$ as follows. For $A \subset \Theta$ let $\mathcal{S}_i(A) = \bigcup_{x \in A} \mathcal{S}_i(x)$. Let $\sigma_i(A) = \Delta(\mathcal{S}_i(A))$. Let $\sigma(A) = \times_{i \in I} \sigma_i(A)$. For $k \geq 2$ let $\sigma^k(A) = \sigma\left(\sigma^{k-1}(A)\right)$. For $A = \Theta$, $\sigma^k(A)$ is a decreasing sequence, and we denote $\sigma^\infty(\Theta) = \bigcap_{k=1}^{\infty} \tau^k(\Theta)$ the set of $\sigma$-rationalizable strategies.

**Proposition 3** *Let $\Gamma \in \mathcal{G}^*$ be the agent normal form of a generic extensive form game of perfect (and complete) information (GEFGOPI). Then only the (unique) subgame-perfect strategy profile is $\sigma$-rationalizable.*

Proof: Consider a final node. A strategy, available to the player, say, $i$ at this node, which is not subgame perfect is weakly dominated. Hence, it can not be in $\mathcal{S}_i(x)$ for any $x \in \Theta$. So it is not in $\sigma(\Theta)$. Now consider an immediate predecessor node to the above final node. A non-subgame perfect strategy at this node can only be a best-reply if the behavior at the following nodes is non-subgame perfect. For any $x \in \Theta$ in a neighborhood of $\sigma(\Theta)$ this is still true. Hence, any such non-subgame perfect strategy at this node can not be in $\sigma^2(\Theta)$. This argument can be reiterated any finite number of times. QED

**Corollary 2** *Let $\Gamma \in \mathcal{G}^*$ be the agent normal form of a GEFGOPI. The only fixed point of $\sigma$ for this game is the (unique) subgame perfect equilibrium.*

Proof: Every fixed point of $\sigma$ must be in the set of $\sigma$-rationalizable strategies. This set, by Proposition 3, only consists of the subgame perfect equilibrium. QED

This corollary also follows from Proposition 1 and the fact that $\sigma$ is a refinement of $\beta$.

## 4 Examples

Our first example demonstrates that not every sequential equilibrium is necessarily a fixed point of $\sigma$. The game given in Figure 3, Figure 13 in Kreps and Wilson (1982), has a sequential equilibrium $(L, r)$ which is not a fixed point of $\sigma$ (it is, in fact, also not perfect). Here the agent normal form and the semi-reduced normal form are the same and given as Game 1. This game demonstrates that some (very) unreasonable sequential equilibria (here one that involves playing a pure weakly dominated strategy) are refined away by appealing to fixed points of $\sigma$.

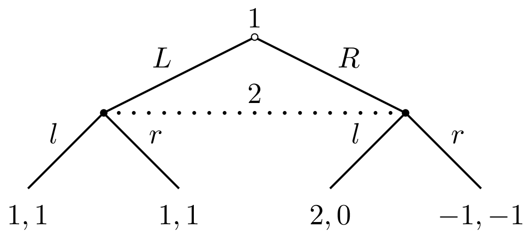


Figure 3: A game with a sequential equilibrium $(L, r)$ which is not a fixed point of $\sigma$.

| | l | r |
|---|---|---|
| L | 1,1 | 1,1 |
| R | 2,0 | -1,-1 |

Game 1: The normal form of the game in Figure 3.

The following example demonstrates that fixed points of $\sigma$, while very strongly idio-justifiable, are not weakly pan-justifiable in the sense that two players, who in a sequential equilibrium have to have the same assessment, are allowed to have different assessments in fixed points of $\sigma$. Consider the game in Figure 4. The Nash equilibrium $(A, R, r)$ is a fixed point of $\sigma$, but is not sequential (and, hence, not extensive form trembling hand perfect).

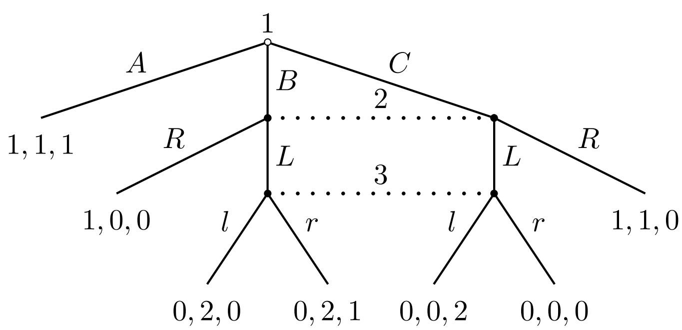


Figure 4: A game in which there is a fixed point of $\sigma$ in the agent normal form which is not sequential (and, hence, not extensive form trembling hand perfect).

To see that $(A, R, r)$ is a fixed point of $\sigma$ we need to check that each strategy choice is a best reply in an open set around $(A, R, r)$. For player 1's choice $A$ this is definitely true as $A$ weakly dominates both $B$ and $C$. Player 2's choice $R$ is best as long as player 1 is sufficiently more likely to tremble to $C$ than to $B$. In fact the probability of $C$ has to be at least twice that of $B$. Player 2's payoffs are unaffected by player 3's choice. Player 3's choice $r$ is best as long as player 1 trembles sufficiently more to $B$ than to $C$. In fact the probability of $B$ has to be at least twice that of $C$. This is true for whatever player 2 does. Hence, for each player's strategy choice there is an open set of strategy profiles around $(A, R, r)$ against which the player's choice is a best reply. Hence, $(A, R, r)$ is indeed a fixed point of $\sigma$. However, these open sets (for players 2 and

3) are mutually exclusive. This in turn means that there is no system of consistent beliefs for players 2 and 3 which make both choices $R$ and $r$ best replies simultaneously. Player 2's belief that sustains the $(A, R, r)$ equilibrium is such that his first node has conditional probability of at most 1/3. Player 3's belief that sustains the $(A, R, r)$ equilibrium is such that her first node has conditional probability of at least 2/3. But in a sequential equilibrium these two beliefs would have to coincide. Thus this $(A, R, r)$ is not sequential (and not trembling-hand perfect).

The next example (Game 5)demonstrates that a mixed fixed-point of $\sigma$ need not be even weak perfect Bayesian. Thus, a very strongly idio-justifiable equilibrium does not even need to be very weakly pan-justifiable. However, we argue that this fixed point of $\sigma$ is very reasonable.

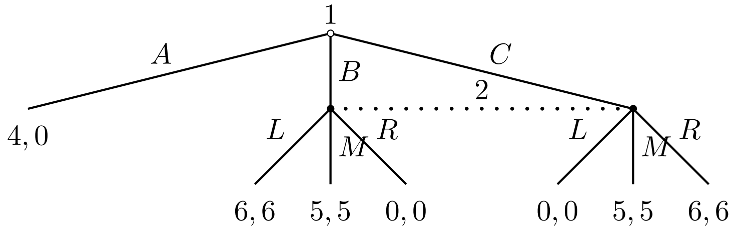


Figure 5: A game in which there is a fixed point of $\sigma$ in the agent normal form which is not weak perfect Bayesian.

Consider the strategy profile $x^*$, in which player 1 plays $A$ and player 2 puts probability $\frac{1}{2}$ each on $L$ and $R$. Note first that this is indeed a Nash equilibrium as $A$ is best when player 2 behaves thus, and player 2 is indifferent between all strategies when player 1 chooses $A$. To see that this is also a fixed point of $\sigma$ note that $A$ continues to be the unique best response of player 1 in a small enough neighborhood of player 2's strategy. Strategy $L$ for player 2 is a unique best reply for some open set of mixed strategy profiles close to $x^*$, all these that put sufficiently higher probability on player 1's $B$ than on $C$. Similarly player 2's strategy $R$ is the unique best reply for some open set of mixed strategy profiles close to $x^*$, all these that put sufficiently higher probability on player 1's $C$ than on $B$. Thus $x* \in \sigma(x^*)$. However, $x^*$ is not weak perfect Bayesian. There is in fact no assessment of player 2's that would make player 1's best response randomizing equally between $L$ and $R$. This is so, because, to induce player 2 to randomize she has to be indifferent between both strategies, which is only true if her assessment is that both nodes in her information set are equally likely. In that case, however, player 2's strategy $M$ dominates.

There is, however, a very good case to be made that $x^*$ is indeed reasonable. This is so, if we interpret a mixed strategy profile not so much as a probability distribution over the actual pure strategies chosen, but rather a belief of opponent players about this players choice of pure strategy. In the case at hand the argument would be as follows. Player 1 chooses $A$ because player 1 does not know how player 2 would interpret a deviation to either $B$ or $C$, which player 2 cannot distinguish. Player 1 might think it is equally likely that player 2 will react by playing $L$ or by playing $R$ (because, presumably player 2 has a clear assessment of player 1's intended choice, only player 1 does not know what that is). Thus, this equilibrium, interpreted as beliefs about opponent strategies, is completely reasonable[4]. In fact this is a weak perfect Bayesian equilibrium in this game if we interchange the two players' essentially simultaneous moves. Consider the thus modified game in Figure 6.

[4]Of course, this game has additional, also reasonable, equilibria, which are also fixed points of $\sigma$.

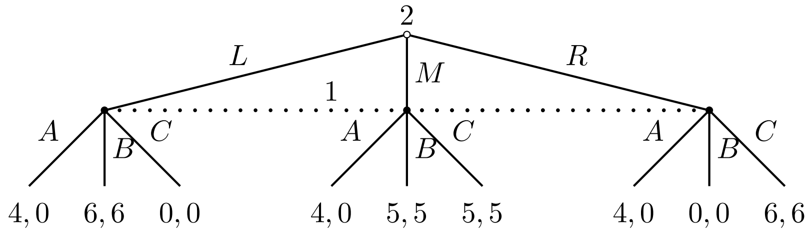


Figure 6: This game is derived from the game in Figure 5 by interchanging the two players' essentially simultaneous moves.

In this game $x^*$ is weak perfect Bayesian. In fact, it is even sequential, given that every information set is reached. This demonstrates the (already mentioned) point made by Battigalli (1996) that weak perfect Bayesian equilibria are not invariant under the interchanging of essentially simultaneous moves. Fixed points of $\sigma$, on the other hand, naturally satisfy this invariance property as they are defined for the agent normal form of the game, which does not change when such essentially simultaneous moves are interchanged.

We should note, however, that as much as we think of fixed points of $\sigma$ in the agent normal form as the "ultimate" backward induction solution concept, it does have its flaw. It is not immune to splitting information sets into parts, as the same example demonstrates.

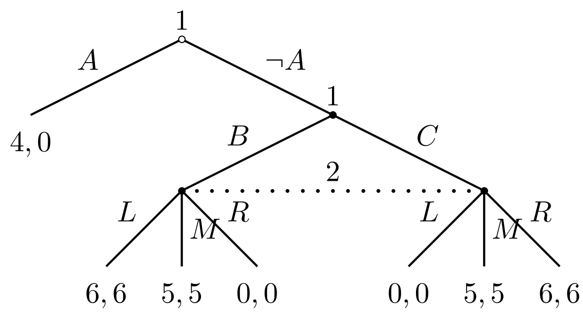


Figure 7: This game is derived from the game in Figure 5 by splitting player 1's move into two sequential moves.

Of course, one problem is that this game has a different agent normal form, as it now has 3 players. However, that alone is not necessarily a problem. The problem is that in any strategy profile of the agent normal form of this game, player 1's choice between B and C now has to be specified. This is equivalent to specifying player 1's possible deviation to B and C in the original game. Thus player 2 in this game is no longer free to interpret how play arrived at her information set. While it may be a deviation of the first player 1 to play $\neg A$, it does not take a deviation of the second player 1 to get to this information set.

Note that $x^*$, the fixed point of $\sigma$ in the original game in Figure 5, is still part of a Nash and Fudenberg and Levine's (1993) self-confirming equilibrium of this game. It is, however, no longer (part of) a fixed point of $\sigma$ in this game (it is not even subgame perfect).

# A Semi-reduced normal form

In this appendix we show by example, that none of the results in this paper extend to the semi-reduced normal form even of a generic extensive form game of perfect information (GEFGOPI).

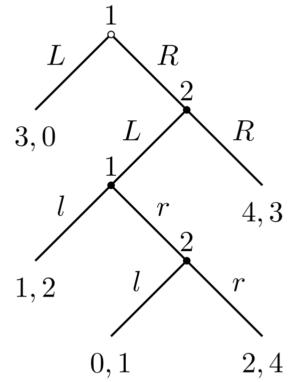


Figure 8: A centipede game.

| | D | E | F |
|---|---|---|---|
| A | 3,0 | 3,0 | 3,0 |
| B | 4,3 | 1,2 | 1,2 |
| C | 4,3 | 0,1 | 2,4 |

Game 2: The normal form game of the centipede game in Figure 8.

Consider the centipede game (Figure 8.2.2 in Cressman (2003)) given here in Figure 8 with semi-reduced normal form given as Game 2, where player 1's strategies are $A = Ll|Lr$, $B = Rl$, and $C = Rr$, while player 2's strategies are $D = Rl|Rr$, $E = Ll$, and $F = Lr$. This game is a GEFGOPI and, hence, has a unique subgame perfect equilibrium, which is $(Lr, Lr)$. The non-subgame perfect, non weak-perfect Bayesian, and, hence, non-sequential, Nash equilibrium $(B, D)$ is a fixed point of $\sigma$. So indeed, fixed points of $\sigma$ in a given normal form game need not induce sequential or even weak perfect Bayesian equilibria in every extensive form game with this semi-reduced normal form.

# B Forward Induction

In this section we provide a (well-known) example that demonstrates that rationalizability based on, and fixed points of, the refined best-reply correspondence have no relation to forward induction solutions.

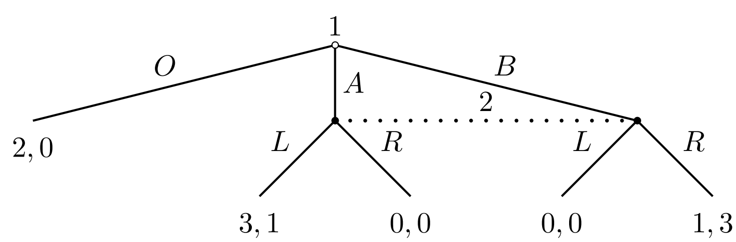


Figure 9: Battle of the Sexes with an outside option.

The strategy profile $(O, R)$ is a fixed point of $\sigma$. This is so because $O$ is best against an open set of strategy profiles close to player 2's $R$ and $R$ is best against an open set of strategy profiles close to $O$ (in which player 1 uses $B$ sufficiently more than $A$). This emphasizes the fact that fixed points of $\sigma$ are exclusively about backward induction, as any deviation from the equilibrium play is essentially interpreted as a tremble or mistake. This is, thus, fundamentally different from forward induction reasoning. According to forward induction reasoning $O, R$ should not be played, because a deviation of player 1 into player 2's information set should be interpreted by player 2 as a clear attempt to go for the other equilibrium $A, L$. This is so, if player 2 tries to maintain as much as possible her original hypothesis that her opponent, player 1, is rational (see

Battigalli and Siniscalchi (2002)). For fixed points of $\sigma$ any such deviation is simply understood as a mistake.

This also implies that $\sigma$-rationalizability, thus, has not much in common with extensive form rationalizability (Pearce (1984) and Battigalli (1997)), which, as shown by Battigalli and Siniscalchi (2002) is almost more related to forward induction than to backward induction.

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