# On Reputational Rents as an Incentive Mechanism in Competitive Markets*

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June 22, 2009

**Abstract**

This paper shows that more intense competition may improve, rather than hamper, the chances that a market for an experience good or service overcomes the problems caused by informational asymmetries. This, in spite of the fact that intensified competition diminishes the reputational rents that–allegedly–provide the incentives for the production of high quality. Our results show that instead, these incentives are created by price differentials not levels.

## 1 Introduction

Klein and Leffler (1981) pioneered the study of the reputational mechanism for the assurance of quality in markets for experience goods. They observed that the existence of a price premium or rent could induce firms to incur the costs of delivering high quality products. This, in spite of the fact that it is impossible to profit from doing so in the short run, because the product's quality is unobservable and hence cannot influence the purchasing decision. However, in a repeat-purchase situation–the argument goes– the risk of consumers taking their business elsewhere if they feel cheated could suffice to keep incentives aligned, as long as the foregone future rents are high enough to compensate for the differential cost of high quality.

Sustaining high quality production through reputation, then, requires the existence of rents. An immediate concern is that as competitive forces push prices down, rents are eroded (Bar-Isaac and Tadelis, 2008). It would seem, then, that competition destroys incentives, thereby rendering the competitive production of high quality unviable. This leads Klein and Leffler to think that sunk investments are necessary for this mechanism to operate, because they stand as barriers to entry to prevent prices from dropping to marginal costs.

This paper argues that there is not necessarily a trade-off between competition and incentives. Indeed, it presents a situation in which, as competition grows fiercer and prices fall, rents are reduced but incentives for high quality production are strengthened, not hampered.

The explanation to this paradox is a two-fold argument. The first one could be summarized as "incentive compatibility is not about levels, but about differences." If the consequence of selling low quality is to be forced to charge lower prices, then the choice between high and low qualities is determined by

*We are grateful to Claudio Fernández and Joseph Ostroy for helpful discussions on some ideas used in the paper, and to seminar participants at Pontificia Universidad Católica de Chile and Universidad de Santiago de Chile for comments.
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the price differential. In this context, the price level relates to the participation decision, not to the quality choice. The second argument is the observation that, if quality is a normal good, then a fall in the prices of all qualities leads to an increase in the price differential between the lowest and the highest qualities. This is so because the positive income effect that the price drop creates increases the willingness to pay for quality at the margin. Taken together, we see that the overall effect of increased competition is to improve incentives (through an increase in the price differential between qualities, triggered by the fall in price of all qualities) for high quality production, not destroy them.

To analyze these issues, we consider a model where a continuum of long-lived firms is imperfectly and publicly monitored by a sequence of continua of short-lived consumers, as in Vial (2008). In the equilibrium we focus on, at any date the price a firm can charge is increasing in its reputation, which in turn is determined by its public history. As a matter of fact, at each date, the mis-behaved or unlucky firms whose public signal turned out to be bad are punished by a drop in the price, reflecting their diminished reputation. The reputational premia (i.e., the price differential between any two reputation levels) are set by competition among consumers, who must be indifferent among all active producers. The price level, on the other hand, is determined by how competitive the industry is. If both, the demand and the supply sides of the market are of the same size, then there is a family of Walrasian price functions. At the other extreme, if the demand side is the thin one–the perfectly competitive case[1]–, then the price level is low, determined by a zero profit condition–in a present value sense–for the firms with no history (i.e., those that are either entering or re-entering). In this context, it is possible to make a clean distinction: incentive compatibility depends on the slope of the price function, while participation depends on its level.

In most literature on competition with heterogeneous reputations, incentive compatibility does depend on the price level. This is so because the punishment to the unlucky or mis-behaved firms who experienced a bad outcome is to go out of business, getting a null utility from then onwards. For instance, in Klein and Leffler (1981) and Diamond (1989) the firms that experience a bad outcome reveal their type, with the consequent drop in reputation down to 0. Similarly, in Hörner (2002)–with imperfect private monitoring–the *focus* is on an equilibrium where the consumers leave the firm after a bad outcome. In all these cases, if, at a given price, incentive compatibility holds, then it also holds at every higher price. By way of contrast, in our model firms stay in business after a bad outcome: their punishment is a reduction in price. Hence, whether incentive compatibility holds or not depends on the size of the price drop following a bad outcome, and not on how high or low the initial price might have been. It follows that incentive compatibility may hold at a given price and not hold at another higher price. This is what we mean when we say that incentive compatibility is about differences, not about levels.

We consider this softer punishment to be more empirically accurate. In practice we do observe firms that continue operations after bad experiences with some of their products; indeed, this seems to be the norm in the professional services market, as Hörner[2] points out. A related concern is the strict association between age and reputation that all the aforementioned papers predict; it turns out that under the kind of punishment we study, this association breaks down.

From a modelling viewpoint, Klein and Leffler (1981) and Diamond (1989) are in the tradition of the first public monitoring models of reputation (e.g., Milgrom and Roberts, 1982; Fudenberg and Levine, 1989), where the normal type wants to mimic the good type. Ours, in turn, is a model where the normal type wants to separate itself from the bad type, as in Mailath and Samuelson (2001). We also consider reputations that dissipate gradually, an asset-like characteristic they highlight. Consequently we are able to emphasize the importance of price differences, as opposed to price levels.

A related literature (Tadelis, 1999, Mailath and Samuelson, 2001, and Tadelis, 2002) looks at reputations (i.e., names) as tradable assets. This paper abstracts from this possibility.

[1]The conditions that Gretsky et al. (1999) identify for perfect competition in the continuous assignment model are met in the ex-ante, uninformed sense.

[2]Hörner (2002), pp. 655.

The rest of the paper is organized as follows. Section 2 lays out the model, and resolves two important technical questions: (1) Is there a steady state distribution of reputations, so that focusing the analysis on the steady state stage game makes sense?, and (2) Is there a reputation level that entrants with null histories may have, that would coincide with the fraction of competents in the population of entrants, so that this belief is a rational expectation? The answer in both cases is affirmative. Section 3 focuses on the case in which entrants would enter with the worse possible reputation, so that no firm would ever want to "wash its name out." This case is particularly interesting because it is the simplest, so that the issues we are interested in are clearly exposed. This section contains our main results. Section 4 extends their application by considering the case where entrants have better reputations and, consequently, there are inflows and outflows of firms. Section 5 concludes.

# 2 The model

## 2.1 Preliminaries

We consider an infinitely repeated game in which, at every round, a market for a given service–formed by a continuum of consumers and firms–opens. Firms are long run players that face an infinite sequence of generations of short-lived consumers.

The service is an experience good as per Nelson (1970): its quality is ex ante unobservable to buyers. Nevertheless, at every round and for each firm, a complete history of an imperfect signal of past qualities is publicly available, i.e., monitoring is imperfect and public. On the other hand, each individual may consume or produce at most one unit.

There are two types of firm, competent ($C$) and inept ($I$). Competent firms are able to choose, at each round, to provide either a low-quality service ($L$) or a high-quality one ($H$), the former being cheaper to produce than the latter. Inept firms, on the other hand, can only provide low quality. Types are privately observed. Hence, the model features adverse selection (unobservable types) and moral hazard (unobservable quality choices). On the other hand, providing high quality makes it more likely that the firm obtains a higher public signal. Note that competent firms can always behave as inept firms do, so that their equilibrium payoffs will always be at least as good as theirs.

Consumers will use each firm's public history to assess the probability that it is a competent one. They have a common prior for each firm and observe the same outcomes. Consequently they assign to each firm the same probability of being competent, which we denote by $\mu$ and refer to as the firm's reputation. Observe that consumers only live for one period, so that the information each one obtained from consuming the service is not transferred to the next generation, but lost altogether.

Active firms are subject to the possibility of dying, case in which they are replaced by a newly born firm.[3] This process is privately observed by each firm. As Mailath and Samuelson (2001) explain, this assumption ensures that along any history there is never almost certainty about any firm's type. The process is assumed to be i.i.d. across time and firms, with $\lambda$ being the probability of dying, and $\theta$ the probability of the newly born being competent. Hence, along any history the probability of being competent is at least $\lambda\theta$ (the probability that a surely inept firm dies and is replaced by a competent one) and at most $1-\lambda(1-\theta)$ (the probability that a surely competent firm is not replaced by an inept one): $\mu \in (\lambda\theta, 1-\lambda+\lambda\theta)$. We further assume that the total availability of competent firms is small relative to the market ($\theta < 1$), because we are interested in the case in which adverse selection is important, i.e., competence is scarce.

We will focus on a high quality equilibrium, by which we mean a situation where all competent firms choose to provide high quality. We will analyze the conditions under which the corresponding incentive

[3]If inactive firms were also subject to this risk, then there would be new-born competents among inactive firms. Section 4 below discusses this case as an extension.

compatibility conditions hold in the repeated game, and we will characterize the stage-game equilibrium as well. Given that inept firms can only provide low quality, the (conditional) probability of being competent coincides in such equilibrium with the (conditional) probability of providing high quality. Hence, consumers will, *ceteris paribus*, strictly prefer to buy from higher-reputed firms. It follows that in a price-taking equilibrium, the price any firm can charge will be solely determined by its reputation: better reputed firms will be able to charge higher prices. We will say that there is a **reputational premium** if the price function is strictly increasing in $\mu$, the firm's reputation. The existence of a reputational premium is a necessary condition to sustain the high-quality equilibrium. The reputational premium causes the firms' equilibrium payoff function to be increasing in their reputation.

We seek to understand how the threat of entry affects market outcomes, in particular the plausibility of reputation building in equilibrium. To that end, we will be comparing two polar cases:

1. Economy with free entry: the mass of firms (to be denoted by $\kappa$) is strictly larger than the mass of consumers (which we normalize to 1): $\kappa > 1$, so that there are always inactive firms, and the threat of entry is credible because there are no costs involved in becoming active.

2. Economy without free entry: the mass of firms (or the mass of firms that have already sunk some prohibitively high entry cost) exactly matches that of consumers ($\kappa = 1$).

In the equilibrium we consider, all competent firms prefer to participate and all consumers are served, so that there is a mass $\theta$ of active competent firms, a mass $(1-\theta)$ of active, inept firms, and a mass $(\kappa - 1)$ of inactive inept firms.

In a steady state the mass of inactive firms will not change from one date to another, but the identity of active or inactive firms may constantly change. On the other hand, each firm's decision making will depend on the consequences of pursuing different reputation levels, or the consequences of producing or exiting. For those reasons, we need to specify what reputation a firm that was previously inactive would have, and what would happen to the reputation of those firms that exit the market.

**Assumption 2.1.** *[anonymity of entrants] Histories are lost upon exiting the market, so that all entrants have an empty history and share the same reputation.*

Thus, we assume that new entrants are perceived alike by consumers: it is not possible to tell one apart from the other and therefore, all entrants have the same reputation (denoted by $\mu_E$). This is true regardless of whether the entrant chose to leave the market in the previous stage or it had been inactive for some time: if a firm leaves the market it effectively destroys its public record and, if it were to re-enter the market, it would do so under the same conditions as a firm that had never entered the market. This rules out interesting phenomena such as umbrella branding; in fact, what this assumption effectively does is to rule out the possibility of firm growth, as the capacity constraint is exogenously set to one unit. In this sense, the model is biased in favor of intense competition and against brand-name capitalizing.

This anonymity-of-entrants assumption implies that as reputation is valuable in the high-quality equilibrium, no active firm may have a lower reputation than entrants, because any firm whose reputation fell below that level may "wash its name out" simply by leaving and re-entering the market. Hence, the entry-level reputation is the lowest one among active firms–and inactive firms do not have reputations at all. In turn, and provided that there are rents for reputations above the entry-level (a fact that will be true in a high-quality equilibrium) and that there are no net inflows or outflows of firms (a fact that will be true in a steady state), this means that:

- If any, only the firms with the lowest reputations will exit;
- All competent firms that leave will re-enter immediately, attracted by the appropriable rents;

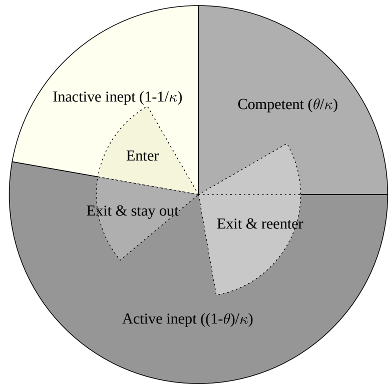



Figure 1: Firms by type and exit-entry decisions

- Among the inept firms, all the inactive and all who left will be indifferent between entering or not, and
- The mass of inept firms that enter (for the first time or otherwise) will be equal to the mass of inept firms that left the market the previous round.

Figure 1 illustrates the composition of competent and inept firms by their exit and entry decisions. What makes possible the fact that competent firms make money and inept firms don't at the entry-level reputation is that the former are more likely to get better public signals than the latter–provided they incur the higher cost of producing high quality–, and therefore their reputations rise "faster," in a stochastic sense.

If expectations are rational, the entry-level reputation $\mu_E$ must coincide with the fraction of competent firms among the group of entrants:

$$\mu_E = \frac{\text{Competent entrants}}{\text{Competent entrants} + \text{Inept entrants}}, \tag{1}$$

provided that there are entrants.

Let $G_t$ denote the cdf of reputations at the beginning of stage $t$ of those firms that were active at $t-1$. Stage $t$ begins with firms' exit and entry decision. $\overline{G}_t$ denotes the cdf of reputations of those firms that chose to be active. Thus, $G_t$ and $\overline{G}_t$ differ because some firms whose reputation fell below $\mu_E$ will re-enter and some firms that were inactive will also enter (with a reputation $\mu_E$ in both cases). After competent firms choose quality and trade takes place, all players observe the public signal $r_t$ for all active firms. With this information at hand, and taking into account the type change process, consumers update their beliefs. $G_{t+1}$ denotes the cdf of these updated reputations. This is summarized in Figure 2, that depicts the time line for the stage game.

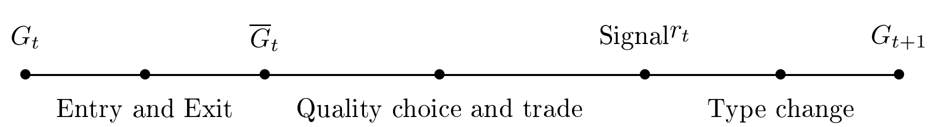



Figure 2: Time line for the stage game

Let the superscripts $C$ and $I$ denote the corresponding sub populations of competent and inept type firms, so that the cdf's are $G_t^C$, $G_t^I$, $\overline{G}_t^C$ and $\overline{G}_t^I$.

The steady state distributions will be individualized by dropping the time subscript. Lemma 2.3 below proves that $G^C$ and $G^I$ are continuous and their support is $[\lambda\theta, 1-\lambda+\lambda\theta]$.

Hence, if $\mu_E > \lambda\theta$ is the steady state entry-level reputation, it must satisfy:

$$\mu_E = \frac{\theta G^C(\mu_E)}{\theta G^C(\mu_E) + (1-\theta) G^I(\mu_E)}. \tag{2}$$

If $\mu_E \leq \lambda\theta$ the denominator on the right hand side of Equation 2 vanishes, so that it fails to be well defined. Lemma 2.4 below proves that the limit of the right hand side of Equation 2 when $\mu$ approaches $\lambda\theta$ from the right is precisely $\lambda\theta$. Indeed, if the entry-level reputation were slightly higher than $\lambda\theta$, we would observe some firms wanting to wash their names. However, at such low reputation levels it is almost certain that the firm has been inept for some time, but since it is always possible that its type changed, its reputation must be close to $\lambda\theta$–the probability that a surely inept firm dies and is replaced by a competent one.

One family of equilibria obtains when the entry-level reputation $\mu_E$ is precisely $\lambda\theta$. In this case, there is neither exit nor entry of firms since it is impossible to obtain a reputation lower than $\lambda\theta$, no matter how bad the history may have been. Therefore, in such equilibria there is no name-washing at all, and $\left(G^C, G^I\right)$ coincides with $\left(\overline{G}^C, \overline{G}^I\right)$. The free entry condition has then only one consequence–albeit an important one–: it puts a ceiling on the price function, which must be such that no inactive firm has an incentive to enter the market. What we seek to understand, then, is how this ceiling on the price function affects the reputational premium and through it the plausibility of the high quality equilibrium.

## 2.2 Strategies and payoffs

Consumers, being short-run players, play static best replies to their beliefs, formed as the Bayesian update of a common prior. They have identical (expected) utility functions and income levels. Utility depends on the consumption of the service, on its quality, and the consumption of other composite, infinitely divisible good $z$, whose price is normalized to 1. The possibilities include purchasing some amount $z$ of the good and one unit of the service that turns out to be of high quality $(H, z)$, or one unit of the service that turns out to be of low quality $(L, z)$, or not purchasing any of the service: $(\varnothing, z)$. The price of the service from a provider of reputation $\mu$ is $p(\mu)$, and the consumer's income is $y$, so that if he purchases the service, he buys $z = y - p(\mu)$ units of the composed good. The associated Bernoulli function is $u(q, z)$, where $q$ is the quality of the service being consumed, which can be high or low, and where not consuming is represented by the empty quality: $q \in \{H, L, \varnothing\}$. We assume that $u(q, z)$ is twice-differentiable in $z$, with a continuous second derivative.

We established previously that in a high-quality equilibrium the probability of getting a high quality service is the same as the probability that its provider is competent. Hence, consumers maximize:

$$E[u] = [\mu u(H, z) + (1-\mu) u(L, z)] d + (1-d) u(\varnothing, z), \tag{3}$$

where $d$ is a variable that takes on the value 1 if the consumer purchases the service ($q \in \{H, L\}$), and 0 otherwise ($q = \varnothing$).

Totally differentiating Equation 3 with respect to $z$ and $\mu$ and setting $d = 1$ we get:

$$\mathrm{d}E[u] = (u(H, z) - u(L, z)) \,\mathrm{d}\mu + \left(\mu \frac{\partial u(H, z)}{\partial z} + (1-\mu) \frac{\partial u(L, z)}{\partial z}\right) \mathrm{d}z. \tag{4}$$

Imposing $\mathrm{d}E[u] = 0$, we obtain the marginal rate of substitution (MRS) between perceived quality (i.e., reputation) and consumption:

$$MRS \equiv \left|\frac{\mathrm{d}z}{\mathrm{d}\mu}\right|_{\mathrm{d}E[u]=0} = \frac{u(H, z) - u(L, z)}{\mu u_z(H, z) + (1-\mu) u_z(L, z)}. \tag{5}$$

In contrast to the consumers' problem, the firms' problem is undoubtedly dynamic. When making a decision, firms must not only take into account the current-stage payoff, but also the long-term consequences of their actions.

Each producer chooses a contingent production strategy (i.e., whether to produce or not, and if so of which quality), and a "branding strategy" (i.e., keep its name or wash it out by exiting and re-entering the market). The fact that the price a firm can charge is increasing in its reputation renders the optimal branding strategy trivial: to wash its name whenever its reputation falls below $\mu_E$, and otherwise to keep it.

As to the production strategy, after every history competent firms choose $q_t$ from $\{H, L, \varnothing\}$ while inept firms from $\{L, \varnothing\}$, so as to maximize:

$$(1-\delta)\sum_{t=0}^{\infty}(1-\lambda)^t \delta^t E\left[\pi_t(\mu_t, q_t)\right], \tag{6}$$

where $\delta$ is a discount factor and $\pi_t$ is the time $t$ profit, given by:

$$\pi(\mu, q) = \begin{cases} p(\mu) - c_H & \text{if } q = H \\ p(\mu) - c_L & \text{if } q = L \\ 0 & \text{if } q = \varnothing \end{cases} \tag{7}$$

where the cost of providing high quality is higher than the cost of providing low quality: $c_H > c_L$.

The policy functions that characterize the high-quality equilibrium have all competent firms choosing $H$ after every history, all inept firms with $\mu > \mu_E$ choosing $L$, and a mass $(1-\theta)\, G_I(\mu_E)$ of the remaining inept firms that come to replace those who exited the market in the last round, also choosing $L$. Section 3 establishes the conditions under which these policy functions are indeed optimal. The associated value functions for competent and inept firms are denoted by $v_C(\mu)$ and $v_I(\mu)$, respectively.

## 2.3 High-quality equilibrium

There are dynamic and static aspects to equilibrium. In the stage game, we look at a price-taking equilibrium where consumers are indifferent among providers, firms behave according to the policies just described in the previous section, and the market clears. The dynamic aspects refer, on one hand, to the connection between histories and beliefs and, on the other, to the optimality of firms' policy functions. In particular, a free-entry condition must hold for inept, inactive firms to stay out: $v_I(\mu_E) = 0$, where $\mu_E$ is a rational expectation.

### 2.3.1 Bayesian updating

The signal is denoted by $r$, and drawn from the open unit interval: $r \in (0,1)$. When a firm provides high quality, then its signal is distributed according to the cumulative density function (cdf) $F_H$; when it provides low quality, then it is distributed according to the cdf $F_L$, where $F_H(r) \leq F_L(r)$ for all $r$ (this is to say, the signal conditional on $H$ first-order stochastically dominates the signal conditional on $L$). For instance, if the firms were schools, $r$ could be the score percentile in a standardized test; if the firms were lawyers, $r$ could be the percentage of cases won in a year; if the firms were doctors, $r$ could be the survival rate of patients in a year, and so on. Both, $F_H$ and $F_L$ are assumed to be smooth functions. In particular, they have densities, which as customary will be denoted by the lower cased letters of the corresponding cdf's.

Moreover, we assume that a higher signal makes it more likely that the firm chose high quality:

**Assumption 2.2.** *[monotone likelihood ratio] The likelihood ratio* $R(r) \equiv \frac{f_H(r)}{f_L(r)}$ *is a monotonically increasing bijection from* $(0,1)$ *to* $(0,\infty)$.

In the absence of the exit-entry process, and if all competent firms provide a high quality service, then Bayes' rule defines the difference equation that each firm's reputation follows:

$$\mu_{t+1} = \lambda\theta + (1-\lambda)\frac{f_H(r)\,\mu_t}{f_H(r)\,\mu_t + f_L(r)(1-\mu_t)}. \tag{8}$$

In effect, the probability of being competent at $t+1$ is made of the probability of being reborn as competent ($\lambda\theta$) and the probability of being competent at $t$ updated according to the signal $r$ and conditional on not having had a type change.

When there is an exit-entry process, the firm replaces its reputation with $\mu_E$ whenever it falls below this threshold, so that the evolution of $\{\mu_t\}$ is actually defined by Equation 8 if $\mu_t$ is replaced by $\max\{\mu_t, \mu_E\}$.

Define the right-hand side of Equation 8 as the function $\varphi : (\lambda\theta,\ 1-\lambda+\lambda\theta)\times(0,1) \to (\lambda\theta,\ 1-\lambda+\lambda\theta)$ given by:

$$\varphi(\mu, r) = \lambda\theta + (1-\lambda)\frac{R(r)\,\mu}{R(r)\,\mu + 1 - \mu}, \tag{9}$$

where the densities have been written in the likelihood ratio form. Assumption 2.2 ensures that $\varphi$ is strictly increasing in $r$, so that higher signal values always improve a firm's reputation. Moreover, the image of $\varphi$ is $(\lambda\theta, \lambda\theta+1-\lambda)$ because the likelihood ratio is surjective: $\varphi((\lambda\theta,\ 1-\lambda+\lambda\theta)\times(0,1)) = (\lambda\theta, \lambda\theta+1-\lambda)$. On the other hand, it is readily seen that $\varphi$ is strictly increasing in $\mu$, and continuous. Hence, being onto, it is a bijection, and the following implicit functions are well defined:

$$r = \tilde{r}(x,\mu) \Leftrightarrow x = \varphi(\mu, r), \tag{10}$$

$$\mu = \tilde{\mu}(x,r) \Leftrightarrow x = \varphi(\mu, r). \tag{11}$$

These equations may be written as:

$$\tilde{r}(x,\mu) = R^{-1}\left(\frac{1-\mu}{\mu}\frac{x-\lambda\theta}{1-\lambda+\lambda\theta-x}\right), \tag{12}$$

$$\tilde{\mu}(x,r) = \frac{x-\lambda\theta}{x-\lambda\theta+R(r)(1-\lambda+\lambda\theta-x)}. \tag{13}$$

The first function, $\tilde{r}(x,\mu)$, says what the signal value should be for a firm of reputation $\mu$ today to have a reputation $x$ tomorrow; by Assumption 2.2 the inverse of the likelihood ratio is well defined. The second function, $\tilde{\mu}(x,r)$, indicates what was the reputation in the previous period of a firm with a signal $r$ that currently enjoys a reputation $x$. Similarly, $\varphi(\mu, r)$ is the next period's reputation of a firm that started off with a reputation $\mu$ and whose signal was $r$.

Appendix A.1 shows that the end-of-period reputation cdf for each type of firm is given by:

$$G^C_{t+1}(x) = (1-\lambda+\lambda\theta)\int_0^{\tilde{r}(x,\mu_E)} \overline{G}^C_t(\tilde{\mu}(x,r))\,dF_H + \lambda(1-\theta)\int_0^{\tilde{r}(x,\mu_E)} \overline{G}^I_t(\tilde{\mu}(x,r))\,dF_L \tag{14a}$$

$$G^I_{t+1}(x) = \lambda\theta\int_0^{\tilde{r}(x,\mu_E)} \overline{G}^C_t(\tilde{\mu}(x,r))\,dF_H + (1-\lambda\theta)\int_0^{\tilde{r}(x,\mu_E)} \overline{G}^I_t(\tilde{\mu}(x,r))\,dF_L \tag{14b}$$

These cdf's are transformed by the entry-exit process as follows:

$$\overline{G}^\tau_{t+1}(x) = \begin{cases} 0 & x < \mu_E \\ G^\tau_{t+1}(x) & x \geq \mu_E \end{cases} \quad \text{for } \tau = C, I, \tag{15}$$

i.e., $\overline{G}^{\tau}_{t+1}$ is a truncated version of $G^{\tau}_{t+1}$ because those firms who fell below the entry-level reputation will re-enter, and will do so with the entry-level reputation $\mu_E$.

The process just described transformed the pair $\begin{pmatrix} \overline{G}^C_t \\ \overline{G}^I_t \end{pmatrix}$ into $\begin{pmatrix} \overline{G}^C_{t+1} \\ \overline{G}^I_{t+1} \end{pmatrix}$, thus defining an operator in the set of pairs of cdf's, which will be denoted by $T$ :

$$\begin{pmatrix} \overline{G}^C_{t+1} \\ \overline{G}^I_{t+1} \end{pmatrix} = T \begin{pmatrix} \overline{G}^C_t \\ \overline{G}^I_t \end{pmatrix}. \tag{16}$$

For future reference, observe that the operator $T$ depends parametrically on $\mu_E$.

### 2.3.2 Steady state and entry-level reputation

In a steady-state stage of the game, the reputation distributions $\left(G^C,\ G^I,\ \overline{G}^C,\ \overline{G}^I\right)$ are constant over time. That is, $\left(\overline{G}^C,\ \overline{G}^I\right)$ is a fixed point of $T$. Moreover, we prove that the fixed point is unique because $T$ is a contraction.

**Lemma 2.3.** *$T$ has a unique fixed point for any given $\mu_E \in [0,1]$. Moreover, if $\mu_E < 1-\lambda+\lambda\theta$ then:*

1. $\overline{G}^C$ and $\overline{G}^I$ are continuous and their common support is $[\max\{\mu_E, \lambda\theta\}, 1-\lambda+\lambda\theta]$,
2. $G^C$ and $G^I$ are continuous with support $[\lambda\theta, 1-\lambda+\lambda\theta]$, and
3. for all $\mu \in (\lambda\theta, 1-\lambda+\lambda\theta)$, $G^C(\mu) \leq G^I(\mu)$.

*Proof.* In Appendix A.2. □

The continuity of the reputation distributions derives from the fact that the operator $T$ maps the set of continuous distribution functions into itself, so that the fixed point must be in it. On the other hand, the reputation of competent firms first-order stochastically dominates the reputation of inept firms because the signal conditional on $H$ first-order stochastically dominates the signal conditional on $L$.

In the previous Lemma, $\mu_E$ is a parameter taking any value in $[0,1]$. However, if $\mu_E$ is a rational expectation, it cannot take on any value in that interval. In particular, given that competent firms have stochastically larger reputations than inept ones, they will be underrepresented in the group of entrants and therefore $\mu_E$ cannot be higher than $\theta$, the unconditional probability of being competent.

If $\mu_E > \lambda\theta$, the rational expectations' constraint is fulfilled whenever $\mu_E$ is a fixed point of the following function:

$$\psi(\mu) \equiv \frac{\theta G^C(\mu)}{\theta G^C(\mu) + (1-\theta) G^I(\mu)}, \tag{17}$$

as in Equation 2. Observe, however, that this function is defined in terms of $G^C$ and $G^I$, which depend themselves on $\mu_E$ as a parameter. Hence, $\psi$ is more properly written $\psi(\mu|\mu_E)$. It turns out that $\psi$ is continuous in its parameter $\mu_E$, because so are the steady state distributions–insofar as they are the fixed points of a contraction (see De la Fuente, 2000, Chapter 2, Theorem 7.18). Then, the rationally expected $\mu_E$ are its fixed points in the following sense:

$$\mu_E = \psi(\mu_E|\mu_E). \tag{18}$$

The following Lemma refers to the case in which there are some exit and entry flows, no matter how small, i.e., where the parameter $\mu_E$ is larger than $\lambda\theta$ :

**Lemma 2.4.** *If $\mu_E > \lambda\theta$, the function $\psi(\mu|\mu_E)$ maps $(\lambda\theta, 1-\lambda+\lambda\theta]$ into $(\lambda\theta, \theta]$. Moreover, if the following condition holds:*

$$\exists N < \infty : \forall n < N,\ f_H^{(n)}(0) = f_L^{(n)}(0) = 0 \text{ and } f_H^{(N)}(0) = 0 \text{ while } f_L^{(N)}(0) \neq 0,$$

*(where $f^{(k)}$ denotes the $k^{th}$-derivative of $f$), then $\lim_{\mu \to \lambda\theta+} \psi(\mu|\mu_E) = \lambda\theta$.*

*Proof.* In Appendix A.3. □

Notice that the condition in Lemma 2.4 is closely related to the assumption that $\lim_{r\to 0} R(r) = 0$,[4] that is, the requirement that the likelihood function be continuous at $r = 0$. In particular, it says that for the very lowest signals possible to be observed, it is infinitely more likely that those signals came from a low quality service than from a high quality one. Under that condition, the following function is a natural extension of $\psi$:

$$\psi^*(\mu|\mu_E) = \begin{cases} \lambda\theta & \text{for } \mu = \lambda\theta \\ \psi(\mu|\mu_E) & \text{for } \mu \in (\lambda\theta, 1-\lambda+\lambda\theta] \end{cases}, \tag{19}$$

where $\psi^*$ is a continuous function since $\lim_{\mu\to\lambda\theta+} \psi(\mu|\mu_E) = \lambda\theta$.

Clearly $\lambda\theta$ is a fixed point of $\psi^*$. As such, $\mu_E = \lambda\theta$ can be thought as a limiting rational expectation: even though it is an out-of-equilibrium belief–since there is no entry–, it is the limit of the Bayesian inference represented by $\psi$, and in that sense it is a consistent belief.

Moreover, there are cases in which it is the unique consistent belief. For instance, if $r$ has a Beta distribution, with parameters $a$ and $b$ when conditional on $H$, and with parameters $c$ and $d$ when conditional on $L$, with $a, b, c, d \in \mathbb{N}$. The density functions are then given by:

$$\begin{aligned} f_H(r) &= \frac{r^{a-1}(1-r)^{b-1}}{B(a,b)} \text{ and} \\ f_L(r) &= \frac{r^{c-1}(1-r)^{d-1}}{B(c,d)}, \end{aligned}$$

where $B$ is the Beta function. Assumption 2.2 is satisfied if $a > c$ and $b < d$, in which case the $N$ in the condition of Lemma 2.4 is $N = c$. Figure 3(a) shows the steady-state reputation distributions, and Figure 3(b), $\psi^*(\mu|\mu_E)$ for $\mu = \lambda\theta$, when $\lambda = 0.1$, $\theta = 0.5$, $a = d = 3$ and $b = c = 2$. It is apparent from 3(b) that $\lambda\theta$ is the unique fixed point of $\psi^*$, and hence the unique consistent belief.

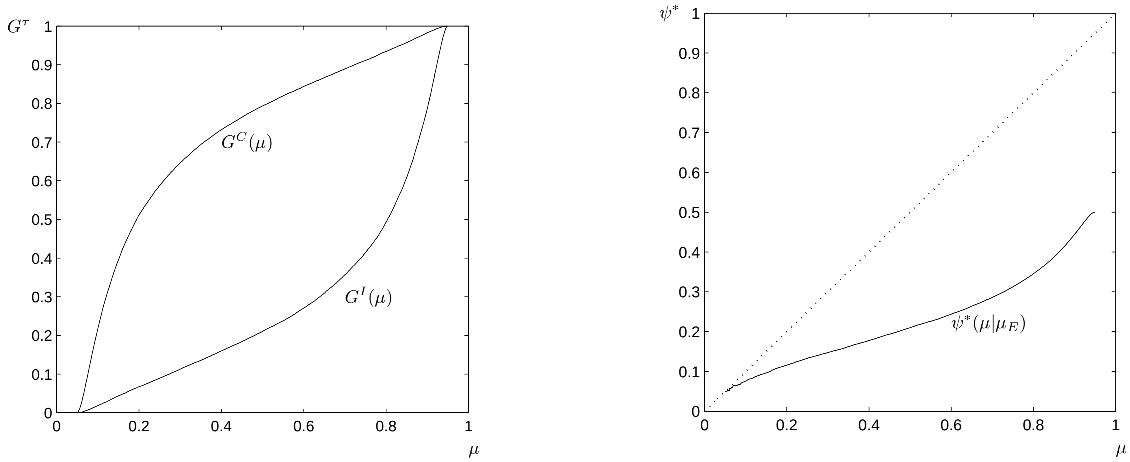
#### 2.3.3 Equilibrium

**Definition 2.5.** A high-quality equilibrium for this game is:

- a price list $p(\mu)$,
- a reputation distribution for each type of firm $(G^C, G^I)$, and
- an entry-level reputation $\mu_E$,

such that:

[4] If the condition in Lemma 2.4 is satisfied, then $\lim_{r\to 0} R(r) = 0$.




(a) $G^C$ and $G^I$ for $\mu_E = \lambda\theta$

(b) $\psi^*$ for $\mu_E = \lambda\theta$

Figure 3: Reputation distributions and rationally-expected entry-level reputation

**(i)** Bayesian updating and Steady State: Firms' reputations evolve by Bayes' rule according to Equation 8, and the population cdfs of reputations are the fixed point of $T$ as defined in Equation 16;

**(ii)** Rational Expectations: $\mu_E$ is a fixed point of $\psi^*$ defined in Equation 19;

**(iii)** Choices and market clearing: At $p(\mu)$ all competent firms choose $q = H$, a mass $(1-\theta)$ of inept firms chooses $q = L$ and the remaining ones choose $q = \varnothing$, and the unit mass of consumers chooses to buy, and are indifferent among active providers.

Fix the firms' behavior as per (iii). Then, the shape of the price function $p(\mu)$ is determined by consumers' preferences. In effect, since all consumers are identical, in equilibrium they must be indifferent among all active producers. The assignment of consumers to firms is not determined. This, in turn, implies that the equilibrium price function does not depend on the reputation distributions–the supply side of the market–; instead, it is determined by the consumer's global indifference, up to the intercept, which is determined by a border condition.

Each consumer chooses a provider $\mu$ and pays a price $p(\mu)$. From the maximization of consumer's utility function (Equation 3) we obtain the first order condition:

$$\frac{\partial E[u]}{\partial \mu} = u(H,z) - u(L,z) - \frac{\partial p(\mu)}{\partial \mu}\left(\mu \frac{\partial u(H,z)}{\partial z} + (1-\mu)\frac{\partial u(L,z)}{\partial z}\right) = 0 \tag{20}$$

Market clearing requires that Equation 20 holds for all $\mu$, since otherwise all consumers–who are identical–will strictly prefer some providers over others, and the latter would be unable to sell their services. Hence, Equation 20 yields a first-order differential equation for $p(\mu)$. Solving for its derivative, we get:

$$\frac{\partial p(\mu)}{\partial \mu} = \frac{u(H, y - p(\mu)) - u(L, y - p(\mu))}{[u_z(H, y-p(\mu))\,\mu + (1-\mu)\,u_z(L, y - p(\mu))]}. \tag{21}$$

In words, the slope of the equilibrium price function *is* the MRS between the service's perceived quality and the consumption good. Notice that this means that if the service's perceived quality $\mu$ is a normal good, then the slope of the price function increases with income and decreases with the price level.

Equation 21 defines $p(\mu)$ up to a border condition, say $p(\mu_E) = p_0$. That is, it determines the slope of the price function, but not its level. Let us denote by $p(\mu, p_0)$ a generic member of the family of price functions that solves Equation 21, a family that becomes indexed by $p_0$. The function $p(\mu, p_0)$ is continuous in $p_0$, and for any two different values $p_0$ and $p_0'$, one of the functions $p(\mu, p_0)$ and $p(\mu, p_0')$ must be strictly higher than the other one, for all $\mu$. This implies that both, $v_I$ and $v_C$ are increasing in $p_0$.

It is well known that in the assignment model with homogeneous valuations, the price is indeterminate in a Walrasian equilibrium if both sides of the market are of the same size (mass): any price that yielded a non-negative surplus to buyers and sellers would be an equilibrium price. If, on the other hand, the mass of sellers were larger than the mass of buyers, then the price would become determinate, i.e., there would be a unique Walrasian equilibrium price, namely the one that yields no surplus to sellers. This is precisely what happens in this model. In the no-entry case the price function is indeterminate: any function $p(\mu, p_0)$ that yields a non negative surplus to each side of the market is an equilibrium price function. However, in the free entry case, the lowest of such functions is the only one that can be observed in equilibrium, namely, the one that yields zero profits to entrants. Otherwise a larger number (mass) of firms than there are consumers would want to participate. Then, out of the family $p(\mu, p_0)$, the free entry condition:

$$v_I(\mu_E) = 0 \tag{22}$$

pins down a particular value for $p_0$, whence the equilibrium price function $p(\mu)$. Any function higher than this may still induce full participation of consumers, but would not clear the market because it would induce more production than is demanded. Indeed, the value function $v_I(\mu)$ is increasing in $\mu$, because $p$ is.[5] This implies that all firms with higher reputations (those to the right of $\mu_E$) will strictly prefer to participate, while those to the left will not. Moreover, since $v_C(\mu) \geq v_I(\mu)$ (because competent firms can always act as inept firms do), participation of the competent is guaranteed by participation of the inept.

Finally, equilibrium also requires incentive compatibility for competent firms: they must prefer to produce high quality rather than low quality. This will be so if the cost savings generated by producing low quality $(c_H - c_L)$ are smaller than the foregone expected profit associated with renouncing a stochastically larger reputation in the future.

# 3 Equilibrium with potential, but not actual, entry

The case in which the entry-level reputation is the lowest possible $(\mu_E = \lambda\theta)$ is particularly interesting because of the complete absence of exit and entry flows. As explained before, there is no signal value for which the reputation can fall below $\lambda\theta$, and thus no firm ever exits the market. The threat of entry keeps prices down, but on the equilibrium path never materializes.[6]

In this case, Equation 8 completely characterizes the evolution of any firm's reputation, and therefore the value functions are defined by the following integral equations:

$$v_C(\mu) = (1-\delta)(p(\mu) - c_H) + \delta(1-\lambda)\int_0^1 v_C(\varphi(\mu, r))\, dF_H \tag{23a}$$

$$v_I(\mu) = (1-\delta)(p(\mu) - c_L) + \delta(1-\lambda)\int_0^1 v_I(\varphi(\mu, r))\, dF_L \tag{23b}$$

[5] Moreover, the value function for competent firms, $v_C(\mu)$, is also increasing (see De la Fuente, 2000, Chapter 12, Theorem 1.12).

[6] Moreover, we conjecture that there is no other consistent entry-level reputation because for $\psi$ to increase faster than 1 (a necessary condition for the existence of a second fixed point), the group of competent firms that would exit would have to grow faster than the group of inept firms that would exit as $\mu_E$ increases, which *contradicts* the fact that the first group first-order stochastically dominates the second one in the steady state. However, we have not been able to prove this claim.

where the stage payoff is given by the profit margin, and the expected continuation payoff is weighted by a discount factor $\delta$ times the survival probability.

The following incentive and participation conditions must hold for all $\mu \in [\lambda\theta, 1-\lambda+\lambda\theta]$:

$$v_C(\mu) \geq (1-\delta)(p(\mu)-c_L) + \delta(1-\lambda)\int_0^1 v_C(\varphi(\mu,r))\,dF_L \tag{24}$$

$$v_I(\mu) \geq 0 \tag{25}$$

The first inequality says that a competent firm with a current reputation $\mu$ should prefer high quality over low quality. The second inequality says that an inept firm should prefer to be active. Finally, the free entry condition is:

$$v_I(\lambda\theta) = 0, \tag{26}$$

whereby the inactive firms (with reputation $\lambda\theta$) expect no gain from entering the market.

Combining Equations 23a and 24, and solving for the differential cost of high and low quality, we obtain:

$$c_H - c_L \leq \frac{\delta(1-\lambda)}{1-\delta}\left(\int_0^1 v_C(\varphi(\mu,r))\,dF_H - \int_0^1 v_C(\varphi(\mu,r))\,dF_L\right). \tag{27}$$

Define the right-hand-side of Equation 27 as:

$$\xi(\mu) \equiv \frac{\delta(1-\lambda)}{1-\delta}\left(\int_0^1 v_C(\varphi(\mu,r))\,dF_H - \int_0^1 v_C(\varphi(\mu,r))\,dF_L\right). \tag{28}$$

The function $\xi(\mu)$ measures the long term net benefit of producing high quality at a given stage as opposed to deviating to low quality for one period. This benefit stems from the (stochastic) increase in future stream of reputations.

Since $v_C$ and $\varphi$ are increasing functions, and since the signal under $H$ first-order stochastically dominates the one under $L$, $\xi(\mu)$ is clearly non-negative. How large it is depends in part on how strong the dominance is, i.e., how good is the signal. It also depends on the shape of the value function $v_C$, which in turn depends on that of the price function $p(\mu)$. To see this more clearly, let $\varphi^n(\mu, \mathbf{r}^n)$ denote the $n-$th iterate of the operator $\varphi$ (as defined by Equation 9), that is, the trajectory of a reputation that starts at the level $\mu$ and after a history $\mathbf{r}^n \equiv \{r_t\}_{t=1}^n$ of $n$ signals. Recursive substitution of Equation 23 and the application of the law of iterated expectations yields:

$$v_C(\mu) = (1-\delta)\sum_{k=1}^{\infty}\delta^{k-1}(1-\lambda)^{k-1}E_{H^{k-1}}\left[p\left(\varphi^{k-1}\left(\mu,\mathbf{r}^{k-1}\right)\right)-c_H\right] \tag{29a}$$

$$v_I(\mu) = (1-\delta)\sum_{k=1}^{\infty}\delta^{k-1}(1-\lambda)^{k-1}E_{L^{k-1}}\left[p\left(\varphi^{k-1}\left(\mu,\mathbf{r}^{k-1}\right)\right)-c_L\right] \tag{29b}$$

where the subscript $H^n$ in the expectation operator means that each of the sequence of signals with respect to which the expectation is taken has been drawn from the distribution $F_H$; similarly for $L^n$ in regard to $F_L$.

The value functions have the form of an expected present value of profits, where the discount factor includes the survival probability $(1-\lambda)$. Replacing $v_C(\mu)$ in Equation 27 as in 29a, and writing $\left(\varphi^{k-1}\left(\varphi(\mu,r),\mathbf{r}^{k-1}\right)\right)$ for the $k$–th iterate of the operator $\varphi$, the function $\xi(\mu)$ becomes:

$$\begin{aligned}\xi(\mu) = &\int_0^1\sum_{k=1}^{\infty}\delta^k(1-\lambda)^k E_{H^{k-1}}\left[p\left(\varphi^{k-1}\left(\varphi(\mu,r),\mathbf{r}^{k-1}\right)\right)-c_H\right]dF_H \\ &-\int_0^1\sum_{k=1}^{\infty}\delta^k(1-\lambda)^k E_{H^{k-1}}\left[p\left(\varphi^{k-1}\left(\varphi(\mu,r),\mathbf{r}^{k-1}\right)\right)-c_H\right]dF_L,\end{aligned} \tag{30}$$

which can be written more compactly as:

$$\begin{aligned}\xi(\mu) &= \int_0^1 \sum_{k=1}^{\infty} \delta^k (1-\lambda)^k E_{H^{k-1}} \left[ p\left(\varphi^k\left(\mu, \mathbf{r}^k\right)\right) - c_H \right] dF_H \\ &\quad - \int_0^1 \sum_{k=1}^{\infty} \delta^k (1-\lambda)^k E_{H^{k-1}} \left[ p\left(\varphi^k\left(\mu, \mathbf{r}^k\right)\right) - c_H \right] dF_L. \end{aligned} \tag{31}$$

In a final step, we introduce the integrals into the sums to obtain:

$$\xi(\mu) = \sum_{k=1}^{\infty} \delta^k (1-\lambda)^k \left( E_{H^k} \left[ p\left(\varphi^k\left(\mu, \mathbf{r}^k\right)\right)\right] - E_{LH^{k-1}} \left[ p\left(\varphi^k\left(\mu, \mathbf{r}^k\right)\right)\right]\right). \tag{32}$$

where $LH^{k-1}$ stands for a sequence of signal draws that starts with $L$ and continues with $H^{k-1}$. Equation 32 shows that the net benefit of producing high quality instead of low quality today is the discounted stream of the expected increase in the prices the firm will be able to charge in virtue of its stochastically higher reputation. The discount factor includes the time discount $\delta$ as well as the survival probability $(1-\lambda)$.

Thus, the incentive compatibility condition can be written as:

$$c_H - c_L \leq \xi(\mu) \qquad \forall \mu \in [\lambda\theta, 1-\lambda+\lambda\theta] \tag{33}$$

The condition says that at any point in time and at any reputation level the expected benefit of producing a higher quality must exceed its higher production cost variable. The incentive compatibility condition thus translates into an upper bound on the cost differential:

$$c_H - c_L \leq \min_{\mu \in [\lambda\theta, 1-\lambda+\lambda\theta]} \xi(\mu) \tag{34}$$

What sustains the provision of high quality in equilibrium is the reputational premium, namely the expected gain associated to achieving a stochastically higher reputation in the future. This expected benefit exists if and only if the price function is increasing in $\mu$. Moreover, the following Lemma shows that the magnitude of this expected benefit is directly related to the slope of $p(\mu)$.

**Lemma 3.1.** *$\xi(\mu)$ is a strictly increasing function of $\frac{\partial p(\mu)}{\partial \mu}$.*

*Proof.* Integrating successively by parts (see Appendix A.4 for details), Equation 32 can be written as:

$$\begin{aligned}\xi(\mu) = \sum_{k=1}^{\infty} \delta^k (1-\lambda)^k E_{H^{k-1}} \Bigg[ \int_{r_1} & \frac{\partial p\left(\varphi^k\left(\mu_1, \mathbf{r}^k\right)\right)}{\partial \mu_k} \frac{\partial \varphi\left(\varphi^{k-1}\left(\mu_1, \mathbf{r}^{k-1}\right), r_k\right)}{\partial \mu_{k-1}} \ldots \\ & \ldots \frac{\partial \varphi(\mu_1, r_1)}{\partial r_1} \left(F_L(r_1) - F_H(r_1)\right) dr_1 \Bigg], \end{aligned} \tag{35}$$

where $\frac{\partial p\left(\varphi^k\left(\mu_1, \mathbf{r}^k\right)\right)}{\partial \mu_k} = \frac{\partial p(\mu_k)}{\partial \mu_k}$ is the slope of $p(\mu)$ at $\mu = \mu^k$.

We know that $(F_L(r) - F_H(r)) > 0$ because the signal conditional on $H$ first-order stochastically dominates the signal conditional on $L$. On the other hand, the higher the prior reputation, the higher the posterior after any history $\mathbf{r}^n$, so that all derivatives involving $\frac{\partial \varphi(\mu)}{\partial \mu}$ are positive. Therefore, for any given $\mu^k$, the right-hand side of Equation 33–i.e., the expected profit associated with achieving a stochastically higher reputation in the future–is increasing in $\frac{\partial p(\mu)}{\partial \mu}$. □

Hence, the steeper the slope of $p(\mu)$, the higher is the upper bound on $(c_H - c_L)$–the differential cost of high and low quality–imposed by the incentive compatibility constraint. That is, the steeper the slope of $p(\mu)$, the easier it is for the incentive compatibility condition to be met.

Notice that since the function $\xi$ depends on the shape of the price function $p(\mu)$, which is in turn determined on the demand side of the market–from consumers' indifference–, as a general rule $\xi$ will indeed depend on $\mu$. Since $\xi$ measures the expected net benefit from choosing high quality, this implies that incentive compatibility will not be binding for all reputation levels.[7]

On the other hand, the free entry condition imposes a ceiling on $p(\mu)$: out of the family of price functions that make consumers indifferent among providers and satisfy the participation constraint for all $\mu \in (\lambda\theta, 1 - \lambda + \lambda\theta)$, the free entry condition pins down the lowest. The following lemma shows that if perceived quality is a normal good–i.e., if the MRS is increasing in $z$–, then the positive income effect associated with imposing this ceiling on the price actually increases the slope of $p(\mu)$.

**Lemma 3.2.** *The lower $p_0$, the higher is $\frac{\partial p(\mu, p_0)}{\partial \mu}$ if and only if the service's perceived quality is a normal good.*

*Proof.* The differential equation in Equation 21 sets the slope of $p(\mu)$ equal to the MRS evaluated at $z = y - p(\mu)$. But for any pair $(p_0, p_0')$, if $p_0 < p_0'$ then $p(\mu, p_0) < p(\mu, p_0')$ for all $\mu$. Then, if the MRS is increasing in $z$, the lower $p_0$, the higher the MRS is, and the steeper the slope of $p(\mu)$. □

Taken together, Lemma 3.2 and Lemma 3.1 imply that:

**Corollary 3.3.** *If the service's perceived quality is a normal good, then $\min_\mu \xi(\mu)$ is increasing in $p_0$.*

All else equal, without free entry the equilibrium price functions are at least as high as the one with free entry. Generically, then, the change from a situation without free entry to another with free entry involves a drop in $p_0$, and in this sense free entry reduces $p_0$. This leads us to our main theorem:

**Theorem 3.4.** *If the service's perceived quality is a normal good, then free entry increases the upper bound on the differential cost of high quality for which a high-quality equilibrium can be sustained.*

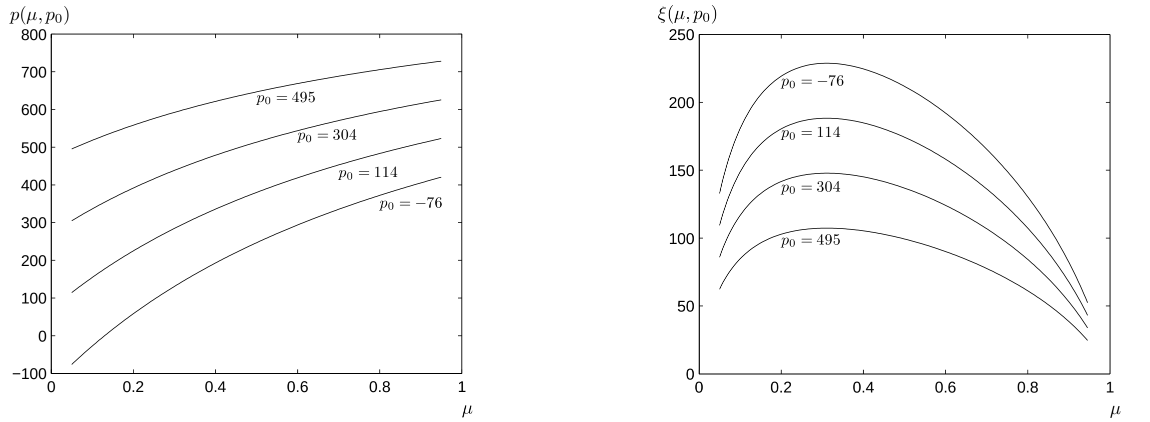
Indeed, as entry is freed, $p_0$ drops. As a consequence, the upper bound on the cost differential $\min_\mu \xi(\mu)$ in Equation 24 increases. This means that the high-quality equilibrium can be sustained for a wider range of differential costs of high quality. In other words, the high-quality equilibrium is easier to sustain.

As an illustration, consider the example in Section 2.3.2 under the same parametrization as Figure 3, namely $a = d = 3$, $b = c = 2$, $\lambda = 0.1$. Figure 4(a) depicts the family of price functions for different levels of $p_0$, and Figure 4(b) the associated $\xi(\mu)$ functions for the case in which consumers' preferences are represented by the function:

$$u(q, z) = z(q + 1)$$

and $y = 1,000$. If we assume $c_L = 0$, the free entry condition picks $p_0 = -76$, the lowest curve in Figure 4(a). In Figure 4(a) we see that the higher the price function, the less steep it is. In Figure 4(b) we confirm that the *smaller* the price function, the *larger* is the gross benefit of choosing high quality as asserted in Theorem 3.4. Figure 5 depicts the value functions for competent and inept firms whith the lowest equilibrium price function, in the free entry case $(v_I(\mu_E) = 0)$; as expected, they are both increasing, and the values for competent firms are always higher than for inept ones.

[7] Hörner (2002) studies a case in which all competent firms, regardless of their reputation, are exactly indifferent between high and low qualities. The conditions that make this possible are, however, very special. In his model, as in ours, the price function is determined by consumer indifference among firms. The variable that makes the adjustment is each firm's production: firms are not capacity constrained. However, firms do not choose output even though they would like to. Instead, they serve just the number of customers that happen to be required for that adjustment. Consumers, being indifferent, play a mixed strategy. Moreover, what prevents firms from attempting to break that indifference by changing prices–cutting prices when margins are positive, and raising them when margins are negative–are consumers' out-of-equilibrium beliefs.




(a) The family of price functions $p(\mu, p_0)$

(b) The family of $\xi(\mu, p_0)$ functions, the gross benefit from producing high quality.

Figure 4: Price functions and gross benefit from high quality

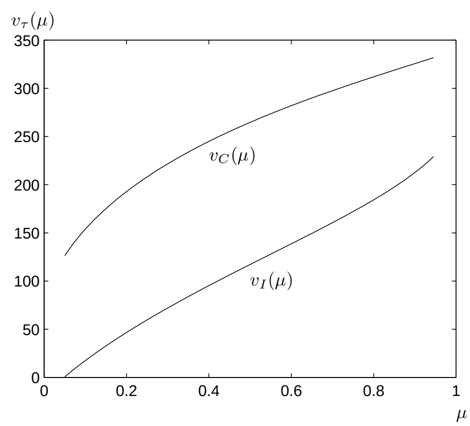



Figure 5: Value functions under free entry ($p_0 = -76$)

# 4 Equilibrium with exit and entry flows

This section discusses the extension of the previous results to the case in which the entry-level reputation is high enough for a positive mass of firms to be washing their names, i.e., $\mu_E > \lambda\theta$.

The first difference to notice is that Equation 8 no longer describes the evolution of an active firm's reputation. Instead, it is described by:

$$\mu_{t+1} = \varphi^* \left(\mu_t, r, \mu_E\right), \tag{36}$$

where the function $\varphi^*$ is defined by:

$$\varphi^* \left(\mu_t, r, \mu_E\right) = \max \left\{\varphi \left(\mu_t, r\right), \mu_E\right\}, \tag{37}$$

and reflects the fact that firms choose to re-enter under a reputation $\mu_E$ whenever the public signal turns out so bad that their reputation falls below the threshold $\mu_E$. The value functions can be written almost as before, simply replacing $\varphi$ by $\varphi^*$. Thus, the function $\xi(\mu)$ is defined by:

$$\xi(\mu) \equiv \frac{\delta(1-\lambda)}{1-\delta}\left(\int_0^1 v_C\left(\varphi^*\left(\mu, r, \mu_E\right)\right) dF_H - \int_0^1 v_C\left(\varphi^*\left(\mu, r, \mu_E\right)\right) dF_L\right) \tag{38}$$

Lemma 3.1 still holds. The thing to notice is that even though $\varphi^*$ is flat in the interval $(\lambda\theta, \mu_E)$ and therefore in it the derivatives of the form $\frac{\partial\varphi^*\left(\varphi^{*k-1}\left(\mu_1, \mathbf{r}^{k-1}\right), r_k\right)}{\partial\mu_{k-1}}$ vanish, there are always intervals with positive mass where those derivatives are strictly positive, yielding a strictly positive expectation. Intuitively, the probability of being able to charge higher prices in the future always increases with the probability of getting better public signals, because there is always a positive probability that the signals will yield reputations higher than $\mu_E$.

As Lemma 3.1 still holds, so does Theorem 3.4, our main result.

There is an interesting case where the entry-level reputation must be greater than $\lambda\theta$, namely, when not only active but also inactive firms are subject to type changes in a random fashion. The main effect of allowing for this is the alleviation of the adverse selection effect that inactive firms face since, in the high quality equilibrium under consideration, only inept firms may prefer to become inactive. The fact that these inept firms that remained inactive may become competent automatically improves their reputation, thereby opening the possibility of an equilibrium in which entry and exit do occur, i.e., $\mu_E > \lambda\theta$.

Let $\phi_a$ be the probability of becoming competent conditional on having being active the preceding date, and $\phi_i$ the corresponding probability conditional on having being inactive. In the previous sections, we had $\phi_a = \theta$ and $\phi_i = 0$; under this condition, the mass of competent firms remained constant at the level $\theta$. Now, however, the mass of competent firms is the sum of:

$$\begin{aligned} \text{active competents that didn't change} &: \quad \theta(1-\lambda+\lambda\phi_a) \\ \text{active inepts that changed} &: \quad (1-\theta)\lambda\phi_a \\ \text{inactive inepts that changed} &: \quad (\kappa-1)\lambda\phi_i \end{aligned}$$

The mass of competent firms will thus remain constant at $\theta$ if:

$$\theta = \theta(1-\lambda+\lambda\phi_a) + (1-\theta)\lambda\phi_a + (\kappa-1)\lambda\phi_i, \tag{39}$$

which is equivalent to:

$$\phi_a = \theta - \phi_i(\kappa-1) \tag{40}$$

We assume that $(1-\lambda+\lambda\phi_a)$, the largest reputation an active firm may reach, is larger than $\theta$; otherwise, most competent firms would come from the group of inactive firms, and the entry-level reputation would be too high for any firm to want to keep its own.

The modified version of Equation 9, is given by:

$$\varphi(\mu, r) = \lambda\phi_a + (1-\lambda)\frac{\mu f_H(r)}{\mu f_H(r) + (1-\mu) f_L(r)}. \tag{41}$$

We still have that the difference equation for the firm's reputation is defined by:

$$\mu_{t+1} = \varphi^*\left(\mu_t, r, \mu_E\right). \tag{42}$$

Likewise, the modified version of Equation 17 is:

$$\psi(\mu) \equiv \frac{\alpha G^C(\mu) + \beta}{\alpha G^C(\mu) + (1-\alpha) G^I(\mu)}, \tag{43}$$

where $\alpha \equiv (\theta(1-\lambda)+\lambda\phi_a)$ and $\beta \equiv \lambda(\theta-\phi_a)$. Observe that $\alpha \leq \theta$ and that $\beta$ can be written as $\beta = \theta - \alpha$ or $\beta = \lambda\phi_i(\kappa-1)$, the mass of newly-born competents among inactive firms. It is apparent that $\lambda\theta$ can no longer be a fixed point of $\psi$, because $\lim_{x\to\lambda\theta^+}\psi(x)=\infty$. In effect, the mass of the newly-born competents among inactive firms becomes disproportionally large when compared with the group of firms that exited as the latter becomes smaller.

On the other hand, since $\psi$ is continuous and $\psi(1-\lambda+\lambda\theta)=\theta$, it has at least one fixed point $\mu_E \in (\lambda\theta, 1-\lambda+\lambda\theta)$. Moreover, Equation 43 has the same properties as its previous version (Equation 17) with regard to its continuity in $\mu_E$ and the existence of a solution to Equation 18. Hence, when types change also among inactive firms, the equilibrium entry-level reputation is strictly higher than the minimum achievable reputation ($\mu_E > \lambda\theta$), so that at all times there are exit and entry flows of both competent and inept firms.

# 5 Further remarks

In the face of two markets, the first with high price dispersion and low mean, and the second with low price dispersion and high mean, which of the two is more likely to be engaged in high quality production? Our results suggest that the first one. Even though the rents are smaller, incentives are better aligned because the reputational premium is larger.

The conclusion that intensified competition lowers prices but increases the price differences among reputation levels can be seen as the result of a comparative static exercise, namely: What would happen with incentives when the thick side of the market switches from being the demand side to being the supply side?

There are other interesting comparative static results as well. Take, for example, the case of an increase in consumers' income. Being a normal good, their willingness to pay for reputations would rise, and consequently the price function would become steeper. Moreover, higher prices would induce entry, so that by this second effect the price function would have to decrease, inducing a further increase in its slope. Hence, incentive compatibility would again unambiguously improve. It turns out that a different source of increase in demand, namely the increase in the mass of consumers to overpass that of producers, would have the opposite effect–simply reverse signs in our central analysis.

As for robustness, we may wonder about the assumption that all consumers are alike, a simplification that has the consequence that the shape of the price function is solely determined by demand, by consumers' indifference. Vial (2008) considers the case of heterogeneous consumers, where the supply side also intervenes in its determination: prices need to induce the assortative matching of consumers to firms. However, by the income effect the lower Walrasian price functions are steeper.

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# A Appendices

## A.1 Computation of the reputation distributions

This section derives Equation 14, which gives the distributions of reputation for competent and inept types, $\Pr(\mu_{t+1} \le x | \tau_{t+1}) = G_{t+1}^{\tau_{t+1}}(x)$.

Let $H$ denote the joint cdf of the random variables $(\mu_{t+1}, \mu_t, \tau_{t+1}, \tau_t, r_t)$ after the time-$t$ truncation process (i.e., the one generated by the exit and entry decisions) and before the time-$t+1$ truncation process. $H$ is a mixed distribution: while $\mu_t$ and $\mu_{t+1}$ are continuous random variables, $\tau_t$ and $\tau_{t+1}$ are discrete. Moreover, $\mu_t$ has a point mass at $\mu_E$. The marginal cdf of $H$ over $\mu_t$ is denoted by $\overline{G}_t(\mu_t)$; notice that it is discontinuous at $\mu_E$. $\overline{G}_{t+1}(\mu_{t+1})$ is the truncated version of $G_{t+1}(\mu_{t+1})$, i.e., after the exit and entry process: for $\mu_{t+1} \ge \mu_E$, $\overline{G}_{t+1}(\mu_{t+1}) = G_{t+1}(\mu_{t+1})$, otherwise, $\overline{G}_{t+1}(\mu_{t+1}) = 0$. Finally, the marginal cdf of $H$ over $r_t$ conditional on $\tau_t$ is denoted by $F_{\tau_t}$, and assumed to have a density. Note that the distribution of $r_t$ only depends on current type $\tau_t$.[8] Hence, the distributions of

[8] In the high-quality equilibrium, all competent firms pay the cost of high quality so that their results are drawn from $F_H$, while inept firms' results are drawn from $F_L$. Hence, we may simplify the notation by writing $F_C$ instead of $F_H$, and $F_I$ instead of $F_L$, so that choice and type are treated as the same variable. That is why we write $F_{\tau_t}$.

reputations and signals relate to $H$ as follows:

$$\begin{array}{ll} G_{t+1}^{\tau_{t+1}} \equiv H\left(\mu_{t+1}|\tau_{t+1}\right) & G_{t+1} \equiv H\left(\mu_{t+1}\right) \\ \overline{G}_t^{\tau_t} \equiv H\left(\mu_t|\tau_t\right) & \overline{G}_t \equiv H\left(\mu_t\right) \\ F_{\tau_t} \equiv H\left(r_t|\tau_t\right) & \end{array}$$

On the other hand, the exogenous type-change process is independent of any other variable, and is fully described by the joint distribution in Table 1:

| | | Before ($\tau_t$) | | |
|---|---|---|---|---|
| | | $C$ | $I$ | Marginal |
| After | $C$ | $\theta\left(1-\lambda+\lambda\theta\right)$ | $\lambda\theta\left(1-\theta\right)$ | $\theta$ |
| ($\tau_{t+1}$) | $I$ | $\lambda\theta\left(1-\theta\right)$ | $\left(1-\lambda\theta\right)\left(1-\theta\right)$ | $1-\theta$ |
| | Marginal | $\theta$ | $1-\theta$ | |

Table 1: Joint distribution of types

The marginal distributions over $\mu_{t+1}$ (before truncation), conditional on $\tau_{t+1}$, is the expectation of $H\left(\mu_{t+1}|\tau_t, r_t, \mu_t, \tau_{t+1}\right)$ over $\tau_t$, $r_t$, and $\mu_t$. For instance, in the case of $\tau_{t+1} = C$, we have:

$$\begin{aligned} G_{t+1}^{C}\left(x\right) &= \Pr\left(\mu_{t+1} \leq x|\tau_{t+1} = C\right) \\ &= H\left(\mu_{t+1}|\tau_{t+1} = C\right) \\ &= E_{\tau_t}\left[E_{r_t}\left[E_{\mu_t}\left[H\left(\mu_{t+1}|\tau_t, r_t, \mu_t, \tau_{t+1} = C\right)\right]\right]\right] \end{aligned}$$

There are different ways in which we can compute this expectation, depending on the order of integration we choose. In any case, notice that there is a functional relation between $\mu_{t+1}$ and $(\mu_t, r_t)$, as established by Equation 9:

$$\mu_{t+1} = \varphi\left(\mu_t, r_t\right)$$

This implies that when conditioning on $(r_t, \mu_t)$, $H$ becomes the indicator function:

$$H\left(\mu_{t+1}|r_t, \mu_t\right) = \mathbf{1}_{\{\varphi(\mu_t, r_t) \leq \mu_{t+1}\}}$$

Thus, we can write:

$$G_{t+1}^{C}\left(x\right) = E_{\tau_t}\left[E_{r_t}\left[E_{\mu_t}\left[\mathbf{1}_{\{\varphi(\mu_t, r_t)\leq x\}}|\tau_t, \tau_{t+1} = C\right]\right]\right]$$

If we start with the expectation over the current type, we get:

$$G_{t+1}^{C}\left(x\right) = \sum_{\tau_t} \Pr\left(\tau_t|\tau_{t+1} = C\right) E_{r_t}\left[E_{\mu_t}\left[\mathbf{1}_{\{\varphi(\mu_t, r_t)\leq x\}}|\tau_t, \tau_{t+1} = C\right]\right]$$

Recall from Equations 13 and 12 that we write:

$$\begin{aligned} \mu_t &= \tilde{\mu}\left(x, r_t\right) \\ r_t &= \tilde{r}\left(x, \mu_t\right) \end{aligned}$$

for the implicit functions for $\mu_t$ and $r_t$ from $x = \varphi\left(\mu_t, r_t\right)$. Hence:

$$E_{\mu_t}\left[\mathbf{1}_{\{\varphi(\mu_t, r_t)\leq x\}}|\tau_t, \tau_{t+1} = C\right] = \overline{G}_t^{C}\left(\tilde{\mu}\left(x, r_t\right)\right)$$

so that we get:

$$G_{t+1}^{C}\left(x\right) = \left(1-\lambda+\lambda\theta\right) E_{r_t}\left[\overline{G}_t^{C}\left(\tilde{\mu}\left(x, r_t\right)\right)|\tau_t = C, r_t\right] + \lambda\left(1-\theta\right) E_{r_t}\left[\overline{G}_t^{I}\left(\tilde{\mu}\left(x, r_t\right)\right)|\tau_t = I, r_t\right]$$

On the other hand, the expectation $E_{r_t}\left[\overline{G}_t^C\left(\tilde{\mu}\left(x,r_t\right)\right)|\tau_t=C,r_t\right]$must be computed bearing in mind that $r$ induces a discontinuous distribution over $\overline{G}_t^C\left(\tilde{\mu}\left(x,r_t\right)\right)$, with point mass at 0. Thus, we split it into the two events separated by the point of discontinuity:

$$\begin{aligned}E_{r_t}\left[\overline{G}_t^C\left(\tilde{\mu}\left(x,r_t\right)\right)|\tau_t=C,r_t\right] &= \Pr\left(r_t>\tilde{r}\left(x,\mu_E\right)\right)E_{r_t}\left[\overline{G}_t^C\left(\tilde{\mu}\left(x,r_t\right)\right)|\tau_t=C,r_t>\tilde{r}\left(x,\mu_E\right)\right] \\ &\quad +\Pr\left(r_t\leq\tilde{r}\left(x,\mu_E\right)\right)E_{r_t}\left[\overline{G}_t^C\left(\tilde{\mu}\left(x,r_t\right)\right)|\tau_t=C,r_t\leq\tilde{r}\left(x,\mu_E\right)\right]\end{aligned}$$

where $E_{r_t}\left[\overline{G}_t^C\left(\tilde{\mu}\left(x,r_t\right)\right)|\tau_t=C,r_t>\tilde{r}\left(x,\mu_E\right)\right]$=0, so that:

$$E_{r_t}\left[\overline{G}_t^C\left(\tilde{\mu}\left(x,r_t\right)\right)|\tau_t=C,r_t\right] = \left(1-F_{\tau_t}\left(\tilde{r}\left(x,\mu_E\right)\right)\right)*0+\int_0^{\tilde{r}(x,\mu_E)}\overline{G}_t^C\left(\tilde{\mu}\left(x,r_t\right)\right)dF_H$$

Plugging into the equation for $G_{t+1}^C\left(x\right)$, we get:

$$G_{t+1}^C\left(x\right)=\left(1-\lambda+\lambda\theta\right)\int_0^{\tilde{r}(x,\mu_E)}\overline{G}_t^C\left(\tilde{\mu}\left(x,r_t\right)\right)dF_H+\lambda\left(1-\theta\right)\int_0^{\tilde{r}(x,\mu_E)}\overline{G}_t^I\left(\tilde{\mu}\left(x,r_t\right)\right)dF_L \tag{44}$$

An analogous equation is obtained for $G_{t+1}^I\left(x\right)$ :

$$G_{t+1}^I\left(x\right)=\lambda\theta\int_0^{\tilde{r}(x,\mu_E)}\overline{G}_t^C\left(\tilde{\mu}\left(x,r_t\right)\right)dF_H+\left(1-\lambda\theta\right)\int_0^{\tilde{r}(x,\mu_E)}\overline{G}_t^I\left(\tilde{\mu}\left(x,r_t\right)\right)dF_L \tag{45}$$

Since $\overline{G}_t^\tau\left(\tilde{\mu}\left(x,r_t\right)\right)=G_t^\tau\left(\tilde{\mu}\left(x,r_t\right)\right)$ for all $r_t\leq\tilde{r}\left(x,\mu_E\right)$, we can replace $\overline{G}_t^\tau\left(\tilde{\mu}\left(x,r_t\right)\right)$ by $G_t^\tau\left(\tilde{\mu}\left(x,r_t\right)\right)$ in Equations 44 and 45, and obtain Equation 14, as desired.

## A.2 Proof of Lemma 2.3

We start by establishing that the operator $T$ defined by Equation 16 is a contraction mapping in the set of pairs of cdf's $(\overline{G}^C,\overline{G}^I)$ endowed with the following metric:

$$\rho\left(\left(\overline{G}^C,\overline{G}^I\right),\left(\overline{H}^C,\overline{H}^I\right)\right) = \max\left\{\rho_\infty\left(\overline{G}^C,\overline{H}^C\right),\rho_\infty\left(\overline{G}^I,\overline{H}^I\right)\right\},$$

where;

$$\rho\left(\overline{G}^\tau,\overline{H}^\tau\right)=\sup_{x\in[\lambda\theta,1-\lambda+\lambda\theta]}\left|\overline{G}^\tau\left(x\right)-\overline{H}^\tau\left(x\right)\right|$$

and $\tau\in\{C,I\}$. The supremum is taken over $x\in[\lambda\theta,1-\lambda+\lambda\theta]$ since the supports of $\overline{G}$ and $\overline{H}$ are always contained in this interval.

Recall that the operator $T$ transforms the pair $\left(\overline{G}_t^C,\overline{G}_t^I\right)$ into a pair $\left(\overline{G}_{t+1}^C,\overline{G}_{t+1}^I\right)$ according to:

$$\overline{G}_{t+1}^\tau\left(x\right) = \begin{cases}0 & x<\mu_E\\ G_{t+1}^\tau\left(x\right) & x\geq\mu_E\end{cases} \quad \text{for } \tau=C,I,$$

with:

$$\begin{aligned}G_{t+1}^C\left(x\right) &= \left(1-\lambda+\lambda\theta\right)\int_0^{\tilde{r}(x,\mu_E)}\overline{G}_t^C\left(\tilde{\mu}\left(x,r_t\right)\right)dF_H+\lambda\left(1-\theta\right)\int_0^{\tilde{r}(x,\mu_E)}\overline{G}_t^I\left(\tilde{\mu}\left(x,r_t\right)\right)dF_L,\\ G_{t+1}^I\left(x\right) &= \lambda\theta\int_0^{\tilde{r}(x,\mu_E)}\overline{G}_t^C\left(\tilde{\mu}\left(x,r_t\right)\right)dF_H+\left(1-\lambda\theta\right)\int_0^{\tilde{r}(x,\mu_E)}\overline{G}_t^I\left(\tilde{\mu}\left(x,r_t\right)\right)dF_L,\end{aligned}$$

where $\mu_E \in (\lambda\theta, 1-\lambda+\lambda\theta)$ is treated as a constant.

First notice that, as there are no firms with reputation either below $\mu_E$ or above $(1-\lambda)+\lambda\theta$, we have:

$$\begin{aligned}\overline{G}_t^\tau(\mu) &= \overline{H}_t^\tau(\mu) = 0 \text{ if } \mu < \mu_E \text{ and} \\ \overline{G}_t^\tau(\mu) &= \overline{H}_t^\tau(\mu) = 1 \text{ if } \mu > (1-\lambda)+\lambda\theta\end{aligned}$$

for any distribution of reputations $\overline{G}_t^\tau$. Moreover, the definitions of $\tilde{r}(x,\mu)$ and $\tilde{\mu}(x,r)$ imply that:

1. $\tilde{\mu}(x,\tilde{r}(x,\mu_E)) = \mu_E$,
2. $\tilde{\mu}(x,r) < \mu_E \Leftrightarrow r > \tilde{r}(x,\mu_E)$ (those who have a reputation $x$ today and had a reputation lower than $\mu_E$ the previous period are those who obtained signals of at least $\tilde{r}(x,\mu_E)$), and
3. $\tilde{\mu}(x,r) > 1-\lambda+\lambda\theta \Leftrightarrow r < \tilde{r}(x, 1-\lambda+\lambda\theta)$ (those who had a higher reputation than $1-\lambda+\lambda\theta$ the previous period and have a reputation $x$ today are those whose signals were lower than $\tilde{r}(x,1-\lambda+\lambda\theta)$).

Hence,

$$\begin{aligned}\overline{G}_t^\tau(\tilde{\mu}(x,r_t)) &= \overline{H}_t^\tau(\tilde{\mu}(x,r_t)) = 0 \quad \text{if } r > \tilde{r}(x,\mu_E) \quad \text{and} \\ \overline{G}_t^\tau(\tilde{\mu}(x,r_t)) &= \overline{H}_t^\tau(\tilde{\mu}(x,r_t)) = 1 \quad \text{if } r < \tilde{r}(x,1-\lambda+\lambda\theta).\end{aligned}$$

Therefore, the distance between $\overline{G}_{t+1}^C$ and $\overline{H}_{t+1}^C$:

$$\rho_\infty\left(\overline{G}_{t+1}^C, \overline{H}_{t+1}^C\right) = \sup_x \left|\overline{G}_{t+1}^C(x) - \overline{H}_{t+1}^C(x)\right|$$

can be rewritten as:

$$\begin{aligned}\rho_\infty\left(\overline{G}_{t+1}^C, \overline{H}_{t+1}^C\right) = \sup_x \Bigg|(1-\lambda+\lambda\theta)\int_{\tilde{r}(x,1-\lambda+\lambda\theta)}^{\tilde{r}(x,\mu_E)} \left(\overline{G}_t^C(\tilde{\mu}(x,r_t)) - \overline{H}_t^C(\tilde{\mu}(x,r_t))\right) dF_H \\ +\lambda(1-\theta)\int_{\tilde{r}(x,1-\lambda+\lambda\theta)}^{\tilde{r}(x,\mu_E)} \left(\overline{G}_t^I(\tilde{\mu}(x,r_t)) - \overline{H}_t^I(\tilde{\mu}(x,r_t))\right) dF_L\Bigg|.\end{aligned}$$

Using the properties of the sup and $|\cdot|$ operators, and the definition of $\rho_\infty\left(\overline{G}_t^\tau, \overline{H}_t^\tau\right)$, we also know that:

$$\begin{aligned}
\rho_\infty\left(\overline{G}^C_{t+1},\overline{H}^C_{t+1}\right) &\leq (1-\lambda+\lambda\theta)\sup_x\left|\int_{\tilde r(x,1-\lambda+\lambda\theta)}^{\tilde r(x,\mu_E)}\left(\overline{G}^C_t\left(\tilde\mu\left(x,r_t\right)\right)-\overline{H}^C_t\left(\tilde\mu\left(x,r_t\right)\right)\right)dF_H\right| \\
&\quad +\lambda\left(1-\theta\right)\sup_x\left|\int_{\tilde r(x,1-\lambda+\lambda\theta)}^{\tilde r(x,\mu_E)}\left(\overline{G}^I_t\left(\tilde\mu\left(x,r_t\right)\right)-\overline{H}^I_t\left(\tilde\mu\left(x,r_t\right)\right)\right)dF_L\right| \\
&\leq (1-\lambda+\lambda\theta)\sup_x\int_{\tilde r(x,1-\lambda+\lambda\theta)}^{\tilde r(x,\mu_E)}\left|\overline{G}^C_t\left(\tilde\mu\left(x,r_t\right)\right)-\overline{H}^C_t\left(\tilde\mu\left(x,r_t\right)\right)\right|dF_H \\
&\quad +\lambda\left(1-\theta\right)\sup_x\int_{\tilde r(x,1-\lambda+\lambda\theta)}^{\tilde r(x,\mu_E)}\left|\overline{G}^I_t\left(\tilde\mu\left(x,r_t\right)\right)-\overline{H}^I_t\left(\tilde\mu\left(x,r_t\right)\right)\right|dF_L \\
&\leq (1-\lambda+\lambda\theta)\sup_x\int_{\tilde r(x,1-\lambda+\lambda\theta)}^{\tilde r(x,\mu_E)}\rho_\infty\left(\overline{G}^C_t,\overline{H}^C_t\right)dF_H \\
&\quad +\lambda\left(1-\theta\right)\sup_x\int_{\tilde r(x,1-\lambda+\lambda\theta)}^{\tilde r(x,\mu_E)}\rho_\infty\left(\overline{G}^I_t,\overline{H}^I_t\right)dF_L \\
&= (1-\lambda+\lambda\theta)\,\rho_\infty\left(\overline{G}^C_t,\overline{H}^C_t\right)\sup_x\int_{\tilde r(x,1-\lambda+\lambda\theta)}^{\tilde r(x,\mu_E)}dF_H \\
&\quad +\lambda\left(1-\theta\right)\rho_\infty\left(\overline{G}^I_t,\overline{H}^I_t\right)\sup_x\int_{\tilde r(x,1-\lambda+\lambda\theta)}^{\tilde r(x,\mu_E)}dF_L
\end{aligned}$$

But $\int_{\tilde r(x,1-\lambda+\lambda\theta)}^{\tilde r(x,\mu_E)}dF = F\left(\tilde r\left(x,\mu_E\right)\right)-F\left(\tilde r\left(x,1-\lambda+\lambda\theta\right)\right) < 1$ for all $x\in[\lambda\theta,1-\lambda+\lambda\theta]$. Let us define $\beta$ as follows:

$$\beta=\max\left\{\sup_{x\in[\lambda\theta,1-\lambda+\lambda\theta]}F_H\left(\tilde r\left(x,\mu_E\right)\right)-F_H\left(\tilde r\left(x,1-\lambda+\lambda\theta\right)\right),\right.$$
$$\left.\sup_{x\in[\lambda\theta,1-\lambda+\lambda\theta]}F_L\left(\tilde r\left(x,\mu_E\right)\right)-F_L\left(\tilde r\left(x,1-\lambda+\lambda\theta\right)\right)\right\}<1.$$

Then,

$$\begin{aligned}
\rho_\infty\left(\overline{G}^C_{t+1},\overline{H}^C_{t+1}\right) &\leq (1-\lambda+\lambda\theta)\,\rho_\infty\left(\overline{G}^C_t,\overline{H}^C_t\right)\beta \\
&\quad +\lambda\left(1-\theta\right)\rho_\infty\left(\overline{G}^I_t,\overline{H}^I_t\right)\beta \\
&\leq \beta\rho\left(\left(\overline{G_t}^C,\overline{G_t}^I\right),\left(\overline{H_t}^C,\overline{H_t}^I\right)\right)
\end{aligned}$$

Following a similar procedure, we can show that the distance between $\overline{G}^I_{t+1}$ and $\overline{H}^I_{t+1}$ satisfies:

$$\rho_\infty\left(\overline{G}^I_{t+1},\overline{H}^I_{t+1}\right)\leq\beta\rho\left(\left(\overline{G_t}^C,\overline{G_t}^I\right),\left(\overline{H_t}^C,\overline{H_t}^I\right)\right),$$

We thus conclude that:

$$\rho\left(\left(\overline{G}^C_{t+1},\overline{H}^C_{t+1}\right),\left(\overline{G}^I_{t+1},\overline{H}^I_{t+1}\right)\right)\leq\beta\rho\left(\left(\overline{G_t}^C,\overline{G_t}^I\right),\left(\overline{H_t}^C,\overline{H_t}^I\right)\right),$$

and hence $T$ is a contraction mapping.

Moreover, the set of distribution functions $G$ with support contained in the interval $[\lambda\theta, \lambda\theta + (1-\lambda)]$ endowed with the sup metric, $\rho_\infty$, is a complete metric space. Therefore, the set of pairs of distribution functions with the metric defined above is also a complete metric space, and hence Banach's Fixed-Point Theorem applies. As a result, we know that $T$ has a unique fixed point in this set, and it can be reached beginning from any pair of distribution functions.

Notice that:

(i) if we had used $\lambda\theta$ instead of $\mu_E$ as the lower bound of reputations in the proof, we would have obtained a higher modulus $\beta$–but we would still be able to prove that $T$ is a contraction mapping–;

(ii) $\mu_E$ not only affects the modulus of the contraction, but also the limiting distribution.

Thus, the proof of the existence of a steady state distribution of reputations is not affected by the endogenous entry-exit process as long as $\mu_E$ is fixed, but the shape of the steady state distribution is. The continuity of $F_H$ an $F_L$ implies that $G^C$ and $G^I$ are continuous with common support $[\lambda\theta, 1-\lambda+\lambda\theta]$. In turn, the fact that $F_H$ first-order stochastically dominates $F_L$ implies that $G^C$ first-order stochastically dominates $G^I$. To see this, notice that if we start at $t=0$ with $G_0^C = G_0^I$, then $G_t^C \leq G_t^I$ for all $t > 0$, and therefore $G^C \leq G^I$.

## A.3 Proof of Lemma 2.4

Remember that we defined $\psi$ as follows:

$$\psi(x) \equiv \frac{\theta G^C(x)}{\theta G^C(x) + (1-\theta) G^I(x)},$$

while the steady state distributions satisfy:

$$\begin{aligned}
G^C(x) &= (1-\lambda+\lambda\theta) \int_0^{\tilde{r}(x,\mu_E)} \overline{G}^C(\tilde{\mu}(x,r))\, dF_H + \lambda(1-\theta) \int_0^{\tilde{r}(x,\mu_E)} \overline{G}^I(\tilde{\mu}(x,r))\, dF_L \\
G^I(x) &= \lambda\theta \int_0^{\tilde{r}(x,\mu_E)} \overline{G}^C(\tilde{\mu}(x,r))\, dF_H + (1-\lambda\theta) \int_0^{\tilde{r}(x,\mu_E)} \overline{G}^I(\tilde{\mu}(x,r))\, dF_L
\end{aligned}$$

Let us define the integrals as:

$$\begin{aligned}
I^C(x) &\equiv \int_0^{\widetilde{r}(x,\mu_E)} G^C(\widetilde{\mu}(x,r))\, dF_H. \\
I^I(x) &\equiv \int_0^{\widetilde{r}(x,\mu_E)} G^I(\widetilde{\mu}(x,r))\, dF_L.
\end{aligned}$$

We use L'Hospital rule to obtain:

$$\begin{aligned}
\lim_{x\to\lambda\theta+} \frac{I^C(x)}{I^I(x)} &= \lim_{x\to\lambda\theta+} \frac{\frac{\partial I^C(x)}{\partial x}}{\frac{\partial I^I(x)}{\partial x}} \\
&= \lim_{x\to\lambda\theta+} \frac{\frac{\partial\widetilde{r}}{\partial x} G^C(\mu_E) f_H(\widetilde{r}(\lambda\theta,\mu_E)) + \int_0^{\widetilde{r}(x,\mu_E)} \frac{\partial G^C(\widetilde{\mu})}{\partial\widetilde{\mu}} \frac{\partial\widetilde{\mu}}{\partial x} dF_H}{\frac{\partial\widetilde{r}}{\partial x} G^I(\mu_E) f_L(\widetilde{r}(\lambda\theta,\mu_E)) + \int_0^{\widetilde{r}(x,\mu_E)} \frac{\partial G^I(\widetilde{\mu})}{\partial\widetilde{\mu}} \frac{\partial\widetilde{\mu}}{\partial x} dF_L},
\end{aligned}$$

where we use the fact that $\tilde{\mu}(x, \tilde{r}(x,\mu_E)) = \mu_E$. However, $\widetilde{r}(\lambda\theta,\mu_E) = 0$. Hence, if $f_H(0) = 0$ and $f_L(0) \neq 0$ we know that $\lim_{x\to\lambda\theta+} \frac{I^C(x)}{I^I(x)} = 0$. If both $f_H(0)$ and $f_L(0)$ are zero, then we can apply

L'Hospital again to obtain:

$$\lim_{x\to\lambda\theta+}\frac{I^{C}(x)}{I^{I}(x)} = \lim_{x\to\lambda\theta+}\left(\frac{\left(\frac{\partial^2\widetilde{r}}{\partial x^2}G^{C}(\mu_E)f_H(\widetilde{r})+\left(\frac{\partial\widetilde{r}}{\partial x}\right)^2G^{C}(\mu_E)\frac{\partial f_H(\widetilde{r})}{\partial r}\right.}{\left(\frac{\partial^2\widetilde{r}}{\partial x^2}G^{I}(\mu_E)f_L(\widetilde{r})+\left(\frac{\partial\widetilde{r}}{\partial x}\right)^2G^{I}(\mu_E)\frac{\partial f_L(\widetilde{r})}{\partial r}\right.}\right.$$

$$\left.\frac{\left.\frac{\partial\widetilde{r}}{\partial x}\frac{\partial G^{C}(\widetilde{\mu})}{\partial\widetilde{\mu}}\frac{\partial\widetilde{\mu}}{\partial x}\right|_{\widetilde{r}(x,\mu_E)}f_H(\widetilde{r})+\int_0^{\widetilde{r}(x,\mu_E)}\left(\frac{\partial^2G^{C}(\widetilde{\mu})}{\partial\widetilde{\mu}^2}\frac{\partial\widetilde{\mu}}{\partial x}+\frac{\partial G^{C}(\widetilde{\mu})}{\partial\widetilde{\mu}}\frac{\partial^2\widetilde{\mu}}{\partial x^2}\right)dF_H\right)}{\left.\frac{\partial\widetilde{r}}{\partial x}\frac{\partial G^{I}(\widetilde{\mu})}{\partial\widetilde{\mu}}\frac{\partial\widetilde{\mu}}{\partial x}\right|_{\widetilde{r}(x,\mu_E)}f_L(\widetilde{r})+\int_0^{\widetilde{r}(x,\mu_E)}\left(\frac{\partial^2G^{I}(\widetilde{\mu})}{\partial\widetilde{\mu}^2}\frac{\partial\widetilde{\mu}}{\partial x}+\frac{\partial G^{I}(\widetilde{\mu})}{\partial\widetilde{\mu}}\frac{\partial^2\widetilde{\mu}}{\partial x^2}\right)dF_L\right)}\right)$$

Then, if $\left.\frac{\partial f_H(\widetilde{r})}{\partial r}\right|_{\widetilde{r}=0}=0$ and $\left.\frac{\partial f_L(\widetilde{r})}{\partial r}\right|_{\widetilde{r}=0}\neq 0$, we obtain the desired result. If not, we apply L'Hospital's rule again, until we have $\left.f_H^{(n)}(\widetilde{r})\right|_{\widetilde{r}=0}=0$ and $\left.f_L^{(n)}(\widetilde{r})\right|_{\widetilde{r}=0}\neq 0$ for some $n$.

## A.4 Derivation of Equation 35 in Lemma 3.1

The incentive compatibility condition is:

$$c_H-c_L\leq\sum_{k=1}^{\infty}\delta^k(1-\lambda)^kE_{H^k}\left[p\left(\varphi^k\left(\mu,\mathbf{r}^k\right)\right)\right]-\sum_{k=1}^{\infty}\delta^k(1-\lambda)^kE_{LH^{k-1}}\left[p\left(\varphi^k\left(\mu,\mathbf{r}^k\right)\right)\right]$$

The first expectation above can be written as:

$$E_{H^k}\left[p\left(\varphi^k\left(\mu,\mathbf{r}^k\right)\right)\right]=\int_{r_k}...\int_{r_1}p\left(\varphi\left(...\varphi\left(\mu_1,r_1\right)...r_k\right)\right)dF_H\left(r_1\right)...dF_H\left(r_k\right).$$

When integrating by parts the first integral one obtains:

$$\begin{aligned}
&\int_{r_k}...\int_{r_2}\left\{\int_{r_1}p\left(\varphi\left(...\varphi\left(\mu_1,r_1\right)...r_k\right)\right)dF_H\left(r_1\right)\right\}...dF_H\left(r_k\right)\\
&\qquad=\int_{r_k}...\int_{r_2}\left\{\left(p\left(\varphi\left(...\varphi\left(\mu_1,r_1\right)...r_k\right)\right)F_H\left(r_1\right)\right)\Big|_0^1\right.\\
&\qquad\left.-\int_{r_1}\frac{\partial p\left(\varphi\left(...\varphi\left(\mu_1,r_1\right)...r_k\right)\right)}{\partial\mu_k}\frac{\partial\varphi\left(\varphi^{k-1}\left(\mu_1,\mathbf{r}^{k-1}\right),r_k\right)}{\partial\mu_{k-1}}...\frac{\partial\varphi\left(\mu_1,r_1\right)}{\partial r_1}F_H\left(r_1\right)dr_1\right\}...dF_H\left(r_k\right)\\
&\qquad=\int_{r_k}...\int_{r_2}\left\{p\left(\varphi\left(...\varphi\left(1-\lambda+\lambda\theta,r_2\right)...r_k\right)\right)\right.\\
&\qquad\left.-\int_{r_1}\frac{\partial p\left(\varphi\left(...\varphi\left(\mu_1,r_1\right)...r_k\right)\right)}{\partial\mu_k}\frac{\partial\varphi\left(\varphi^{k-1}\left(\mu_1,\mathbf{r}^{k-1}\right),r_k\right)}{\partial\mu_{k-1}}...\frac{\partial\varphi\left(\mu_1,r_1\right)}{\partial r_1}F_H\left(r_1\right)dr_1\right\}...dF_H\left(r_k\right)
\end{aligned}$$

Proceeding with the second one:

$$\begin{aligned}
&\int_{r_k}...\int_{r_3}\left\{\int_{r_2}p\left(\varphi\left(...\varphi\left(1-\lambda+\lambda\theta,r_2\right)...r_k\right)\right)dF_H\left(r_2\right)\right\}...dF_H\left(r_k\right)\\
&\qquad=\int_{r_k}...\int_{r_3}\left\{\left(p\left(\varphi\left(...\varphi\left(1-\lambda+\lambda\theta,r_2\right)...r_k\right)\right)F_H\left(r_2\right)\right)\Big|_0^1\right.\\
&\qquad\left.-\int_{r_2}\frac{\partial p\left(\varphi\left(...\varphi\left(1-\lambda+\lambda\theta,r_2\right)...r_k\right)\right)}{\partial\mu}...\frac{\partial\varphi\left(1-\lambda+\lambda\theta,r_2\right)}{\partial r_2}F_H\left(r_2\right)dr_2\right\}...dF_H\left(r_k\right)
\end{aligned}$$

and similarly for the complete sequence:

$$
\begin{aligned}
E_{H^k}\left[p\left(\varphi^k\left(\mu,\mathbf{r}^k\right)\right)\right] = & \\
& -\int_{r_k}\ldots\int_{r_1}\frac{\partial p\left(\varphi\left(\ldots\varphi\left(\mu_1,r_1\right)\ldots r_k\right)\right)}{\partial\mu_k}\ldots\frac{\partial\varphi\left(\mu_1,r_1\right)}{\partial r_1}F_H\left(r_1\right)dr_1\ldots dF_H\left(r_k\right) \\
& -\int_{r_k}\ldots\int_{r_2}\frac{\partial p\left(\varphi\left(\ldots\varphi\left(1-\lambda+\lambda\theta,r_2\right)\ldots r_k\right)\right)}{\partial\mu_k}\ldots\frac{\partial\varphi\left(1-\lambda+\lambda\theta,r_2\right)}{\partial r_2}F_H\left(r_2\right)dr_2\ldots dF_H\left(r_k\right) \\
& -\ldots+p\left(1-\lambda+\lambda\theta\right)
\end{aligned}
$$

Applying the same procedure to $E_{LH^{k-1}}\left[p\left(\varphi^k\left(\mu,\mathbf{r}^k\right)\right)\right]$ and collecting terms, the incentive compatibility condition can be written as:

$$
\begin{aligned}
c_H-c_L & \leq\sum_{k=1}^{\infty}\delta^k\left(1-\lambda\right)^k\int_{r_k}\ldots\int_{r_1}\frac{\partial p\left(\varphi\left(\ldots\varphi\left(\mu_1,r_1\right)\ldots r_k\right)\right)}{\partial\mu_k}\frac{\partial\varphi\left(\varphi^{k-1}\left(\mu_1,\mathbf{r}^{k-1}\right),r_k\right)}{\partial\mu_{k-1}}\ldots \\
& \qquad \ldots\frac{\partial\varphi\left(\mu_1,r_1\right)}{\partial r_1}\left(F_L\left(r_1\right)-F_H\left(r_1\right)\right)dr_1\ldots dF_H\left(r_k\right) \\
& =\sum_{k=1}^{\infty}\delta^k\left(1-\lambda\right)^k E_{H^{k-1}}\left[\int_{r_1}\frac{\partial p\left(\varphi^k\left(\mu_1,\mathbf{r}^k\right)\right)}{\partial\mu_k}\frac{\partial\varphi\left(\varphi^{k-1}\left(\mu_1,\mathbf{r}^{k-1}\right),r_k\right)}{\partial\mu_{k-1}}\ldots\right. \\
& \qquad\qquad \left.\frac{\partial\varphi\left(\mu_1,r_1\right)}{\partial r_1}\left(F_L\left(r_1\right)-F_H\left(r_1\right)\right)dr_1\right]
\end{aligned}
$$