A Relationship between Risk and Time Preferences *

Kota SAITO †
Department of Economics, Northwestern University
kotasaito@northwestern.edu

June 12, 2009

Abstract

This paper investigates a general relationship between risk and time preferences. I consider a decision maker who chooses a one-time consumption. I assume however, that he believes both that today's good is certain, and that, as the promised date for future goods becomes increasingly distant, the probability of his consuming the goods decreases continuously to zero. For example, the probability might be the subjective mortality rate of a decision maker or the objective hazard rate of future goods. Under the assumptions specified above, the present paper shows that (i) a decision maker exhibits the common ratio effect if and only if he discounts hyperbolically; (ii) he exhibits the certainty effect if and only if he discounts quasi-hyperbolically; and (iii) he exhibits the expected utility if and only if he is temporally unbiased (an exponential discounter).

Keywords: Allais paradox; hyperbolic discounting; preference over lotteries; intertemporal consumption.

*This paper is a generalization of Saito, K. (2008) "Hyperbolic Discounting: It's Just Allais Paradox," University of Tokyo, COE Discussion Papers, No. F-218.

†I wish to thank Eddie Dekel, Jeff Ely and Michihiro Kandori for their excellent guidance. I am also grateful to Kazuki Baba, Colin Camerer, Faruk Gul, Kazuya Kamiya, David Levine, Akihiko Matsui, Shinjiro Miyazawa, Daisuke Nakajima, Arthur Robson, Balazs Szentes and seminar audiences at Northwestern.

1 Introduction

Conventional wisdom recognizes that the future is uncertain in many respects. Several researchers, therefore, have claimed that there should be a relationship between risk and time preferences. Based on this intuition, they have tried to explain future discounting on the basis of the uncertainty associated with future payoffs. These approaches are not completely satisfactory, however, because each paper imposes different restrictive assumptions on utility functions and the probability functions representing the future uncertainty. It is therefore difficult to identify the fundamental relationship between risk and time preferences.[1]

The purpose of the present paper is to establish a general theory on the relationship between risk and time preferences, without assuming specific forms of utility functions and probability distributions. To achieve this purpose as simply as possible, I consider a decision maker who chooses a one-time consumption. However, he discounts the future payoff because it is uncertain whether he can consume it or not. I assume a weak condition on the probability function representing the uncertainty. This condition means that today's good is certain, but as the promised date for future goods becomes increasingly distant, the probability of consuming the goods decreases continuously to zero. I call the above property *regular*. For example, if the probability reflects the decision maker's subjective mortality rate or an objective hazard rate for future goods, the regular condition would be reasonable.

The theorem of this paper shows the following: (i) a decision maker exhibits the common ratio effect if and only if he discounts hyperbolically; (ii) he exhibits the certainty effect if and only if he discounts quasi-hyperbolically; and (iii) he exhibits the expected utility if and only if he is temporally unbiased (an exponential discounter). One implication of the theorem would be that the certain delivery of present goods makes subjects present biased. This implication is compatible with the experimental evidence found by Keren and Roelofsma (1995), which finds that a present bias disappears when the outcome become uncertain.

Since most of the conventional research satisfies the regularity condition, the present paper may be viewed as a generalization of that research. In particular, parts (i) and (ii) of the theorem are a significant generalization of Halevy (2008) and Epper et al. (2009).[2] Although some authors (such as, Loewenstein and Prelec (1992)) had suggested an analogy between risk and time preferences, the precise relationship between the two had not been formally studied until Halevy (2008). Both Halevy (2008) and Epper et al. (2009) assume a constant Poisson mortality rate. Halevy (2008) shows, within a class of Yarri (1987)'s rank-dependent utilities, that the common ratio effect

[1] For example, many papers are only interested in finding the specific probability (hazard) function to describe hyperbolic discounting, under the implicit assumption of expected utility theory. So in this sense, they study only the one-way relationship from expected utility theory to hyperbolic discounting.

[2] I learned of Epper et al. (2009) after the present paper had been completed and presented.

implies quasi-hyperbolic discounting.[3] Epper et al.(2009) shows the same result within a class of utilities of Prelec (1998)'s prospect theory. In the present paper, I drop any restriction on the preferences and the assumption of a constant Poisson mortality rate. Nevertheless, thanks to the flexibility of the model, I can generalize the conclusion. Under the aforementioned regularity condition on probability function, I obtain three relationships between risk and time preferences. The "only if" component of part (ii) of the theorem implies the conclusion about the relationship between risk and time preferences drawn in both Halevy (2008) and Epper et al. (2009), because hyperbolic discounting entails quasi-hyperbolic discounting.

Part (iii) of the theorem implies that the hazard function approach of "explaining" deviations from exponential discounting by assuming that prizes are not received with some probability but otherwise using a standard expected utility model, as is currently prevalent in psychology and biology, cannot succeed. For example, Kagel et al. (1986) and Green and Myerson (1996) argue that the decreasing rate of the Poisson hazard rate over time leads to hyperbolic discounting.[4] However, part (iii) of the theorem shows that this approach must lead to temporally unbiased preferences, i.e., to dynamic consistency. That is because the probability (survival) function defined from the hazard function satisfies the regularity condition of this paper. In fact, most researchers who adopt the hazard function approach describe a preference reversal in *static* decision making, but not in dynamic decision making.[5] In contrast, the theorem herein suggests two ways of using a hazard function approach to successfully describe dynamic inconsistency. One is to assume non-regular uncertainty, such as uncertainty about the timing of consumption (as in, Dasgupta and Maskin (2005)). The other is to assume nonexpected utility, such as rank dependent utilities (as in, Halevy (2008)) or prospect theory (as in, Epper et al. (2009)), or-as part (ii) of the theorem shows more generally-assuming the common ratio effect.

The present paper also sheds light on certain aspects of static decision making, as discussed in Baucells and Heukamp (2008), in which a specific representation of preferences over lotteries with delay is obtained. Their representation suggests a relationship between the common ratio effect and the common difference effect (a preference reversal

[3]Halevy (2008) claims that the common ratio effect is equivalent to quasi-hyperbolic discounting. However, one direction of the equivalence (quasi-hyperbolic discounting $\Rightarrow$ the common ration effect) turns out to be false. I will explain this point in the appendix.

[4]Sozou (1998) offers an alternative theory in which that the hazard rate is constant but unknown to the decision maker.

[5]As Dasgupta and Maskin (2005) point out, there are two distinct meanings for the term "hyperbolic discounting." One applies to dynamic decision making with variable decision times. The other applies to static decision making with fixed decision times. Most of the theoretical works, Laibson(1997), O'Donoghue and Rabin (1999), and Dasgupta and Maskin (2005) for example, are interested in the dynamic concept because of dynamically inconsistent behavior. I also focus on the dynamic concept.

in static decision making).[6] In contrast, the present paper focuses on dynamic decision making and obtains a more general relationship between risk and time preferences.

The rest of the paper is organized as follows. Section 2 formally defines preferences exhibiting the Allais paradox. Section 3 defines preferences exhibiting hyperbolic and quasi-hyperbolic discounting. Section 4 shows the theorem. Section 5 constitutes the appendix.

2 The Allais Paradox

In this section, I consider a risk preference $\succsim^r$ on the set of binary lotteries, defined as follows:

$$\Delta = \Big\{(x, p; 0, 1-p) \;\Big|\; x \in X \text{ and } p \in [0,1]\Big\},$$

where X is a non-degenerate closed interval on $\mathbb{R}$ including 0. I formally define the common ratio effect and the certainty effect on the preference $\succsim^r$, which are typical effects of the Allais paradox. The common ratio effect is characterized as follows: Suppose that subjects choose either a safer option which gives a smaller gain x with a higher probability η, or a riskier option which gives a larger gain y with a lower probability $\eta\mu$, where $\mu < 1$. As η falls, subjects switch their choice from the safe option to the risky option. Note that for both options, reducing η means increasing the risk of getting nothing. Formally, the common ratio effect is defined as follows:

Definition: $\succsim^r$ is said to exhibit *the common ratio effect*[7] if, for any $x, y \in X$ and $\mu, \tilde{\eta} \in [0,1]$ such that $(x, \tilde{\eta}) \sim^r (y, \tilde{\eta}\mu)$,

$$(x, \eta) \prec^r (y, \eta\mu) \text{ for all } \eta \in (0, \tilde{\eta}) \text{ and } (x, \eta) \succ^r (y, \eta\mu) \text{ for all } \eta \in (\tilde{\eta}, 1].$$

This definition appears in Starmer (2000, p. 337). The general definition provided by Machina (1982, p. 305) also becomes equivalent to the above definition within the set of simple binary lotteries. This tendency is called the certainty effect specifically in regard to the choice between a sure option and a risky option. So the condition characterizing the certainty effect is the special case of the common ratio effect, when $\tilde{\eta} = 1$:

[6] In the appendix, I will examine relationship between risk and time preferences in static decision making and show a corollary which is analogous to the theorem of the present paper. The corollary includes equivalence between the common ratio effect and the common difference effect.

[7] Under the standard assumption of monotonicity and continuity axioms, for any $x, y \in X$ and $\tilde{\eta} \in [0,1]$, there exists μ such that $(x, \tilde{\eta}) \sim^r (y, \tilde{\eta}\mu)$. So the condition cannot be satisfied by any trivial way.

Definition: $\succsim^r$ is said to exhibit *the certainty effect* if, for any $x, y \in X$ and $\mu \in [0,1]$ such that $(x,1) \sim^r (y,\mu)$,

$$(x,\eta) \prec^r (y,\eta\mu) \text{ for all } \eta \in (0,1].$$

By definition, if a decision maker exhibits the common ratio effect, then he exhibits the certainty effect.[8]

Finally, in the set Δ of binary lotteries, the independence axiom reduces to the following:

Definition: $\succsim^r$ is said to satisfy *the independence axiom* if, for any $x, y \in X$ and $\mu, \eta, \tilde{\eta} \in [0,1]$,

$$(x,\eta) \succsim^r (y,\eta\mu) \Leftrightarrow (x,\tilde{\eta}) \succsim^r (y,\tilde{\eta}\mu).$$

3 The Present Bias

In this section, I define how to derive time preferences from risk preferences; I also define preferences exhibiting hyperbolic and quasi-hyperbolic discounting. Consider a decision maker who chooses a one-time consumption. He discounts the future goods because it is uncertain whether or not he can consume it. To capture the uncertainty, let $p(t)$ be the probability that the decision maker can consume the good at a promised time $t \in \mathbb{R}_+$. One interpretation of $p(t)$ corresponds to the probability that the decision maker is alive at time t.[9]

Consider the decision maker's time preference $\succsim_0$ at time 0. The preference $\succsim_0$ is on the set of one-time consumptions after time 0; this set is defined as $T_0(X) = \{[x,t] \mid x \in X \text{ and } t \in \mathbb{R}_+ \text{ such that } t \geq 0\}$. Suppose that the decision maker is still alive at date $d \geq 0$. Then the probability that he is still alive and able to consume the good at date $t \geq d$ is the conditional probability $p(t|d) = p(t)/p(d)$. Therefore,

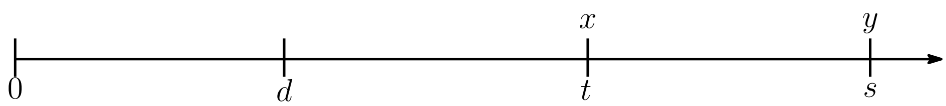

Figure 1: Time Structure

the decision maker at time d prefers prize x at time t, denoted by $[x,t]$, to another

[8] Several experimental studies on the common ratio effect and the certainty effect have found that the preference is reversed by changing the prizes from gains into losses. I can define these preferences by just switching strict preferences from $\succ$ to $\prec$, and vice versa. Henceforth, I will mention the case of negative payoffs only in footnotes.

[9] Another interpretation of $p(t)$, as seen in biology and psychology, is the probability that the goods have not been stolen by other animals by time t. These two interpretations are representative of most of the research ascribing future discounting to future uncertainty.

future payoff $[y,s]$ if and only if he prefers the binary lottery $(x, p(t|d))$, which gives x with the probability $p(t|d)$, to the lottery $(y, p(s|d))$. Thus, the decision maker's time preferences $\{\succsim_d\}_{d\in\mathbb{R}_+}$ for each decision time $d \in \mathbb{R}_+$, is defined as follows:

Definition: For all $d \in \mathbb{R}_+$ and $[x,t], [y,s] \in T_d(X)$,

$$[x,t] \succsim_d [y,s] \Leftrightarrow \big(x, p(t|d)\big) \succsim^r \big(y, p(s|d)\big).$$

Henceforth, I will denote this time preferences by $\{\succsim_d\}$. I am now in a position to define preferences exhibiting hyperbolic and quasi-hyperbolic discounting. Hyperbolic discounting is characterized as follows: Suppose that subjects choose either an earlier, smaller payoff which gives a payoff x at a date t or a later, larger payoff which gives a payoff y at a date s, where $x < y$ and $t < s$. Many subjects want to wait for the later, larger payoff, that is, they prefer $[y,s]$ to $[x,t]$. After some time $\tilde{d}$, however, they do not want to wait any longer, and consequently reverse their preferences as described in Figure 2.[10]

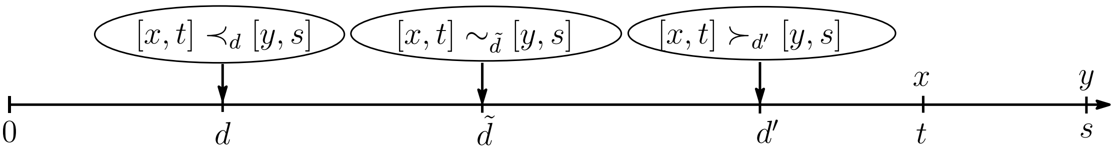

Figure 2: Preferences Exhibiting Hyperbolic Discounting

Hyperbolic discounting is therefore defined as follows:

Definition: $\{\succsim_d\}$ is said to exhibit *hyperbolic* discounting if, for any $x, y \in X$ and $\tilde{d}, t, s \in \mathbb{R}_+$ such that $[x,t] \sim_{\tilde{d}} [y,s]$ and $\tilde{d} \le t \le s$,

$$[x,t] \prec_d [y,s] \text{ for all } d < \tilde{d} \text{ and } [x,t] \succ_d [y,s] \text{ for all } d > \tilde{d}.$$

Note that the characterization of hyperbolic discounting in Proposition 1 of Dasgupta and Maskin (2005, p. 1293) is exactly the same as above.

Quasi-hyperbolic discounting focuses on present-biased behavior specifically when the promised date for the payoff is close at hand. (See Laibson (1997), for example.) Hence, the condition characterizing quasi-hyperbolic discounting is defined as a special case of hyperbolic discounting, where $\tilde{d} = t$ as follows:

[10]In the following three definitions of time preferences, I focus on positive payoffs for simplicity. For the case of negative payoffs, present biasness appear as procrastination and is defined by the same way just by switching strict preference from $\succ$ to $\prec$, and vice versa. O'Donoghue and Rabin (1999) offers examples of procrastination.

DEFINITION: $\{\succsim_d\}$ is said to exhibit *quasi-hyperbolic* discounting if, for any $x, y \in X$ and $t, s \in \mathbb{R}_+$ such that $[x,t] \sim_t [y,s]$ and $t \leq s$,

$$[x,t] \prec_d [y,s] \text{ for all } d < t.$$

By definition, if a decision maker exhibits hyperbolic discounting, then he exhibits quasi-hyperbolic discounting, but the converse is not true.[11]

Finally, I define *temporally unbiased* preferences, which corresponds to exponential discounting, as follows:

DEFINITION: $\{\succsim_d\}$ is said to be *temporally unbiased* if, for any $x, y \in X$ and $d, d', s, t \in \mathbb{R}_+$,

$$[x,t] \succsim_d [y,s] \Leftrightarrow [x,t] \succsim_{d'} [y,s].$$

4 The Theorem

In this section, I will prove the theorem. To establish the result, I assume a regularity condition on future uncertainty which means that today's good is certain, but, as the promised date for future goods becomes increasingly distant, the probability of consuming the good continuously decreases to zero:

ASSUMPTION 1: $p(0) = 1$, p is continuous and strictly decreasing, and $p(\infty) = 0$.

THEOREM: *Under Assumption 1, the following three equivalences hold:*[12]
(i) $\succsim^r$ *exhibits the common ratio effect if and only if* $\{\succsim_d\}$ *exhibits hyperbolic discounting.*
(ii) $\succsim^r$ *exhibits the certainty effect if and only if* $\{\succsim_d\}$ *exhibits quasi-hyperbolic discounting.*
(iii) $\succsim^r$ *satisfies the independence axiom if and only if* $\{\succsim_d\}$ *is temporally unbiased.*

The proof is in the appendix. The proof crucially relies on two structural similarities between risky choices and intertemporal choices. One is the similarity that relates safe outcomes to earlier ones, and risky outcomes to later ones. The other is the similarity that relates increasing risk to moving a decision time forward as well as decreasing risks to moving a decision time backward.

[11] It is easy to see the above definition is equivalent to Halevy (2008)'s characterization of quasi-hyperbolic discounting. Halevy (2008, p. 1150) characterizes quasi-hyperbolic discounting by $\forall t \in \mathbb{Z}_+$ s.t. $t \geq 1 \left[\frac{D(0)}{D(1)} > \frac{D(t)}{D(t+1)}\right]$ (Diminishing Impatience).

[12] In the section above, I have defined the Allais paradox and present bias for positive payoffs. However, as I mentioned in footnote, I can define these concepts for negative payoffs just by switching strict preferences. Hence, the equivalence here also holds for negative payoffs as well.

As explained in detail in the introduction, the theorem of the paper may be viewed as a generalization of most of the conventional research on hyperbolic discounting (for example, Kagel et al. (1986), Green and Myerson (1996), Sozou (1998), Halevy (2008), Epper et al. (2009)), because most of them adopt Assumption 1. In other words, to obtain a relation which is not included in the theorem, it is necessary to violate Assumption 1. As far as I know, Dasgupta and Maskin (2005) is the unique example of such approach. They assume not only a constant Poisson mortality rate, but also uncertainty regarding the timing of the payoffs. Accordingly they violate Assumption 1 and describe dynamically inconsistent behavior, despite assuming the expected utility.

The theorem may also answer the question as to what causes hyperbolic discounting. There exist only three possible answers which are compatible with the theorem presented here. The first answer is that the Allais paradox causes hyperbolic discounting (see, for example, Halevy (2008) and Epper et al. (2009)). The second answer is that non-regular uncertainty causes it (see, for example, Dasgupta and Maskin (2005)). The third answer is that a third factor may cause both the Allais paradox and hyperbolic discounting; for example, Fudenberg and Levine (2008) claim that temptation caused by either certainty or presentness would be the common factor. The choice among these three must await future research.

5 Appendix

5.1 Proof of The Theorem

Proof of Theorem: I will prove (i) $\succsim^r$ exhibits the common ratio effect if and only if $\{\succsim_d\}$ exhibits hyperbolic discounting for the case of positive payoffs. Part (ii) and (iii) of the theorem can be proved in the same way. Analogous theorem for negative payoffs also can be proved in the same way.

Step 1: If $\succsim^r$ exhibits the common ratio effect, then $\{\succsim_d\}$ is hyperbolic.

Proof of Step 1: Choose any $x, y \in X$ and $\tilde{d}, t, s \in \mathbb{R}_+$ such that $[x,t] \sim_{\tilde{d}} [y,s]$ and $\tilde{d} \leq t \leq s$. Then by definition, $(x, p(t|\tilde{d})) \sim^r (y, p(s|\tilde{d})) = (y, p(s|t)p(t|\tilde{d}))$. Fix $d < \tilde{d}$ to show $[x,t] \prec_d [y,s]$. Since p is strictly decreasing, $p(t|d) < p(t|\tilde{d})$. So the common ratio effect implies that $(x, p(t|d)) \precsim^r (y, p(s|t)p(t|d)) = (y, p(s|d))$. Then by definition, $[x,t] \prec_d [y,s]$. The case where $d > \tilde{d}$ can be proved in the same way.

□

Step 2: If $\{\succsim_d\}$ exhibits hyperbolic discounting, then $\succsim^r$ exhibits the common ratio effect.

Proof of Step 2: Choose any $x, y \in X$ and $\mu, \tilde{\eta} \in [0,1]$ such that $(x, \tilde{\eta}) \sim^r (y, \tilde{\eta}\mu)$. Fix $\eta \in (0, \tilde{\eta})$ to show $(x, \eta) \prec^r (y, \eta\mu)$. Since p is strictly decreasing bijection, there exist t and $\tilde{d}$ such that $t \geq \tilde{d} > 0$ and $p(t) = \eta$ and $p(t|\tilde{d}) = \tilde{\eta}$. Also, there exists $s \geq t$ such that $p(s|t) = \mu$. Hence, $(x, p(t|\tilde{d})) \sim^r (y, p(s|t)p(t|\tilde{d})) = (y, p(s|\tilde{d}))$, so that $[x,t] \sim_{\tilde{d}} [y,s]$, by definition. Therefore, if $\{\succsim_d\}$ is hyperbolic, then $[x,t] \prec_0 [y,s]$. So the definition shows that $(x, \eta) \prec^r (y, \eta\mu)$ again. The case where $\eta > \tilde{\eta}$ can be proved in same way. $\square$

$\blacksquare$

5.2 Static Present Bias

In the section above, I focused on dynamic decision making. In this section, I will examine static decision making. I first define preferences exhibiting hyperbolic and quasi-hyperbolic discounting in the static sense. Then I explore the relationship between these preferences and the preferences exhibiting the Allais paradox defined in Section 2. Most of the experimental work on time-discounting focus on the static concept. In typical experiments, subjects are supposed to choose the earlier, smaller payoff $[x, t+\alpha]$ or the later, larger payoff $[y, s+\alpha]$ by changing common delay α, at a fixed decision time. Hyperbolic and quasi-hyperbolic discounting in the static sense are defined analogously to those in the dynamic sense which are defined in Section 3. The only difference is that the variable here is a common delay and the decision time is fixed at some $\tilde{d} \in \mathbb{R}_+$;

Definition:
(i) $\succsim_{\tilde{d}}$ is said to exhibit *hyperbolic discounting in a static sense* [13] if, for any $x, y \in X$ and $\tilde{d}, t, s \in \mathbb{R}_+$ such that $[x, t] \sim_{\tilde{d}} [y, s]$ and $\tilde{d} \le t \le s$,

$[x, t+\alpha] \prec_{\tilde{d}} [y, s+\alpha]$ for all $\alpha \in (0, \infty)$ and $[x, t-\alpha] \succ_{\tilde{d}} [y, s-\alpha]$for all $\alpha \in [0, t-\tilde{d}]$.

(ii) $\succsim_{\tilde{d}}$ is said to exhibit *quasi-hyperbolic discounting in a static sense* if, for any $x, y \in X$ and $\tilde{d}, t, s \in \mathbb{R}_+$ such that $[x, t] \sim_{\tilde{d}} [y, s]$ and $\tilde{d} \le t \le s$,

$$[x, t+\alpha] \prec_{\tilde{d}} [y, s+\alpha] \text{ for all } \alpha \in (0, \infty).$$

(iii) $\succsim_{\tilde{d}}$ is said to be *temporally unbiased in a static sense* if, for any $x, y \in X$ and $d, \tilde{d}, s, t \in \mathbb{R}_+$ such that $\tilde{d} \le t \le s$,

$$[x, t] \succsim_{\tilde{d}} [y, s] \Leftrightarrow [x, t+\alpha] \succsim_{\tilde{d}} [y, s+\alpha] \text{ for all } \alpha \in [\tilde{d}-t, \infty).$$

Assumption 1 must be strengthened in order to link preferences exhibiting hyperbolic and quasi-hyperbolic discounting in the static sense with preferences exhibiting the Allais paradoxes:

Assumption 2: There exists a positive real number r such that $p(t) = \exp(-rt)$ for all $t \in \mathbb{R}_+$.

[13]This effect is often called the common difference effect.

COROLLARY: *Under Assumption 2, the following three equivalences hold:*
(i) $\succsim^r$ *exhibits the common ratio effect if and only if* $\succsim_{\tilde{d}}$ *exhibits hyperbolic discounting in a static sense.*
(ii) $\succsim^r$ *exhibits the certainty effect if and only if* $\succsim_{\tilde{d}}$ *exhibits quasi-hyperbolic discounting in a static sense.*
(iii) $\succsim^r$ *satisfies the independence axiom if and only if* $\succsim_{\tilde{d}}$ *is temporally unbiased in a static sense.*

Since Assumption 2 implies Assumption 1, the theorem also holds under Assumption 2. Hence, each static preference is equivalent to corresponding dynamic preference.

5.3 A Counterexample to Theorem 1 of Halevy (2008)

Halevy (2008) assumes that the decision maker has rank-dependent utilities. He characterizes hyperbolic discounting in terms of *Diminishing Impatience*:

$$\forall t \in \mathbb{Z}_+ \text{ s.t. } t \geq 1 \left[\frac{D(0)}{D(1)} > \frac{D(t)}{D(t+1)} \right],$$

where $D(\cdot)$ is a discount function. In his model, $D(t) = \beta^t g((1-r)^t)$, where β is a pure time-discount factor, g is a rank-dependent probability-weights function, r is a constant hazard probability per period. In Theorem 1, Halevy (2008, p. 1150) shows the following two equivalences: [14]

$$\begin{aligned} &\text{Diminishing Impatience} \\ &\Leftrightarrow \forall t \in \mathbb{Z}_+, \forall r \in (0,1) \ \left[g((1-r)^{t+1}) > g(1-r) g((1-r)^t) \right] \\ &\Leftrightarrow \forall p, q \in (0,1) \ \left[g(pq) > g(p) g(q) \right]. \end{aligned}$$

Then Halevy (2008) cites Segal (1987a, b). Let $\varepsilon_g(p) = \frac{g'(p)p}{g(p)}$ be the elasticity of g.

$$\begin{aligned} \forall p, q \in (0,1) \ \left[g(pq) > g(p) g(q) \right] \ &\Leftrightarrow \ \varepsilon_g(p) \text{is strictly increasing} \\ &\Leftrightarrow \ \text{Common Ratio Effect}, \end{aligned}$$

where the first equivalence is claimed in Lemma 4.1 of Segal (1987a) and the second one is proved by Theorem 1 of Segal (1987b). However, in the proof of Lemma 4.1, Segal (1987a) implicitly assumes that $\varepsilon_g(p)$ is monotone.

[14] The "only if" part ($\Rightarrow$) of the second equivalence may need some additional condition on g. To explain the point, suppose that $g((1-r)^{t+1}) > g(1-r)g((1-r)^t)$ for all $t \in \mathbb{Z}_+, r \in]0,1[$. Fix $p > q$. If we let $1-r = p$, then it should hold that $(1-r)^t = q$ for some $t \in \mathbb{R}$. However, it is not necessarily true that $t \equiv \log_{(1-r)} p$ is an integer. Hence, it is not necessarily true that $g(pq) > g(p)g(q)$. We may eliminate the problem by extending the model into continuous time. However, the definition of diminishing impatience would no longer be an appropriate description of quasi-hyperbolic preferences. Since, in continuous time, the " next" period of period t is no longer period $t+1$, it would be difficult to interpret the ratio $\frac{D(t)}{D(t+1)}$ as the impatience at the period t.

Indeed, a probability-weights function g of prospect theory, proposed in Kahneman and Tversky (1992), satisfies $g(pq) > g(p)g(q)$ for all $p, q \in (0, 1)$, but $\varepsilon_g(p)$ is strictly decreasing on some interval. Hence, only this partial result of Halevy (2008) is true:

$$\text{Diminishing Impatience} \Leftarrow \text{Common Ratio Effect.}$$

Actually, this result is implied by the "only if" component of part (ii) of the theorem in the present paper.

The following is the function g proposed in Kahneman and Tversky (1992). Let $a \in [0, 1]$. For all $p \in [0, 1]$, define

$$g(p) = \frac{p^a}{(p^a + (1-p)^a)^{1/a}}.$$

Claim: *For $a = 0.5$, rank-dependent decision maker with the above probability-weights function g exhibits the diminishing impatience but does not exhibit the common ratio effect.*

Camerer and Ho (1994) estimate the parameter a as 0.52 based on their experiments, so $a = 0.5$ would be a reasonable estimate. The above claim is true for other parameters too, such as $0.4, 0.9$.[15]

Step 1: The decision maker exhibits the diminishing impatience.
Proof of Step 1: I will show that $\forall p, q \in (0, 1)\big[g(pq) > g(p)g(q)\big]$. For all $p, q \in [0, 1]$, define

$$f(p, q) = g(pq) - g(p)g(q).$$

I will show that $f(p, q) > 0$ for all $p, q \in (0, 1)$. Choose any $b \in (0, 1)$ to show that $f(p, b-p) > 0$ for all $p \in (0, b)$. By the symmetry of f, without loss of generality, it suffices to show that $f(p, b-p) > 0$ for all $p \in (0, \frac{b}{2})$. By calculation,

$$\begin{aligned} \tfrac{df(p,b-p)}{dp} &= \tfrac{dg(p(b-p))}{dp} - \tfrac{dg(p)g(b-p)}{dp} \\ &= g'(p(b-p))(b-2p) - g'(p)g(b-p) + g'(b-p)g(p). \end{aligned}$$

For $a = 0.5$, it can be shown that the derivative is 0 if and only if $p = \frac{b}{2}$. Since $f(\frac{b}{2}, \frac{b}{2}) > f(0, b) = 0$, then f attains its maximum when $p = \frac{b}{2}$ and its minimum when $p = 0$. Hence, $f(p, b-p) > 0$ for all $p \in (0, \frac{b}{2})$

□

Step 2: The decision maker does not exhibit the common ratio effect.
Proof of Step 2: By Segal (1987 b), it suffices to show that $\varepsilon_g(p)$ is strictly decreasing for all $p < 0.14$. By calculation,

$$\varepsilon_g'(p) = \frac{(1-p)^{-2+a}((1-p)^a p - a p^a + p^{1+a})}{p((1-p)^a + p^a)^2}.$$

[15] However, the claim does not hold when a is too small.

Hence, for $a = 0.5$,

$$\begin{aligned} \varepsilon_g(p) \text{ is strictly decreasing } &\Leftrightarrow p\sqrt{1-p} - .5\sqrt{p} + p\sqrt{p} < 0 \\ &\Leftarrow p < 0.14. \end{aligned}$$

□

■

References

[1] Baucells, M. and F. Heukamp (2008), "Probability and Time Trade-off," Working paper.

[2] Camerer, C. F. and T. H. Ho (1994), "Violations of the Betweenness Axiom and Nonlinearity in Probability," Journal of Risk and Uncertainty, 8, 167-196.

[3] Dasgupta, P. and E. Maskin (2005), "Uncertainty and Hyperbolic Discounting," American Economic Review, 95, 1290-1299.

[4] Epper, T., H. Fehr-Duda, and A. Bruhin (2009), "Uncertainty Breeds Decreasing Impatience: The Role of Risk Preferences in Time Discounting," Working paper.

[5] Fudenberg, D. and D. Levine (2008), "Self-control, Risk Aversion, and the Allais Paradox," Working paper.

[6] Green, L. and J. Myerson (1996), "Exponential Versus Hyperbolic Discounting of Delayed Outcomes: Risk and Waiting Time," American Zoologist, 36, 496-505.

[7] Halevy, Y. (2008), "Stroz meets Allais: Diminishing Impatience and the Certainty Effect," American Economic Review, 98, 1145-62.

[8] Kagel, J., L. Green, and T. Carcos (1986), "When Foragers Discount The Future: Constraint or Adaptation?" Animal Behavior, 34, 271-283.

[9] Kahneman, D. and A. Tversky (1992), "Advances in Prospect Theory: Cumulative Representation of Uncertainty," Journal of Risk and Uncertainty, 5, 297-323.

[10] Keren, G. and P. Roelofsma (1995), "Immediacy and Certainty in Intertemporal Choice," Organizational Behavior and Human Decision Making, 63, 287-297.

[11] Laibson, D. (1997), "Golden Eggs and Hyperbolic Discounting," Quarterly Journal of Economics, 112, 443-477.

[12] Loewenstein, G. and D. Prelec (1992), "Anomalies in Intertemporal Choice: Evidence and Interpretation," Quarterly Journal of Economics, 107, 573-597.

[13] Machina, M. (1982), "Expected Utility Analysis Without the Independence Axiom," Econometrica, 50, 277-323.

[14] O'Donoghue, T. and M. Rabin (1999), "Doing It Now or Later," American Economic Review, 89, 103-124.

[15] Prelec, D. (1998), "The Probability Weighting Function", Econometrica, 66, 497-527.

[16] Segal, U. (1987a), "The Ellsberg Paradox and Risk Aversion: An Anticipated Utility Approach," International Economic Review , 28, 175-202.

[17] Segal, U. (1987b), "Some Remarks on Quiggin's Anticipated Utility," Journal of Economic Behavior and Organization, 8, 145-154.

[18] Sozou, P. (1998), "On Hyperbolic Discounting and Uncertain Hazard Rates," Proceedings of the Royal Society London(Series B), 265, 2015-2020.

[19] Starmer, C. (2000), "Developments in Non-expected Utility Theory: The Hunt for a Descriptive Theory of Choice under Risk," Journal of Economic Literature, 38, 332-382.

[20] Yarri, M. (1987), "The Dual Theory of Choice Under Risk," Econometrica, 55, 95-115.