Uphill Self-Control*

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August 8, 2009

Abstract

This paper is motivated by the idea that self-control is more difficult to exert the more it is exerted. We extend the theory of temptation and self-control introduced by Gul and Pesendorfer [8] to allow for an increasing marginal cost of resisting temptation, that is, convex self-control costs. We also prove a general representation theorem that admits a general class of self-control cost functions. Both models maintain Gul and Pesendorfer's Order, Continuity and Set-Betweenness axioms but violate Independence.

1 Introduction

Gul and Pesendorfer [8] (henceforth GP) introduce a theory of choice under temptation. They model an agent who experiences temptation at the moment of choice, and anticipates this in an ex-ante period where he selects what choice problem to face. In this ex-ante period he has a particular perspective on what he should choose, embodied in a 'normative preference'. He understands that his choice from menus will not necessarily respect normative preference, but rather will seek to balance his normative preference with the cost of resisting temptation.

GP axiomatize the following model (1)-(2). Denote the space of alternatives (lotteries) by Δ and the space of menus (nonempty subsets of Δ) by Z. The primitive is a preference $\succsim$ over menus Z, and reflects the ex-ante choice between menus prior to ex-post (unmodelled)

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choice from a menu. The general class of models that captures an abstract version of the story in GP is reflected in the following representation for $\succsim$:

$$W(x) = \max_{\mu \in x} \{u(\mu) - c(\mu, x)\}, \quad x \in Z, \tag{1}$$

where $u : \Delta \to \mathbb{R}$ represents a vNM normative preference and $c(\mu, x)$ reflects the self-control cost of choosing μ from the menu x. The representation suggests that the utility of a menu is its indirect utility: the maximum of normative utility less self-control costs. GP's model is a specialization that tells a very specific story about the self-control cost function c. Their model identifies a vNM function $v : \Delta \to \mathbb{R}$ that represents the temptation perspective, and measures self-control costs c in terms of the difference between the maximum temptation utility achievable in a menu x and the actual temptation utility achieved by a choice $\mu \in x$:

$$c(\mu, x) = \max_{\eta \in x} v(\eta) - v(\mu). \tag{2}$$

That is, the self-control cost of choosing μ from x is identified with the corresponding degree of 'frustration' of the temptation perspective.

A peculiar feature of GP's cost function is its *linearity* in the degree to which temptation preferences are frustrated, that is, in $\max_{\eta \in x} v(\eta) - v(\mu)$. Intuitively, the agent has a constant marginal cost of exerting self-control. This paper is based on the idea that the marginal cost may not be constant. Introspection suggests that the exertion of self-control involves an *uphill battle*: the marginal cost appears to increase with the exertion of self-control. This is supported by research in psychology that demonstrates that self-control is a limited resource.[1] Motivated by the idea of uphill self-control, this paper axiomatizes two models.

General Self-Control Representation: The first model takes the form

$$W(x) = \max_{\mu \in x} \left\{ u(\mu) - c(\mu, \max_{\eta \in x} v(\eta)) \right\}, \quad x \in Z,$$

where u and v are linear and c satisfies some minimal regularity properties that support its interpretation as the cost of self-control. This expunges from GP's model all but the basic linearity required for the existence of linear normative and temptation utilities, without departing from the basic qualitative story underlying GP's model. Thus, the agent maximizes normative utility net of self-control costs, and the cost $c(\mu, \cdot)$ is increasing for any given possible choice μ.

Convex Self-Control Representation: The second model is a nonlinear extension of GP's model given by

$$W(x) = \max_{\mu \in x} \left\{ u(\mu) - \varphi \left(\max_{\eta \in x} v(\eta) - v(\mu) \right) \right\}, \quad x \in Z, \tag{3}$$

[1] This is noted by Fudenberg and Levine [6, 7]; see their paper for references.

for some increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ that is convex on the part of the domain that 'matters for behavior'. This model enriches the general model by requiring self-control costs to depend on the degree of frustration of temptation preference, as in the GP model, but without forcing this dependence to be linear.

The convex self-control model behaviorally differs from the GP model in significant ways. As expressed by one of our axioms, a peculiar feature of convex self-control is that randomization never makes it harder to exert self-control. For instance, if an agent can choose a risky lottery r despite being tempted by a safe lottery s, then he can also choose the risky lottery when both are mixed with a common third lottery, that is, he can choose $\alpha r + (1-\alpha)\nu$ over $\alpha s + (1-\alpha)\nu$ even if the latter is tempting. Intuitively, randomization reduces the difference in temptation utility between the alternatives which in turn reduces the marginal cost of exerting self-control, thereby enhancing self-control. Notably, this feature generates the Allais paradox: an agent who may choose s over r may also choose $\alpha r + (1-\alpha)\nu$ over $\alpha s + (1-\alpha)\nu$. Overlapping with our companion paper (Noor and Takeoka [14]) is also the behavioral implication that convex self-control costs will typically cause ex-post choice to violate the Weak Axiom of Revealed Preference: an agent may resist temptation and choose $\alpha r + (1-\alpha)\nu$ over $\alpha s + (1-\alpha)\nu$ in a direct comparison, but when the more tempting option s is available, the increased marginal cost of self-control may make $\alpha s + (1-\alpha)\nu$ an attractive compromise.

Further motivation for studying the foundations of the convex self-control can be derived from Fudenberg and Levine [6, 7]. These authors study a (non-axiomatic) dual-self model that features convex self-control costs. While independently making some of the above observations, they also show that their model can explain a wide range of behavioral anomalies, such the Allais paradox, Rabin's paradox, intertemporal preference reversals, the relationship between time and risk preferences observed in experiments, and the relationship between cognitive load and risk preferences. They also show that plausible parameter values allow them to quantitatively fit their model to data on a range of behaviors.

To offer some additional perspective on our paper, we point out where it stands relative to the current development of the axiomatic literature on temptation. GP's axiomatization of their model makes use of four axioms: Order, Continuity, Independence and Set-Betweenness. The first three are natural extensions of the von Neumann-Morgenstern axioms to a sets of lotteries setting, and the fourth expresses the agent's temptation and anticipated choice from menus. Of the four axioms, Set-Betweenness is clearly a substantive axiom for a model of decision under temptation. Indeed, existing generalizations of GP's model (Chatterjee and Krishna [3], Dekel, Lipman and Rustichini [4], Stoval [17]) have focused on relaxing Set-Betweenness while maintaining Independence. This paper (and the companion paper [14]) seeks to understand what is potentially lost if one maintains Independence, a convenient and standard axiom in the literature that seems less important than Set-Betweenness from the point of view of decision under temptation. We show that the axiom is not auxiliary in nature in that it rules out intuitive qualitative stories about decision under temptation, even if Set-Betweenness is retained. In fact, both the models axiomatized in this paper satisfy Set-Betweenness.

The remainder of the paper is organized as follows. This Introduction concludes with a mention of related literature. Sections 2 and 3 axiomatize our general and convex models respectively. Section 4 concludes with some observations of the convex model's implied properties for ex post choice. All proofs are contained in appendices.

1.1 Related Literature

In a game-theoretic setting, Fudenberg and Levine [6, 7] study the interaction of a long-run patient self and a sequence of short-run impulsive selves, each of which is a primitive of their model. They show that the equilibria of the game played by those selves can be regarded as the solution to a maximization problem analogous to (1). Their general setup allows for cases where the cost function might be convex, which would then correspond to specializations of (1) which include the convex self-control model. In [7] they construct and analyze a model with convex self-control costs that explains and quantitatively fits a range of experimental findings.

In the temptation literature, Chatterjee and Krishna [3], Dekel, Lipman and Rustichini [4] and Stoval [17] generalize GP's model. They model agents who are uncertain about temptation (e.g. uncertain about the temptation preference itself, or uncertainty regarding the strength of self-control, etc.), and Dekel et al also axiomatize a model where multiple temptations are experienced by the agent. These models relax Set-Betweenness but maintain Independence. This paper explores an alternative direction where Set-Betweenness is maintained and Independence relaxed. We interpret violations of Independence in terms of non-linear self-control costs. In a companion paper (Noor and Takeoka [14]), we focus on another possible source of violations of Independence, specifically the possibility of menu-dependent self-control.

Nehring [12] is interested in a more careful description of the notion of self-control, which he interprets in terms of a preference over preferences (second order preferences). Olszewski [15] relaxes the single-dimensionality of temptation in GP's model by permitting different alternatives in a menu to be tempted by different alternatives in the menu. Though not specifically motivated by the idea of uphill self-control, these authors provide foundations for functional forms that can accommodate uphill self-control. On a technical level these papers differ substantially from ours in that they focus on *discrete settings* whereas we provide an axiomatic generalization of GP's model in a sets-of-lotteries setting.

Finally, we mention Gul and Pesendorfer [10] who, also in a discrete setting, axiomatize a general model. Their representation for preference over menus is of the form

$$W(x) = f(\max_{\mu\in x} w(\mu), \max_{\eta\in x} v(\eta)),$$

which admits the interpretation that the agent is tempted to maximize some temptation utility v but choice is determined by the maximization of some function w. The two utilities are then aggregated by the function f. To compare, we note that our general model corresponds to the form

$$W(x) = \max_{\mu\in x} f(\mu, \max_{\eta\in x} v(\eta)).$$

Thus, while ex post choice in the Gul and Pesendorfer [10] model maximizes a utility w, in our model ex post choice from x maximizes the *menu-dependent* utility $f(\mu, \max_{\eta\in x} v(\eta))$. Indeed, ex post choice in their model satisfies the Weak Axiom of Revealed Preference, and this in turn suggests that the model is not suitably interpreted as one involving non-linear self-control.

2 General Model

For any compact metric space C, $\Delta(C)$ denotes the set of all probability measures on the Borel σ-algebra of C, endowed with the weak convergence topology; $\Delta(C)$ is compact and metrizable [1, Thm 14.11], and we often write it simply as Δ. Let $Z = \mathcal{K}(\Delta)$ denote the set of all nonempty compact subsets of Δ. When endowed with the Hausdorff topology, Z is a compact metric space [1, Thm 3.71(3)]. An element $x \in Z$ is referred to as a menu. Generic elements of Z are x, y, z whereas generic elements of Δ are μ, η, ν. For $\alpha \in [0,1]$, $\mu\alpha\eta \in \Delta$ is the α-mixture that assigns $\alpha\mu(A) + (1-\alpha)\eta(A)$ to each A in the Borel $\sigma-$algebra of C. Similarly, $x\alpha y \equiv \{\mu\alpha\eta : \mu \in x, \eta \in y\} \in Z$ is an α-mixture of menus x and y.

As in GP, the primitive is a preference $\succsim$ over Z.

2.1 Axioms

The first three axioms are familiar from GP.

Axiom 1 (Order) $\succsim$ *is complete and transitive.*

Axiom 2 (Continuity) *The sets* $\{y \in Z : y \succsim x\}$ *and* $\{y \in Z : x \succsim y\}$ *are closed for each* $x \in Z$.

Axiom 3 (Set-Betweenness) *For all* $x, y \in Z$,

$$x \succsim y \Longrightarrow x \succsim x \cup y \succsim y.$$

We refer the reader to GP for a more complete discussion of Set-Betweenness, which involves interpreting the ranking of x and $x \cup y$ as indicating whether there temptation lies in y, and the ranking of $x \cup y$ and y as indicating whether (unmodelled) ex post choice from the menu $x \cup y$ lies in y. What needs to be noted for the purpose of this paper is that the interpretation of Set-Betweenness does not hinge on any precise properties of how exertion of self-control in the menu $x \cup y$ affects its desirability. This suggests that Set-Betweenness is not inconsistent with generalizations of GP that relax the structure on self-control costs.

GP's fourth axiom formulates the standard vNM independence axiom to the menus-setting: for all x, y, z and $\alpha \in (0,1)$,

$$x \succ y \Longrightarrow x\alpha z \succ y\alpha z.$$

We relax Independence so as to impose vNM structure on commitment preference and temptation preference *only*.

Axiom 4 (Commitment Independence) *For any μ, η, ν and $\alpha \in (0,1)$,*

$$\{\mu\} \succ \{\eta\} \Longrightarrow \{\mu\alpha\nu\} \succ \{\eta\alpha\nu\}.$$

Axiom 5 (Temptation Independence) *For any μ, η, ν and $\alpha \in (0,1)$ s.t. $\{\mu\} \succ \{\eta\}$,*

$$\{\mu\} \succsim \{\mu, \eta\} \Longleftrightarrow \{\mu\alpha\nu\} \succsim \{\mu\alpha\nu, \eta\alpha\nu\}.$$

Moreover, for any μ, η, η' and $\alpha \in (0,1)$ s.t. $\{\mu\} \succ \{\eta\}, \{\eta'\}$,

$$\begin{aligned} \{\mu\} &\succ \{\mu, \eta\} \text{ and } \{\mu\} \succ \{\mu, \eta'\} \Longrightarrow \{\mu\} \succ \{\mu, \eta\alpha\eta'\} \\ \{\mu\} &\sim \{\mu, \eta\} \text{ and } \{\mu\} \sim \{\mu, \eta'\} \Longrightarrow \{\mu\} \sim \{\mu, \eta\alpha\eta'\}. \end{aligned}$$

Commitment Independence is readily interpreted. The first part of Temptation Independence states that η tempts (resp. does not tempt) μ if and only if $\eta\alpha\nu$ tempts (resp. does not tempt) $\mu\alpha\nu$. The second part states that if η and η' both tempt (resp. do not tempt) μ, then the mixture $\eta\alpha\eta'$ tempts (resp. does not tempt) μ. These are properties that would be expected from a vNM temptation preference.

To introduce the next axiom, consider some rankings of menus that are presumably associated with temptation preference. Say that η *is at least as tempting as* μ if either $\{\mu\} \succ \{\mu, \eta\}$ or $\{\eta\} \sim \{\mu, \eta\} \succ \{\mu\}$ holds. As in the previous axiom, the first condition is a typical behavior revealing that η is more tempting than μ. The second condition says that η is normatively superior to μ, and the agent does not exhibit preference for commitment to η. This preference pattern reveals that μ is not more tempting than η, in other words, η is at least as tempting as μ.

The key axiom we adopt for our general model is:

Axiom 6 (Temptation Dependence) *If $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$ for some μ, η, then for any ν,*

$$\eta \text{ is at least as tempting as } \nu \Longrightarrow \{\mu, \nu\} \succsim \{\mu, \eta\}.$$

The axiom simply says that if we replace η with something less tempting, then the agent is not worse-off. Intuitively, the lower the temptation in a menu, the lower the self-control costs associated with resisting temptation. However, the axiom covers also the following possibility: if ν is less tempting than η and also normatively superior, then ex-post the agent may optimally choose to submit to temptation, rather than incur any self-control cost. However, even in this case, the agent would be better off with $\{\mu, \nu\}$ than $\{\mu, \eta\}$, as he would submit to temptation in $\{\mu, \nu\}$ only because the normative cost of doing so would be smaller than the self-control cost of resisting, which itself is smaller than the self-control cost incurred in $\{\mu, \eta\}$.

2.2 Representation Theorem

The most general representation result in this paper is:

Theorem 1 *A preference $\succsim$ satisfies Order, Continuity, Set-Betweenness, Commitment Independence, Temptation Independence and Temptation Dependence if and only if there exists a representation $W : Z \to \mathbb{R}$ for $\succsim$ defined by:*

$$W(x) = \max_{\mu \in x} \left\{ u(\mu) - c(\mu, \max_{\eta \in x} v(\eta)) \right\},$$

where $u, v : \Delta \to \mathbb{R}_+$ are continuous linear functions and $c : \Delta \times v(\Delta) \to \mathbb{R}_+$ is a continuous function that is weakly increasing in its second argument, and satisfies:
(i) if $v(\mu) \geq l$ then $c(\mu, l) = 0$;
(ii) if $u(\mu) > u(\eta)$ and $l = \max_{\{\mu,\eta\}} v$ then $v(\mu) < v(\eta) \iff c(\mu, l) > 0$.

A preference $\succsim$ that satisfies the noted axiom is referred to as a *temptation-dependent self-control preference*, and the representation is a *temptation-dependent self-control representation*. The function c possesses minimal properties required to interpret it as a self-control cost function. Monotonicity in its second argument reflects the fact that choosing any given alternative μ is more costly from menus with greater temptation. Condition (i) says that the self-control cost of submitting to temptation is zero. Condition (ii) says that the self-control cost of resisting temptation is strictly positive.

The model accommodates cost functions which embody the idea that any deviation from the most tempting alternative is costly:

$$c(\mu, l) > 0 \iff v(\mu) < l.$$

However, the model's uniqueness properties are such that this is not ensured by the axioms for all μ, l. If (μ, l) is such that $v(\mu) < l$ and there is no η with $v(\eta) = l$ such that

$$\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\},$$

then $c(\mu, l)$ is unrestricted. The intuition is that alternatives that are always dominated in both normative and temptation terms are never chosen. Note that the only way that unchosen alternatives 'affect' the ex-ante preference $\succsim$ is if they are most tempting, and in particular, unchosen alternatives that are never most tempt have no impact on $\succsim$. Preferences over menus are not rich enough data in order to recover the cost of choosing such unchosen alternatives, and consequently *any* cost can be assigned to them – this is reflected formally in the following theorem. This lack of uniqueness is a minimal detraction, if at all: alternatives that are never most tempting nor ever chosen are also not of interest either from a descriptive standpoint or a normative one.

Theorem 2 *Suppose that (u, v, c) and (u', v', c') are both representations of a nondegenerate temptation-dependent self-control preference. Then there exist constants $\alpha_u, \alpha_v > 0$*

and β_u, β_v *such that* $u' = \alpha_u u + \beta_u$ *and* $v' = \alpha_v v + \beta_v$. *Moreover,* $c'(\mu, l) = \alpha_u c(\mu, \frac{l-\beta_v}{\alpha_v})$ *on the set:*

$$\{(\mu, l) : v'(\mu) \geq l \text{ or } \{\mu\} \succ \{\mu, \eta\} \succ \{\eta\} \text{ for some } \eta \text{ with } v'(\eta) = l\}.$$

The straightforward proof is omitted.

3 Convex Model

In this section we present a specialization of our general model that reflects two things: First, the cost of self-control depends on the difference between temptation utility from choice and maximum temptation utility possible. Second, the cost of self-control is convex, thereby capturing the idea of uphill self-control. We first formally describe the functional form and then present its axiomatization.

3.1 Functional Form

As in the general model, let u be a normative utility function and v be a temptation utility function over lotteries. Both functions are continuous and mixture linear. If μ is chosen with self-control in $\{\mu, \eta\}$, we refer to the difference $w = v(\eta) - v(\mu)$ as the magnitude of temptation frustration, or frustration for short. We now describe a functional form where the cost of self-control is a convex transformation of w. This requires us to define the maximum benefit from self-control among binary menus where the frustration is equal to w: for all $w > 0$, let

$$F(w) \equiv \sup\{u(\mu) - u(\eta) \,|\, w = v(\eta) - v(\mu) \text{ for some } \mu, \eta \text{ s.t. } v(\eta) - v(\mu) > 0 > u(\eta) - u(\mu)\}.$$

Note that $v(\eta) - v(\mu) > 0 > u(\eta) - u(\mu)$ states that μ is normatively preferred to η, yet η is more tempting.[2] Since $u(\mu) - u(\eta)$ is the benefit from self-control, $F(w)$ is the maximum benefit from self-control among binary menus $\{\mu, \eta\}$ where frustration, $v(\eta) - v(\mu)$, equals w.

Definition 1 *A preference* $\succsim$ *is a convex self-control preference if there exists a representation* $W : Z \rightarrow \mathbb{R}$ *for* $\succsim$ *defined by:*

$$W(x) = \max_{\mu \in x} \left\{ u(\mu) - \varphi\left(\max_{\mu' \in x} v(\mu') - v(\mu) \right) \right\}, \tag{4}$$

where $u, v : \Delta \rightarrow \mathbb{R}_+$ *are continuous mixture linear functions and* $\varphi : [0, \max_\Delta v - \min_\Delta v] \rightarrow \mathbb{R}_+$ *is a continuous strictly increasing function such that* $\varphi(0) = 0$ *and* φ *is convex on a non-degenerate interval* $[0, \overline{w}]$ *and satisfies* $\varphi(w) \geq F(w)$ *for* $w > \overline{w}$.

[2]This condition is equivalent to $\{\mu\} \succ \{\mu, \eta\}$.

Identify any convex self-control representation (4) with the corresponding tuple (u, v, φ). This model describes an agent for whom the costs of self-control increase at an *increasing rate* as more self-control is exerted. This is unlike the GP model, where the marginal cost of exerting self-control is constant. The restriction on φ says simply that the agent does not exercise self-control when frustration exceeds a threshold level $\overline{w}$. Since F defines an upper bound on the benefit of self-control, it is the case that whenever η tempts μ and $v(\eta) - v(\mu) = w > \overline{w}$, the normative benefit $u(\mu) - u(\eta)$ of self-control is always less than the self-control cost $\varphi(w)$:

$$\varphi(w) \geq F(w) \geq u(\mu) - u(\eta).$$

Hence self-control is never exerted outside the interval $[0, \overline{w}]$. Indeed, the condition ensures that self-control costs are convex *where it is meaningful.*

3.2 Axioms

We augment the general model with three axioms. Each are implied by Independence and are therefore weaker.

Axiom 7 (Weak Binary Independence) *For all* $\mu, \mu', \eta, \eta', \nu, \nu' \in \Delta$ *and all* $\alpha \in (0, 1)$,

$$\{\mu, \mu'\}\alpha\{\nu\} \succsim \{\eta, \eta'\}\alpha\{\nu\} \implies \{\mu, \mu'\}\alpha\{\nu'\} \succsim \{\eta, \eta'\}\alpha\{\nu'\}.$$

This is a weakening of an axiom introduced by Ergin and Sarver [5], who impose the axiom on all menus rather than just binary menus. Note that Independence implies that for all $\mu, \mu', \eta, \eta', \nu, \nu' \in \Delta$ and all $\alpha, \beta \in (0, 1)$,

$$\{\mu, \mu'\}\alpha\{\nu\} \succsim \{\eta, \eta'\}\alpha\{\nu\} \implies \{\mu, \mu'\}\beta\{\nu'\} \succsim \{\eta, \eta'\}\beta\{\nu'\}.$$

Weak Binary Independence is the implication of Independence in which the mixing coefficients α, β are equal. The axiom states that the ranking of binary menus $\{\mu, \mu'\}$ and $\{\eta, \eta'\}$ when mixed with a common singleton $\{\nu\}$ is independent of the lottery in the singleton. Intuitively, this reflects a 'translation invariance' property that states that if a common 'translation' is applied to the elements of both the menus $\{\mu, \mu'\}$ and $\{\eta, \eta'\}$, then the ranking of the menus is unaffected.[3] This behavior arises when self-control cost is measured in terms of the deviation from the most tempting alternative.[4]

[3]For any signed measure θ and lottery $\mu \in \Delta$ such that $\mu+\theta \in \Delta$ is a lottery, $\mu+\theta$ is a translate of μ in the direction θ. Translation Invariance is the property that $\{\mu, \mu'\} \succsim \{\eta, \eta'\} \implies \{\mu+\theta, \mu'+\theta\} \succsim \{\eta+\theta, \eta'+\theta\}$ for all suitable θ.

[4]To see this in terms of representations, note that choice from a menu is determined by a comparison of the cost and benefit of self-control. If these costs and benefits are 'differences in utilities', common translations of all elements leaves these differences unchanged, and thus leaves choice unaffected. This in turn ensures that the ranking of any two menus is unaffected by common translations of all elements in the menus.

Axiom 8 (Mixing Preserves Self-Control (MPSC)) *For all μ, η and $\alpha \in (0,1)$,*

$$\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\} \implies \{\mu\} \succ \{\mu, \eta\alpha\mu\} \succ \{\eta\alpha\mu\}.$$

MPSC says that, if the agent exhibits self-control at $\{\mu, \eta\}$, then he does so at $\{\mu, \eta\alpha\mu\}$. Notice that the temptation frustration between μ and $\eta\alpha\mu$ is smaller than that between μ and η, as the first pair of alternatives are 'closer' to each other. Since we are modelling an agent whose self-control ability is greater for small deviations from the tempting alternative, it follows that self-control must be preserved as stated in the axiom.

While the previous axiom describes an implication of uphill self-control on anticipated choice from menus, the next axiom is a direct expression of uphill self-control in the ranking of menus. For any menu x, define its *singleton equivalent* $e_x \in \Delta$ by $\{e_x\} \sim x$. Under Order, Continuity and Commitment Independence, every menu has a singleton equivalent.

Axiom 9 (Self-Control Concavity) *For all μ, μ', η, η' and $\alpha \in (0,1)$,*

$$\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\} \text{ and } \{\mu'\} \succ \{\mu', \eta'\} \succ \{\eta'\}$$
$$\implies \{\mu\alpha\mu', \eta\alpha\eta'\} \succsim \{e_{\{\mu,\eta\}}\alpha e_{\{\mu',\eta'\}}\}.$$

Observe that, due to the vNM structure imposed on underlying temptation preference, the temptation frustration in the mixed menu $\{\mu\alpha\mu', \eta\alpha\eta'\}$ is an *average* of the temptation frustration in each of the two menus. Thus, the evaluation of $\{\mu\alpha\mu', \eta\alpha\eta'\}$ contains the self-control cost associated with this average frustration. On the other hand, due to the linearity of commitment preference, the mixture $\{e_{\{\mu,\eta\}}\alpha e_{\{\mu',\eta'\}}\}$ of singleton equivalents embodies an average of the self-control costs at the two levels of temptation frustration. Convexity of self-control costs implies that the self-control cost of the average must be lower than the average of the self-control costs. The axiom reflects precisely this.

3.3 Representation Theorem

Say that $\succsim$ on Z is a *self-control preference* if there exist $\mu, \mu' \in \Delta$ with $\{\mu\} \succ \{\mu, \mu'\} \succ \{\mu'\}$. The main result of this section is:

Theorem 3 *A self-control preference $\succsim$ satisfies all the axioms of Theorem 1, Weak Binary Independence, MPSC, and Self-Control Concavity if and only if $\succsim$ is a convex self-control preference.*

This establishes the behavioral foundations of the convex self-control model. A discussion of the proof of the result is deferred to the next subsection.

Observe that the convex self-control representation requires φ to be convex only on an interval $[0, \overline{w}]$. Our axioms do not guarantee that a convex extension to $\mathbb{R}_+$ exists. The issue is technical: in order for the extension to exist it is necessary that φ be Lipschitz continuous on $[0, \overline{w}]$. It is possible to describe restrictions on preference that guarantee this, but we omit them because of their lack of transparency.

As a corollary of the theorem, we obtain the GP model when Self-Control Concavity is strengthened to a Self-Control Linearity condition:

Corollary 1 *A convex self-control preference* $\succsim$ *admits a GP representation if and only if it satisfies Self-Control Linearity: For all* μ, μ', η, η' *and* $\alpha \in (0,1)$ *s.t.* $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$ *and* $\{\mu'\} \succ \{\mu', \eta'\} \succ \{\eta'\}$,

$$\{\mu\alpha\mu', \eta\alpha\eta'\} \sim \{e_{\{\mu,\eta\}}\alpha e_{\{\mu',\eta'\}}\}.$$

Thus, we see that it is Self-Control Linearity that forces the linear self-control costs property in GP's model. This alternative axiomatization of GP's model provides perspective on the behavioral foundations of their model by highlighting the various implications of Independence in the presence of Order, Continuity and Set-Betweenness. Indeed, this permits a more transparent evaluation of that axiom and, in turn, of the model.

Now turn to the uniqueness properties of the convex self-control representation. Given a representation (u, v, φ), the *self-control subdomain* is defined as follows:

$$\begin{aligned} R \;=\; & \{w \in [0, \max_\Delta v - \min_\Delta v] \,|\, w = v(\eta) - v(\mu) \\ & \qquad \text{for some } \mu, \eta \text{ s.t. } \{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}\}. \end{aligned}$$

If $v(\eta) - v(\mu) \notin R$, self-control is never exerted at $\{\mu, \eta\}$. Thus, the actual shape of φ outside R is immaterial in the description of choice behavior. Note that since preference satisfies the MPSC axiom, R is an interval with $\inf R = 0$. Notice also that the threshold level $\overline{w}$ associated with the representation must satisfies $R \subset [0, \overline{w}]$ because self-control is never exerted when $v(\eta) - v(\mu) > \overline{w}$.

The uniqueness properties of the representation mirror those of the general representation.

Theorem 4 *Suppose that* (u, v, φ) *and* $(\tilde{u}, \tilde{v}, \tilde{\varphi})$ *are both representations of a convex self-control preference. Then there exist constants* $\alpha_u, \alpha_v > 0$ *and* β_u, β_v *such that* $\tilde{u} = \alpha_u u + \beta_u$ *and* $\tilde{v} = \alpha_v v + \beta_v$. *Moreover, when* R *and* $\tilde{R}$ *are the self-control subdomains for* φ *and* $\tilde{\varphi}$ *respectively,* $\tilde{R} = \alpha_v R$ *and* $\tilde{\varphi}(\alpha_v w) = \alpha_u \varphi(w)$ *for all* $w \in R$.

The theorem states that u and v are unique up to positive affine transformation. When $\varphi, \tilde{\varphi}$ are differentiable, the stated condition on φ and $\tilde{\varphi}$ implies that for $\tilde{w} = \alpha_v w$ and $w \in R$,

$$\frac{\tilde{w}\tilde{\varphi}''(\tilde{w})}{\tilde{\varphi}'(\tilde{w})} = \frac{w\varphi''(w)}{\varphi'(w)}$$

where f' and f'' denote the first and the second derivatives of f, respectively. Thus the curvature of φ is uniquely determined within the self-control subdomain.

3.4 Proof Outline for Theorem 3

The main technical difficulty is not establishing convexity, but rather showing that the self-control cost function in the general model takes the form

$$c(\mu, \max_{\eta \in x} v(\eta)) = \varphi\left(\max_{\eta \in x} v(\eta) - v(\mu)\right).$$

We explain next how the function φ is derived from preference.

The functions u, v, and W are determined as in the general model. Take any μ, η satisfying $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$. This ranking suggests that μ is chosen with self-control in $\{\mu, \eta\}$. Hence, the difference $u(\mu) - W(\{\mu, \eta\})$ should exactly express the cost of self-control at $\{\mu, \eta\}$. On the other hand, the temptation frustration is $w = v(\eta) - v(\mu)$. Define

$$\varphi(v(\eta) - v(\mu)) := u(\mu) - W(\{\mu, \eta\}). \tag{5}$$

The key step in the proof is to show that φ is indeed well-defined. This is demonstrated by establishing that for all μ, μ', η, η' such that $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$ and $\{\mu'\} \succ \{\mu', \eta'\} \succ \{\eta'\}$,

$$v(\eta) - v(\mu) \geq v(\eta') - v(\mu') \implies u(\mu) - W(\{\mu, \eta\}) \geq u(\mu') - W(\{\mu', \eta'\}). \tag{6}$$

The bulk of the proof for this claim concerns the case where μ, μ', η, η' have finite supports and belong to the interior of $\Delta_{\mu,\mu',\eta,\eta'}$, the finite-dimensional set of lotteries over the union of the supports. The result for general lotteries then obtains by a continuity argument together with the fact that the set of lotteries with finite supports is dense in Δ under the weak convergence topology. So take such lotteries μ, μ', η, η' that satisfy the hypothesis of (6). An implication of MPSC and Self-Control Concavity is that for any $\alpha \in [0, 1]$,

$$\{\mu\alpha\mu'\} \succ \{\mu\alpha\mu', \eta\alpha\eta'\} \succ \{\eta\alpha\eta'\}.$$

That is, since η tempts μ and η' tempts μ', it is also true that the mixture $\eta\alpha\eta'$ tempts the mixture $\mu\alpha\mu'$. We observe that there is an open neighborhood $O(\alpha) \subset [0, 1]$ of α such that for any $\tilde{\alpha} \in O(\alpha)$,

$$\tilde{\alpha} \geq \alpha \iff u(\mu\tilde{\alpha}\mu') - W(\{\mu\tilde{\alpha}\mu', \eta\tilde{\alpha}\eta'\}) \geq u(\mu\alpha\mu') - W(\{\mu\alpha\mu', \eta\alpha\eta'\}).$$

That is, small movements from α respect the implication (6).[5] This observation makes use of Weak Binary Independence and in particular Temptation Dependence. But now observe that $\{O(\alpha)\}_{\alpha\in[0,1]}$ is an open cover of $[0, 1]$. Therefore there exists a finite subcover, and indeed, we can find a finite number of mixing coefficients $1 = \alpha^0 \geq \alpha^1 \geq .. \geq \alpha^I = 0$ such that

$$\begin{aligned} u(\mu) - W(\{\mu, \eta\}) &= u(\mu\alpha^0\mu') - W(\{\mu\alpha^0\mu', \eta\alpha^0\eta'\}) \\ &\geq \; ... \geq u(\mu\alpha^I\mu') - W(\{\mu\alpha^I\mu', \eta\alpha^I\eta'\}) = u(\mu') - W(\{\mu', \eta'\}). \end{aligned}$$

Thus, via this 'chain' linking $u(\mu) - W(\{\mu, \eta\})$ and $u(\mu') - W(\{\mu', \eta'\})$ we are able to prove (6).

An immediate implication of (6) is that

$$v(\eta) - v(\mu) = v(\eta') - v(\mu') \implies u(\mu) - W(\{\mu, \eta\}) = u(\mu') - W(\{\mu', \eta'\}).$$

It then follows that φ as defined in (5) is indeed well-defined (in fact it also follows that it is increasing).

[5] By linearity of v, if $v(\eta) - v(\mu) \geq v(\eta') - v(\mu')$, then $\tilde{\alpha} \geq \alpha \iff v(\eta\tilde{\alpha}\eta') - v(\mu\tilde{\alpha}\mu') \geq v(\eta\alpha\eta') - v(\mu\alpha\mu')$

4 Concluding Remarks: Ex post Choice

While the convex self-control model is a representation for an ex ante preference over menus, it suggests that ex post choice is given by the choice correspondence defined by:

$$\mathcal{C}_\varphi(x) = \arg\max_{\mu \in x} \left\{ u(\mu) - \varphi\left(\max_{\mu' \in x} v(\mu') - v(\mu) \right) \right\}. \tag{7}$$

We conclude this paper with some observations about this choice correspondence in the context of choice under risk.

An immediate observation is that $\mathcal{C}_\varphi$ is menu-dependent via its dependence on the most tempting alternative in the menu. If φ is convex, for instance, this would imply that while an agent can pick a 'good' alternative over a moderately tempting alternative, adding an even more tempting alternative to the menu may induce the agent to choose the moderately tempting alternative, thereby violating the Weak Axiom of Revealed Preference. The intuition for such choice is that the loss of self-control ability due to the presence of a great temptation may make the agent unable to choose the 'good' alternative, but he may nevertheless have enough self-control to resist the great temptation. He chooses the moderately tempting alternative as a compromise. An analysis of the notion of menu-dependent self-control can be found in a companion paper (Noor and Takeoka [14]).

The choice structure as given by (7) has an interesting implication for choice under risk: choice between risky prospects may not be explicable by expected utility theory. In fact, the model may accommodate the *common ratio effect* (Allais [2], Kahneman and Tversky [11]). To illustrate, consider an agent who normatively prefers a risky lottery r to a riskless one s but is tempted by the latter. Moreover, suppose he exhibits

$$\{r\} \succ \{r, s\} \sim \{s\}, \quad \text{and} \quad \{r\alpha s\} \succ \{r\alpha s, s\} \succ \{s\} \text{ for some } \alpha \in (0, 1).$$

The former ranking suggests that the agent yields to temptation at $\{r, s\}$ and ends up with choosing tempting option s, while the latter ranking says that he exercises self-control at $\{r\alpha s, s\}$ and chooses $r\alpha s$ over s, that is, mixing r with s induces self-control. Since $\{r\alpha s, s\}$ is obtained by mixing $\{r, s\}$ and $\{s\}$ with proportion α, this preference reversal has the spirit of the common ratio effect. In the convex self-control model, this choice pattern is rationalized if both

$$u(s) \geq u(r) - \varphi(v(s) - v(r)), \quad \text{and} \quad u(r\alpha s) - \varphi(v(s) - v(r\alpha s)) > u(s)$$

hold. That is,

$$\varphi(v(s) - v(r)) \geq u(r) - u(s) > \frac{1}{\alpha}\varphi(\alpha(v(s) - v(r))).$$

These inequalities can hold when φ is convex.

Observe that the utility function in (7) is concave in μ when φ is convex. This feature distinguishes choice behavior of this model from that of other non-expected utility models satisfying monotonicity with respect to first-order stochastic dominance. Given the fact that s is preferred to r, the monotonicity condition requires that s should be preferred to any mixed options $r\alpha s$, whereas some $r\alpha s$ may be strictly preferred to both r and s when φ is convex. Intuitively, this is because the mixed option $r\alpha s$ is a good compromise for the conflict between normative and temptation utilities. In terms of preference over menus, this property implies that $\{r, r\alpha s, s\}$ may be strictly preferred to $\{r, s\}$.

To conclude, we note that the convex model lends itself to an infinite horizon extension in the spirit of [9, 13]. In this setting, convexity potentially has interesting implications for the interaction of risk and time preferences and also for the timing of resolution of risk.

A Appendix: Proof of Theorem 1

The proof of necessity of the axioms is routine. For Temptation Dependence, observe that since $v(\nu) \leq v(\eta)$ and $c(\mu, \cdot)$ is weakly increasing, $W(\{\mu, \eta\}) = u(\mu) - c(\mu, v(\eta)) \leq u(\mu) - c(\mu, v(\nu)) \leq W(\{\mu, \nu\})$.

Suffiency of the axioms is established in a sequence of lemmas.

Lemma 1 *(i) There exists a continuous linear function $u : \Delta \to \mathbb{R}_+$ such that*

$$\{\mu\} \succsim \{\eta\} \Longleftrightarrow u(\mu) \geq u(\eta)$$

(ii) There exists a continuous function $W : Z \to \mathbb{R}_+$ that represents $\succsim$ and satisfies $W(\{\mu\}) = u(\mu)$ for all $\mu \in \Delta$.

(iii) There exists a continuous linear function $v : \Delta \to \mathbb{R}_+$ such that if $\{\mu\} \succ \{\eta\}$ then

$$\{\mu\} \succ \{\mu, \eta\} \Longleftrightarrow v(\eta) > v(\mu).$$

Proof. (i) The first assertion follows from Order, Continuity, Commitment Independence, and the mixture space theorem.

(ii) Since u is continuous on Δ, there exist a maximal and a minimal lottery $\mu^\Delta, \mu_\Delta \in \Delta$ with respect to u. Without loss of generality, we can assume $u(\mu^\Delta) = 1$ and $u(\mu_\Delta) = 0$. From Continuity and Set Betweenness, $\{\mu^\Delta\} \succsim x \succsim \{\mu_\Delta\}$ for all $x \in Z$. By a standard argument, for all $x \in Z$, there exists a unique number $\alpha(x) \in [0, 1]$ such that $x \sim \{\mu^\Delta \alpha(x) \mu_\Delta\}$. Define

$$W(x) \equiv u(\mu^\Delta \alpha(x) \mu_\Delta) \in [0, 1].$$

Then W represents $\succsim$. Moreover, $W(\{\mu\}) = u(\mu)$ for all $\mu \in \Delta$.

To show continuity of W, let $x^n \to x$. Since $u(\mu^\Delta) = 1$ and $u(\mu_\Delta) = 0$, $W(x) = \alpha(x)$. So we want to show $\alpha(x^n) \to \alpha(x)$. By contradiction, suppose otherwise. Then, there exists a neighborhood $B(\alpha(x))$ of $\alpha(x)$ such that $\alpha(x^m) \notin B(\alpha(x))$ for infinitely many m. Let $\{x^m\}$ denote the corresponding subsequence of $\{x^n\}$. Since $x^n \to x$, $\{x^m\}$ also converges to x. Since $\{\alpha(x^m)\}$ is a sequence in $[0, 1]$, there exists a convergent subsequence $\{\alpha(x^\ell)\}$ with a limit $\alpha \neq \alpha(x)$. On the other hand, since $x^\ell \to x$ and $x^\ell \sim \{\mu^\Delta \alpha(x^\ell)\mu_\Delta\}$, Continuity implies $x \sim \{\mu^\Delta \alpha \mu_\Delta\}$. Since $\alpha(x)$ is unique, $\alpha(x) = \alpha$, which is a contradiction.

(iii) See Noor and Takeoka [14, Lemma 9]. ■

Without loss of generality, assume that $v(\Delta) = [0,1]$. By construction, if $\{\mu\} \succ \{\mu, \eta\}$, then $v(\eta) > v(\mu)$. If $\{\mu\} \sim \{\mu, \eta\} \succ \{\eta\}$, then $v(\mu) \geq v(\eta)$.

Lemma 2 *For all $\mu, \eta, \nu \in \Delta$, if $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$ and $v(\nu) \leq v(\eta)$, then $\{\mu, \nu\} \succsim \{\mu, \eta\}$.*

Proof. The first case is where $\{\nu\} \succsim \{\eta\}$. Since $\{\mu\} \succ \{\mu, \eta\}$, we know $u(\mu) > u(\eta)$ and $v(\mu) < v(\eta)$. For all $\alpha \in (0,1)$, $v(\eta) > v(\nu\alpha\mu)$ and $u(\nu\alpha\mu) > u(\eta)$. Thus $\{\nu\alpha\mu\} \succ \{\nu\alpha\mu, \eta\}$. By Temptation Dependence, $\{\mu, \nu\alpha\mu\} \succsim \{\mu, \eta\}$. By Continuity, we have $\{\mu, \nu\} \succsim \{\mu, \eta\}$ as $\alpha \to 1$.

Next suppose $\{\eta\} \succ \{\nu\}$. If $\{\eta\} \succ \{\eta, \nu\}$, we have $v(\nu) > v(\eta)$, which contradicts the assumption. Hence Set Betweenness implies $\{\eta\} \sim \{\eta, \nu\} \succ \{\nu\}$. By Temptation Dependence, $\{\mu, \nu\} \succsim \{\mu, \eta\}$. ■

Define the correspondence $L : v(\Delta) \rightsquigarrow \Delta$ by:

$$L(l) := \{\eta : v(\eta) \leq l\}.$$

By continuity and linearity of v, it is clear that $L(l)$ is a nonempty compact convex set for each l. Define the self-control cost function by:

$$c(\mu, l) = \max \left[0, \max_{\nu \in L(l)} \{u(\mu) - W(\{\mu, \nu\})\}\right].$$

The following Lemma clarifies various properties of c. Properties (iii)-(vi) correspond to the properties in the statement of the Theorem.

Lemma 3 *(i) For any μ, l, if $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$ for some η with $v(\eta) = l$, then $c(\mu, l) = u(\mu) - W(\{\mu, \eta\}) > 0$.*

(ii) For any μ, l, if $\{\mu\} \succ \{\mu, \eta\}$ for some $\eta \in L(l)$, then $c(\mu, l) > 0$.

(iii) For any μ, l, if $l \leq v(\mu)$ then $c(\mu, l) = 0$.

(iv) If $u(\mu) > u(\eta)$ and $l = \max_{\mu,\eta} v$ then

$$v(\mu) < v(\eta) \Longleftrightarrow c(\mu, l) > 0.$$

(v) For any μ, $c(\mu, \cdot)$ is weakly increasing.

(vi) The function c is continuous.

Proof. (i) For any $\nu \in L(l)$, $v(\nu) \leq v(\eta)$, and thus by Lemma 2, $u(\mu) - W(\{\mu, \nu\}) \leq u(\mu) - W(\{\mu, \eta\})$. Since $\eta \in L(l)$, it follows that $\max_{\nu \in L(l)}\{u(\mu) - W(\{\mu, \nu\})\} = u(\mu) - W(\{\mu, \eta\}) > 0$ and thus $c(\mu, l) = u(\mu) - W(\{\mu, \eta\})$.

(ii) Obvious from the definition of c.

(iii) Under the hypothesis, $\{\mu\} \not\succ \{\mu, \eta\}$ for all $\eta \in L(l)$. Consequently $\max_{\nu \in L(l)}\{u(\mu) - W(\{\mu, \nu\})\} \leq 0$ and so $c(\mu, l) = 0$.

(iv) Sufficiency obtains from part (ii). For the converse, note that if $v(\mu) \geq v(\eta)$ then $l = v(\mu)$, and thus part (iii) implies $c(\mu, l) = 0$.

(v) For any $l, l' \in v(\Delta)$,
$l' < l$
$\implies L(l') \subset L(l)$
$\implies \max_{\nu \in L(l')}\{u(\mu) - W(\{\mu, \nu\})\} \leq \max_{\nu \in L(l)}\{u(\mu) - W(\{\mu, \nu\})\}$
$\implies c(\mu, l') \leq c(\mu, l)$.

(vi) We show below that $L : v(\Delta) \rightsquigarrow \Delta$ is a continuous correspondence. The assertion then follows from the following argument: Since u and W are continuous, the Maximum Theorem implies that $(\mu, l) \mapsto \max_{\nu \in L(l)}\{u(\mu) - W(\{\mu, \nu\})\}$ is continuous. Moreover, since the upper envelope of two continuous functions is continuous, the function c is continuous.

To show that L is upper hemicontinuous, take any sequence $\{l_n\} \subset v(\Delta)$ that converges to some $l \in v(\Delta)$, and suppose that $\eta_n \in L(l_n)$ for each n. We must show that there is a subsequence $\{l_{n(m)}\}$ s.t. $\eta_{n(m)} \to \eta$ for some $\eta \in L(l)$. Since $\{\eta_n\}$ is a sequence in a compact set Δ, it has a convergent subsequence $\eta_{n(m)} \to \eta$ for some η. Since $v(\eta_{n(m)}) \leq l_{n(m)}$ for each m, and since v is continuous, it follows that $v(\eta) \leq l$, and thus $\eta \in L(l)$, as desired.

To show that L is lower hemicontinuous, take any sequence $\{l_n\} \subset v(\Delta)$ that converges to some $l \in v(\Delta)$, and suppose that $\eta \in L(l)$. We must show that there exists a subsequence $\{l_{n(m)}\}$ s.t. $\eta_{n(m)} \to \eta$, where $\eta_{n(m)} \in L(l_{n(m)})$ for each m. Consider two possibilities:

i - There exists N s.t. $l_n \geq v(\eta)$ for all $n \geq N$.

Then $\eta \in L(l_n)$ for each $n \geq N$. In particular, lower hemicontinuity is established by taking the subsequence $\{l_N, l_{N+1}, ...\}$ and the corresponding trivial sequence $\{\eta\}$ that converges to η.

ii - For all N there exists $n_N \geq N$ s.t. $l_n < v(\eta)$.

Take the subsequence $\{l_{n(m)}\}$ satisfying $l_{n(m)} < v(\eta)$ for all m. Construct $\{\eta_{n(m)}\}$ as follows: Let η_* be the minimizer of v over Δ (normalized so that $v(\eta_*) = 0$) and let $\alpha_{n(m)}$ satisfy $v(\eta\alpha_{n(m)}\eta_*) = l_{n(m)}\frac{v(\eta)}{l} \leq l_{n(m)}$.[6] Then $\eta_{n(m)} \in L(l_{n(m)})$, where $\eta_{n(m)} := \eta\alpha_{n(m)}\eta_*$ for each m. To see that $\eta_{n(m)} \to \eta$, observe that
$v(\eta\alpha_{n(m)}\eta_*) = l_{n(m)}\frac{v(\eta)}{l}$
$\implies \alpha_{n(m)}v(\eta) = l_{n(m)}\frac{v(\eta)}{l}$ (since v is linear and $v(\eta_*) = 0$)
$\implies \alpha_{n(m)} = \frac{l_{n(m)}}{l}$ (note that $v(\eta) > 0$ since $v(\eta) > l_{n(m)} \geq 0$, and also note that $\frac{l_{n(m)}}{l} < 1$ since $l_{n_m} < v(\eta) \leq l$).

Since $l_{n(m)} \to l$, it follows that $\alpha_{n(m)} \to 1$, and in turn, $\eta_{n(m)} \to \eta$, as desired. ■

Lemma 4 *For all $\mu, \eta \in \Delta$,*

$$W(\{\mu, \eta\}) = \max_{\nu \in \{\mu,\eta\}} \left\{ u(\nu) - c\left(\nu, \max_{\{\mu,\eta\}} v\right) \right\}.$$

[6]Note that $l > 0$, otherwise $0 \leq l_{n(m)} < v(\eta) \leq l = 0$ is a contradiction. Recall also that $\eta \in L(l)$ implies $v(\eta) \leq l$, and thus $l_{n(m)}\frac{v(\eta)}{l} \leq l_{n(m)}$.

Proof. Consider the various cases. In each case, let $l = \max_{\{\mu,\eta\}} v$.

(i) $\{\mu\} \succ \{\mu,\eta\} \succ \{\eta\}$.

Since $v(\mu) < v(\eta) = l$, Lemma 3(i) implies $W(\{\mu,\eta\}) = u(\mu) - c(\mu, l) = u(\mu) - c(\mu, \max_{\{\mu,\eta\}} v)$. Since $c(\eta, l) = 0$, we have $u(\eta) - c(\eta, \max_{\{\mu,\eta\}} v) = u(\eta)$, and since $\{\mu,\eta\} \succ \{\eta\}$, it follows that

$$u(\mu) - c(\mu, \max_{\{\mu,\eta\}} v) > u(\eta) - c(\eta, \max_{\{\mu,\eta\}} v).$$

Indeed, $W(\{\mu,\eta\}) = \max_{\nu\in\{\mu,\eta\}} u(\nu) - c(\nu, \max_{\{\mu,\eta\}} v)$, as desired.

(ii) $\{\mu\} \succ \{\mu,\eta\} \sim \{\eta\}$.

By definition of $c(\mu, l)$,

$$c(\mu, l) \geq \max_{\nu\in L(l)} \{u(\mu) - W(\{\mu,\nu\})\} \geq u(\mu) - W(\{\mu,\eta\}).$$

In particular, $W(\{\mu,\eta\}) \geq u(\mu) - c(\mu, l) = u(\mu) - c(\mu, \max_{\{\mu,\eta\}} v)$. Then

$$u(\eta) - c(\eta, \max_{\{\mu,\eta\}} v) = u(\eta) = W(\{\mu,\eta\}) \geq u(\mu) - c(\mu, \max_{\{\mu,\eta\}} v),$$

and hence $W(\{\mu,\eta\}) = \max_{\nu\in\{\mu,\eta\}} u(\nu) - c(\nu, \max_{\{\mu,\eta\}} v)$.

(iii) $\{\mu\} \sim \{\mu,\eta\} \succ \{\eta\}$ or $\{\eta\} \sim \{\eta,\mu\} \succsim \{\mu\}$.

Suppose $\{\mu\} \sim \{\mu,\eta\} \succ \{\eta\}$. Then $l = v(\mu) \geq v(\eta)$, and in particular, $c(\mu, l) = 0$. Since $c(\eta, l) \geq 0$,

$W(\{\mu,\eta\}) = u(\mu)$

$= u(\mu) - c(\mu, \max_{\{\mu,\eta\}} v)$

$= u(\eta)$ since $\{\mu\} \succ \{\eta\}$ and $c(\mu, l) = 0$

$\geq u(\eta) - c(\eta, l)$ since $c(\eta, l) \geq 0$. This establishes the result.

For the case where $\{\eta\} \sim \{\eta,\mu\} \succsim \{\mu\}$, we have $l = v(\eta) \geq v(\mu)$ (this is wlog when $\{\mu\} \sim \{\mu,\eta\} \sim \{\eta\}$), $c(\eta, l) = 0$ and $c(\mu, l) \geq 0$. Arguing as above yields the result.

(iv) $\{\eta\} \succ \{\eta,\mu\} \succsim \{\mu\}$.

The argument is analogous to that in cases (i) and (ii). ■

Lemma 5 *For all finite menus $x \in Z$,*

$$W(x) = \max_{\nu\in x} \left\{ u(\nu) - c\left(\nu, \max_x v\right) \right\}.$$

Proof. The argument is similar that used in the conclusion of the proof [8, Thm 1]. Gul and Pesendorfer [8, Lemma 2] show that if $\succsim$ satisfies Set Betweenness, for all finite menus $x \in Z$,

$$W(x) = \max_{\mu\in x} \min_{\eta\in x} W(\{\mu,\eta\}) = \min_{\eta\in x} \max_{\mu\in x} W(\{\mu,\eta\}). \tag{8}$$

Fix $\mu \in x$ arbitrarily. Since $c(\nu, \cdot)$ is weakly increasing for all ν,

$$\begin{aligned} \min_{\eta\in x} W(\{\mu,\eta\}) &= \min_{\eta\in x} \max_{\nu\in\{\mu,\eta\}} u(\nu) - c\left(\nu, \max_{\{\mu,\eta\}} v\right) \geq \min_{\eta\in x} \max_{\nu\in\{\mu,\eta\}} u(\nu) - c\left(\nu, \max_x v\right) \\ &= \max_{\nu\in\{\mu,\eta^\mu\}} u(\nu) - c\left(\nu, \max_x v\right), \end{aligned}$$

where η^μ is a minimizer of the associated minimization problem. Since the above inequality holds for all $\mu \in x$, if follows from (8) that

$$W(x) \geq \max_{\mu\in x} \max_{\nu\in\{\mu,\eta^\mu\}} u(\nu) - c\left(\nu, \max_x v\right) = \max_{\nu\in x}\left\{u(\nu) - c\left(\nu, \max_x v\right)\right\}. \quad (9)$$

On the other hand, fix $\eta \in x$ arbitrarily. Since $c(\nu,\cdot)$ is weakly increasing,

$$\begin{aligned}\max_{\mu\in x} W(\{\mu,\eta\}) &= \max_{\mu\in x} \max_{\nu\in\{\mu,\eta\}} u(\nu) - c\left(\nu, \max_{\{\mu,\eta\}} v\right) \leq \max_{\mu\in x} \max_{\nu\in\{\mu,\eta\}} u(\nu) - c\left(\nu, \min_{\mu\in x}\max_{\{\mu,\eta\}} v\right) \\ &= \max_{\nu\in x} u(\nu) - c\left(\nu, \min_{\mu\in x}\max_{\{\mu,\eta\}} v\right) = \max_{\nu\in x} u(\nu) - c\left(\nu, \max_{\{\mu^\eta,\eta\}} v\right),\end{aligned}$$

where μ^η is a minimizer of the associated minimization problem. Since $c(\nu,\cdot)$ is weakly increasing and the above inequality holds for all $\eta \in x$, if follows from (8) that

$$\begin{aligned}W(x) &\leq \min_{\eta\in x}\max_{\nu\in x}\left\{u(\nu) - c\left(\nu, \max_{\{\mu^\eta,\eta\}} v\right)\right\} = \max_{\nu\in x}\left\{u(\nu) - \max_{\eta\in x} c\left(\nu, \max_{\{\mu^\eta,\eta\}} v\right)\right\} \\ &= \max_{\nu\in x}\left\{u(\nu) - c\left(\nu, \max_x v\right)\right\}. \quad (10)\end{aligned}$$

Taking (9) and (10) together, the desired result holds. ■

Lemma 6 *For all $x \in Z$, W can be written as the desired form.*

Proof. By Lemma 0 of Gul and Pesendorfer [8, p.1421], there exists a sequence of subsets x^n of x such that each x^n is finite and $x^n \to x$ in the Hausdorff metric. By Lemma 5,

$$W(x^n) = \max_{\nu\in x^n}\left\{u(\nu) - c\left(\nu, \max_{x^n} v\right)\right\}. \quad (11)$$

Since c is continuous by Lemma 3 (vi), the maximum theorem implies that the RHS of (11) converges to

$$\max_{\nu\in x}\left\{u(\nu) - c\left(\nu, \max_x v\right)\right\}.$$

On the other hand, by Lemma 1 (ii), $W(x^n) \to W(x)$. This completes the proof. ■

B Appendix: Proof of Theorem 3

Proof of necessity of axioms is omitted. The proof of suffiency is as follows.

Let (u, v, W) be the objects guaranteed by Lemma 1. Since u and v are mixture linear, assume that $u(\Delta) = v(\Delta) = [0,1]$.

Let

$$A \equiv \{w \in [0,1] \,|\, w = v(\eta) - v(\mu), \text{ for some } \mu, \eta \text{ such that } \{\mu\} \succ \{\mu,\eta\} \succ \{\eta\}\}.$$

Since $\succsim$ is a self-control preference, A is non-empty.

Lemma 7 *(i) A is an interval with* $\inf A = 0$*, and (ii) if* $\sup A \in A$*, then* $\sup A = 1$.

Proof. (i) It suffices to show that for all $w \in A$, $\alpha w \in A$ for all $\alpha \in (0, 1)$. Let $w \in A$. There exist μ, η such that $w = v(\eta) - v(\mu)$ and $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$. By MPSC, $\{\mu\} \succ \{\mu, \eta\alpha\mu\} \succ \{\eta\alpha\mu\}$. Thus $\alpha w = \alpha(v(\eta) - v(\mu)) = v(\eta\alpha\mu) - v(\mu) \in A$.

(ii) Since $\sup A \in A$, there exist μ, η such that $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$ and $v(\eta) - v(\mu) = \sup A$. By contradiction, suppose $\sup A < 1$. Then, either $\max_\Delta v > v(\eta)$ or $\min_\Delta v < v(\mu)$. In case of the former, Continuity implies that there exists ν sufficiently close to η such that $\{\mu\} \succ \{\mu, \nu\} \succ \{\nu\}$ and $v(\nu) > v(\eta)$. Thus $\sup A < v(\nu) - v(\mu) \in A$, which is a contradiction. The symmetric argument can be applied to the latter case. ■

Define $\varphi : A \to (0, 1]$ by

$$\varphi(w) \equiv u(\mu) - W(\{\mu, \eta\}),$$

where μ, η satisfy $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$ and $w = v(\eta) - v(\mu)$.

The lemmas below establish that φ is well-defined.

Lemma 8 *For all* $\mu, \eta, \mu', \eta' \in \Delta$ *and* $\alpha \in (0, 1)$*, If* $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$ *and* $\{\mu'\} \succ \{\mu', \eta'\} \succ \{\eta'\}$*, then* $\{\mu\alpha\mu'\} \succ \{\mu\alpha\mu', \eta\alpha\eta'\} \succ \{\eta\alpha\eta'\}$ *for all* $\alpha \in [0, 1]$.

Proof. Since $\{\mu\} \succ \{\mu, \eta\}$ and $\{\mu'\} \succ \{\mu', \eta'\}$, we have $u(\mu) > u(\eta)$, $v(\eta) > v(\mu)$, $u(\mu') > u(\eta')$, and $v(\eta') > v(\mu')$. Since u and v are mixture linear, $u(\mu\alpha\mu') > u(\eta\alpha\eta')$ and $v(\eta\alpha\eta') > v(\mu\alpha\mu')$, and, hence, $\{\mu\alpha\mu'\} \succ \{\mu\alpha\mu', \eta\alpha\eta'\}$. As shown in Lemma1 (ii), there exist $\nu, \nu' \in \Delta$ such that $\{\mu, \eta\} \sim \{\nu\}$ and $\{\mu', \eta'\} \sim \{\nu'\}$. By Self-Control Concavity, $\{\mu\alpha\mu', \eta\alpha\eta'\} \succsim \{\nu\alpha\nu'\}$. Since $\{\nu\} \succ \{\eta\}$ and $\{\nu'\} \succ \{\eta'\}$, Commitment Independence implies that $\{\nu\alpha\nu'\} \succ \{\eta\alpha\eta'\}$ for all $\alpha \in [0, 1]$. Therefore, we have $\{\mu\alpha\mu'\} \succ \{\mu\alpha\mu', \eta\alpha\eta'\} \succ \{\eta\alpha\eta'\}$. ■

Take any finite subset $\mathbf{c} = \{c_1, \cdots, c_N\} \subset C$. Define

$$\Delta_{(N,\mathbf{c})} \equiv \left\{ \nu \in \mathbb{R}^N_+ \;\middle|\; \sum_{i=1}^{N} \nu(c_i) = 1 \right\} \subset \Delta,\ \Theta_{(N,\mathbf{c})} \equiv \left\{ \theta \in \mathbb{R}^N \;\middle|\; \sum_{i=1}^{N} \theta(c_i) = 0 \right\}.$$

For all $\mu \in \Delta_{(N,\mathbf{c})}$ and $\theta \in \Theta_{(N,\mathbf{c})}$, if $\mu + \theta \in \Delta_{(N,\mathbf{c})}$, we can view $\mu + \theta$ as the lottery obtained by shifting μ toward θ. For all $\mu \in \Delta_{(N,\mathbf{c})}$, say that $\theta \in \Theta_{(N,\mathbf{c})}$ is admissible for μ if $\mu + \theta \in \Delta_{(N,\mathbf{c})}$.

Lemma 9 *Given any two menus* $x, y \subset \Delta_{(N,\mathbf{c})}$*, the following statements are equivalent:*

(a) For all $\alpha \in [0, 1]$ *and* $\mu, \eta \in \Delta_{(N,\mathbf{c})}$, $x\alpha\{\mu\} \succsim y\alpha\{\mu\} \Longrightarrow x\alpha\{\eta\} \succsim y\alpha\{\eta\}$.

(b) For all $\theta \in \Theta_{(N,\mathbf{c})}$ *that are admissible for* x, y, $x \succsim y \Longleftrightarrow x + \theta \succsim y + \theta$.

Proof. Inspecting the proof of Ergin and Sarver [5, Lemma 6] reveals that the proof works for any two fixed menus x, y. ■

The preceding lemma yields that Weak Binary Independence is equivalent to the condition that for all $\mu, \mu', \eta, \eta' \in \Delta_{(N,\mathbf{c})}$ and admissible translations $\theta \in \Theta_{(N,\mathbf{c})}$ for these lotteries,

$$\{\mu, \mu'\} \succsim \{\eta, \eta'\} \implies \{\mu + \theta, \mu' + \theta\} \succsim \{\eta + \theta, \eta' + \theta\}, \tag{12}$$

which is referred to as *Translation Invariance.*

For all $\theta \in \Theta_{(N,\mathbf{c})}$, let $u(\theta)$ denote $\sum_i u(c_i)\theta(c_i)$.

Lemma 10 *For all $\mu, \mu' \in \Delta_{(N,\mathbf{c})}$ and $\theta \in \Theta_{(N,\mathbf{c})}$, if $\mu+\theta, \mu'+\theta \in \Delta_{(N,\mathbf{c})}$, then $W(\{\mu+\theta, \mu'+\theta\}) = W(\{\mu,\mu'\}) + u(\theta)$.*

Proof. By Set Betweenness, assume that $\{\mu\} \succsim \{\mu,\mu'\} \succsim \{\mu'\}$. Since u is continuous, there exists $\alpha \in [0,1]$ such that $W(\{\mu,\mu'\}) = u(\mu\alpha\mu')$. If $\mu+\theta, \mu'+\theta \in \Delta_{(N,\mathbf{c})}$, $\mu\alpha\mu'+\theta = (\mu+\theta)\alpha(\mu'+\theta) \in \Delta_{(N,\mathbf{c})}$. Hence Translation Invariance implies that

$$W(\{\mu+\theta, \mu'+\theta)\}) = u(\mu\alpha\mu'+\theta) = u(\mu\alpha\mu') + u(\theta) = W(\{\mu,\mu'\}) + u(\theta).$$

■

Lemma 11 *Take all $\mu, \mu', \eta, \eta' \in \Delta$ with finite supports. Assume that $\{\mu\} \succ \{\mu,\eta\} \succ \{\eta\}$ and $\{\mu'\} \succ \{\mu',\eta'\} \succ \{\eta'\}$. Then,*

$$v(\eta) - v(\mu) \geq v(\eta') - v(\mu') \quad \Rightarrow \quad u(\mu) - W(\{\mu,\eta\}) \geq u(\mu') - W(\{\mu',\eta'\}).$$

Proof. Let $\mathbf{c} \equiv \{c_1, \cdots, c_N\} \subset C$ be the union of the supports of μ, μ', η, η'. Hence, these lotteries belong to $\Delta_{(N,\mathbf{c})}$.

Step 1: We claim that if $\theta \equiv \mu' - \mu \in \Theta_{(N,\mathbf{c})}$ is admissible for η, then $u(\mu) - W(\{\mu,\eta\}) \geq u(\mu') - W(\{\mu',\eta'\})$. Since v is mixture linear,

$$v(\eta+\theta) - v(\mu') = v(\eta+\theta) - v(\mu+\theta) = v(\eta) - v(\mu) \geq v(\eta') - v(\mu').$$

Thus $v(\eta+\theta) \geq v(\eta')$. Furthermore, by Translation Invariance as given in (12), $\{\mu+\theta\} \succ \{\mu+\theta, \eta+\theta\} \succ \{\eta+\theta\}$, that is, $\{\mu'\} \succ \{\mu', \eta+\theta\} \succ \{\eta+\theta\}$. By Lemma 2, $\{\mu',\eta'\} \succsim \{\mu',\eta+\theta\}$. Thus, from Lemma 10,

$$\begin{aligned} & u(\mu') - W(\{\mu',\eta'\}) \leq u(\mu') - W(\{\mu',\eta+\theta\}) \\ \Leftrightarrow\ & u(\mu') - W(\{\mu',\eta'\}) \leq u(\mu+\theta) - W(\{\mu+\theta,\eta+\theta\}) \\ \Leftrightarrow\ & u(\mu') - W(\{\mu',\eta'\}) \leq u(\mu) + u(\theta) - W(\{\mu,\eta\}) - u(\theta) \\ \Leftrightarrow\ & u(\mu') - W(\{\mu',\eta'\}) \leq u(\mu) - W(\{\mu,\eta\}). \end{aligned}$$

Take a lottery ν in the interior of $\Delta_{(N,\mathbf{c})}$. For all $\alpha \in (0,1)$ sufficiently close to one, let $a \equiv \mu\alpha\nu$, $b \equiv \eta\alpha\nu$, $a' \equiv \mu'\alpha\nu$, $b' \equiv \eta'\alpha\nu \in \Delta_{(N,\mathbf{c})}$. Continuity implies $\{a\} \succ \{a,b\} \succ \{b\}$ and $\{a'\} \succ \{a',b'\} \succ \{b'\}$. Furthermore, $v(b) - v(a) = v(\eta\alpha\nu) - v(\mu\alpha\nu) = \alpha(v(\eta) - v(\mu)) \geq \alpha(v(\eta') - v(\mu')) = v(b') - v(a')$. From Lemma 8, for all $\beta \in [0,1]$, $\{a\beta a'\} \succ \{a\beta a', b\beta b'\} \succ \{b\beta b'\}$. Notice also that $a\beta a'$, $b\beta b' \in \Delta_{(N,\mathbf{c})}$ for all $\beta \in [0,1]$.

Step 2: We claim that for all $\beta \in [0,1]$, there exists a relative open interval $O(\beta)$ containing β such that for all $\tilde{\beta} \in O(\beta)$,

$$\tilde{\beta} \geq \beta \iff u(a\tilde{\beta}a') - W(\{a\tilde{\beta}a', b\tilde{\beta}b'\}) \geq u(a\beta a') - W(\{a\beta a', b\beta b'\}). \tag{13}$$

Since $v(b)-v(a)\geq v(b')-v(a')$, we have, for all $\tilde{\beta}\in(0,1)$ with $\tilde{\beta}\geq\beta$,

$$\begin{aligned} v(b\beta b')-v(a\beta a') &= \beta(v(b)-v(a))+(1-\beta)(v(b')-v(a')) \\ &\leq \tilde{\beta}(v(b)-v(a))+(1-\tilde{\beta})(v(b')-v(a'))=v(b\tilde{\beta}b')-v(a\tilde{\beta}a'). \end{aligned}$$

Let $\theta\equiv a\beta a'-a\tilde{\beta}a'\in\Theta_{(N,\mathbf{c})}$. Notice that

$$b\tilde{\beta}b'+\theta=(\eta\tilde{\beta}\eta')\alpha\nu+(\beta-\tilde{\beta})(a-a').$$

Since $(\eta\beta\eta')\alpha\nu$ is in the interior of $\Delta_{(N,\mathbf{c})}$, there exists a relative open interval $O(\beta)$ containing β such that $(\eta\tilde{\beta}\eta')\alpha\nu+(\beta-\tilde{\beta})(a-a')\in\Delta_{(N,\mathbf{c})}$ for all $\tilde{\beta}\in O(\beta)$. That is, for all $\tilde{\beta}\in O(\beta)$, θ is admissible for $b\tilde{\beta}b'$. Thus, by Step 1, we have (13).

Step 3: We claim that $u(a)-W(\{a,b\})\geq u(a')-W(\{a',b'\})$. Let $O(\beta)$ be an open interval containing $\beta\in[0,1]$ guaranteed by Step 2. Since $\{O(\beta)|\beta\in[0,1]\}$ is an open cover of $[0,1]$, there exists a finite subcover, denoted by $\{O(\beta^i)|i=1,\cdots,I\}$. Without loss of generality, assume $\beta^i\leq\beta^{i+1}$. Define $\beta^0=0$ and $\beta^{I+1}=1$. Since $\beta^0\in O(\beta^1)$ and $\beta^{I+1}\in O(\beta^I)$, from Step 2,

$$\begin{aligned} u(a')-W(\{a',b'\}) &\leq u(a\beta^1a')-W(\{a\beta^1a',b\beta^1b'\})\leq \\ \ldots &\leq u(a\beta^Ia')-W(\{a\beta^Ia',b\beta^Ib'\})=u(a)-W(\{a,b\}). \end{aligned}$$

From Step 3, for all $\alpha\in(0,1)$ sufficiently close to one,

$$u(\mu\alpha\nu)-W(\{\mu\alpha\nu,\eta\alpha\nu\})\geq u(\mu'\alpha\nu)-W(\{\mu'\alpha\nu,\eta'\alpha\nu\}).$$

Continuity ensures that $u(\mu)-W(\{\mu,\eta\})\geq u(\mu')-W(\{\mu',\eta'\})$ as $\alpha\to1$. ■

Lemma 12 *For all $\mu,\mu',\eta,\eta'\in\Delta$ such that $\{\mu\}\succ\{\mu,\eta\}\succ\{\eta\}$ and $\{\mu'\}\succ\{\mu',\eta'\}\succ\{\eta'\}$,*

$$v(\eta)-v(\mu)\geq v(\eta')-v(\mu')\ \Rightarrow\ u(\mu)-W(\{\mu,\eta\})\geq u(\mu')-W(\{\mu',\eta'\}).$$

Proof. Let μ^+ and μ^- be a maximal and a minimal lottery in Δ with respect to v. By continuity and mixture linearity of v, for all α sufficiently close to one, $v(\eta\alpha\mu^+)-v(\mu)>v(\eta'\alpha\mu^-)-v(\mu')$. Since the set of lotteries with finite supports is dense in Δ under the weak convergence topology (Aliprantis and Border [1, p.513, Theorem 15.10]), there exist sequences $\{\mu_n\}$, $\{\eta_n\}$, $\{\mu'_n\}$, and $\{\eta'_n\}$ with finite supports such that $\mu_n\to\mu$, $\eta_n\to\eta\alpha\mu^+$, $\mu'_n\to\mu'$, and $\eta'_n\to\eta'\alpha\mu^-$. Moreover, by continuity of W, $W(\{\mu_n\})>W(\{\mu_n,\eta_n\})>W(\{\eta_n\})$ and $W(\{\mu'_n\})>W(\{\mu'_n,\eta'_n\})>W(\{\eta'_n\})$, and, by continuity of v, $v(\eta_n)-v(\mu_n)>v(\eta'_n)-v(\mu'_n)$. By Lemma 11, for all n, we have $u(\mu_n)-W(\{\mu_n,\eta_n\})\geq u(\mu'_n)-W(\{\mu'_n,\eta'_n\})$. Continuity of u and W implies that $u(\mu)-W(\{\mu,\eta\alpha\mu^+\})\geq u(\mu')-W(\{\mu',\eta'\alpha\mu^-\})$ as $n\to\infty$. Again, by continuity, $u(\mu)-W(\{\mu,\eta\})\geq u(\mu')-W(\{\mu',\eta'\})$ as $\alpha\to1$. ■

Lemma 13 *$\varphi:A\to(0,1]$ is (i) well-defined, (ii) weakly increasing, and (iii) continuous.*

Proof. (i) Take any μ, μ', η, η' such that $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$ and $\{\mu'\} \succ \{\mu', \eta'\} \succ \{\eta'\}$. From Lemma 12, if $v(\eta) - v(\mu) = v(\eta') - v(\mu')$, $u(\mu) - W(\{\mu, \eta\}) = u(\mu') - W(\{\mu', \eta'\})$. Hence, φ is well-defined.

(ii) Take $w, w' \in A$ such that $w' < w$. There exist μ, μ', η, η' such that $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$, $\{\mu'\} \succ \{\mu', \eta'\} \succ \{\eta'\}$, $w = v(\eta) - v(\mu)$, and $w' = v(\eta') - v(\mu')$. By Lemma 12, $\varphi(w) = u(\mu) - W(\{\mu, \eta\}) \geq u(\mu') - W(\{\mu', \eta'\}) = \varphi(w')$.

(iii) Take any $w^0 \in A$. For any sequence $w^n \to w^0$, $n = 1, 2, \cdots$, we want to show that $\varphi(w^n) \to \varphi(w^0)$. First suppose $w^0 < \sup A$. Take any $w \in (w^0, \sup A)$. There exist μ, η such that $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$ and $w = v(\eta) - v(\mu)$. Since $w^n \to w$, $w^n < w$ for all sufficiently large n. Define $\alpha^n \equiv \frac{w^n}{w}$ for $n = 0$ and all sufficiently large n. By MPSC, $\{\mu\} \succ \{\mu, \eta\alpha^n\mu\} \succ \{\eta\alpha^n\mu\}$ and $w^n = v(\eta\alpha^n\mu) - v(\mu)$. By continuity of W,

$$\lim_{n\to\infty} \varphi(w^n) = \lim_{n\to\infty} u(\mu) - W(\{\mu, \eta\alpha^n\mu\}) = u(\mu) - W(\{\mu, \eta\alpha^0\mu\}) = \varphi(w^0).$$

Next suppose $w^0 = \sup A$. Since $w^0 \in A$, There exist μ, η such that $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$ and $w^0 = v(\eta) - v(\mu)$. Define $\alpha^n \equiv \frac{w^n}{w^0} \in (0, 1]$. By MPSC, $\{\mu\} \succ \{\mu, \eta\alpha^n\mu\} \succ \{\eta\alpha^n\mu\}$. Moreover, $w^n = v(\eta\alpha^n\mu) - v(\mu)$. By continuity of W,

$$\lim_{n\to\infty} \varphi(w^n) = \lim_{n\to\infty} u(\mu) - W(\{\mu, \eta\alpha^n\mu\}) = u(\mu) - W(\{\mu, \eta\}) = \varphi(w^0).$$

■

Denote the closure of A by $\overline{A}$. By Lemma 7 (i), $\overline{A}$ is a closed non-degenerate interval including 0. Let $\overline{w} = \sup A$. Define $\varphi(0) = \inf\{\varphi(w) \mid w \in A\}$ and $\varphi(\overline{w}) = \sup\{\varphi(w) \mid w \in A\}$.

Lemma 14 *$\varphi : \overline{A} \to [0, 1]$ is a unique continuous and weakly increasing extension of φ. Moreover, (i) $\varphi(0) = 0$, (ii) φ is weakly convex, and (iii) strictly increasing.*

Proof. Since φ is continuous and weakly increasing, the former statement holds.

(i) We show that $\varphi(0) = 0$. Take any $w \in A$. There exist μ, η such that $w = v(\eta) - v(\mu)$ and $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$. By MPSC, $\{\mu\} \succ \{\mu, \eta\alpha\mu\} \succ \{\eta\alpha\mu\}$ for all $\alpha \in (0, 1)$. Thus,

$$\varphi(0) = \lim_{\alpha\to 0} \varphi(\alpha w) = \lim_{\alpha\to 0} u(\mu) - W(\{\mu, \eta\alpha\mu\}) = 0.$$

(ii) We show that φ is convex on A. Then, by continuity, φ is convex on $\overline{A}$. Take any $w_i \in (0, \overline{w})$, $i = 1, 2$. Without loss of generality, assume $w_1 < w_2$. There exists $\mu, \eta_2 \in \Delta$ such that $\{\mu\} \succ \{\mu, \eta_2\} \succ \{\eta_2\}$ and $w_2 = v(\eta_2) - v(\mu)$. Let $\eta_1 = \eta_2 \frac{w_1}{w_2} \mu$. Then, $w_1 = v(\eta_1) - v(\mu)$. Moreover, by MPSC, $\{\mu\} \succ \{\mu, \eta_1\} \succ \{\eta_1\}$. Since v is mixture linear, $\alpha w_1 + (1-\alpha) w_2 = v(\eta_1\alpha\eta_2) - v(\mu)$ for all $\alpha \in (0, 1)$. By MPSC, $\{\mu\} \succ \{\mu, \eta_1\alpha\eta_2\} \succ \{\eta_1\alpha\eta_2\}$. In the proof of Lemma 1 (ii), we show that for all $x \in Z$, there exists $\nu \in \Delta$ such that $\{\nu\} \sim x$. Let $\nu_i \in \Delta$ satisfy $\{\nu_i\} \sim \{\mu, \eta_i\}$. By Self-Control Concavity, $\{\mu, \eta_1\alpha\eta_2\} \succsim \{\nu_1\alpha\nu_2\}$. Thus, we have

$$\begin{aligned} \varphi(\alpha w_1 + (1-\alpha) w_2) &= u(\mu) - W(\{\mu, \eta_1\alpha\eta_2\}) \\ &\leq u(\mu) - u(\nu_1\alpha\nu_2) = \alpha(u(\mu) - u(\nu_1)) + (1-\alpha)(u(\mu) - u(\nu_2)) \\ &= \alpha(u(\mu) - W(\{\mu, \eta_1\})) + (1-\alpha)(u(\mu) - W(\{\mu, \eta_2\})) \\ &= \alpha\varphi(w_1) + (1-\alpha)\varphi(w_2). \end{aligned}$$

(iii) First of all, since $\varphi(0) = 0$, $\varphi(0) < \varphi(w)$ for all $w \neq 0$. Next, take $w, w' \in \overline{A}$ such that $w' > w > 0$. There exists $\alpha \in (0,1)$ with $w = \alpha w'$. Since φ is convex,

$$\varphi(w) = \varphi(\alpha w') \leq \alpha\varphi(w') + (1-\alpha)\varphi(0) < \varphi(w'),$$

as desired. ■

Lemma 15 *Let* $\{\mu\} \succ \{\mu, \eta\} \sim \{\eta\}$*. If* $v(\eta) - v(\mu) \in A$*, then* $u(\eta) \geq u(\mu) - \varphi(v(\eta) - v(\mu))$*.*

Proof. There exist μ', η' such that $\{\mu'\} \succ \{\mu', \eta'\} \succ \{\eta'\}$ and $v(\eta') - v(\mu') = v(\eta) - v(\mu)$. Since $\varphi(v(\eta) - v(\mu)) = \varphi(v(\eta') - v(\mu')) = u(\mu') - W(\{\mu', \eta'\})$, it suffices to show that $u(\mu') - W(\{\mu', \eta'\}) \geq u(\mu) - u(\eta)$.

We will claim that $u(\mu') - u(\eta') > u(\mu) - u(\eta)$. Suppose otherwise, that is, $u(\mu) - u(\eta) \geq u(\mu') - u(\eta')$. Let

$$L \equiv \{\alpha \in [0,1] \,|\, \{\mu\alpha\mu'\} \succ \{\mu\alpha\mu', \eta\alpha\eta'\} \succ \{\eta\alpha\eta'\}\}.$$

By assumption, $0 \in L$ and $1 \notin L$. Moreover, by Continuity, L is open in $[0,1]$. Let $\bar{\alpha} \equiv \sup L \in (0,1]$. By Continuity, $\bar{\alpha} \notin L$, and hence

$$\{\mu\bar{\alpha}\mu'\} \succ \{\mu\bar{\alpha}\mu', \eta\bar{\alpha}\eta'\} \sim \{\eta\bar{\alpha}\eta'\}. \tag{14}$$

Since $u(\mu) - u(\eta) \geq u(\mu') - u(\eta') > \varphi(v(\eta') - v(\mu'))$ and $v(\eta') - v(\mu') = v(\eta) - v(\mu)$,

$$u(\mu\alpha\mu') - u(\eta\alpha\eta') > \varphi(v(\eta\alpha\eta') - v(\mu\alpha\mu')) \tag{15}$$

for all $\alpha \in [0,1]$. On the other hand, since $\bar{\alpha}$ is a supremum of L, there exists a sequence $\{\alpha^n\}$ in L converging to $\bar{\alpha}$. We have $\{\mu\alpha^n\mu'\} \succ \{\mu\alpha^n\mu', \eta\alpha^n\eta'\} \succ \{\eta\alpha^n\eta'\}$, and hence $u(\mu\alpha^n\mu') - u(\eta\alpha^n\eta') > \varphi(v(\eta\alpha^n\eta') - v(\mu\alpha^n\mu')) = u(\mu\alpha^n\mu') - W(\{\mu\alpha^n\mu', \eta\alpha^n\eta'\})$. Continuity and (15) imply $u(\mu\bar{\alpha}\mu') - u(\eta\bar{\alpha}\eta') > \varphi(v(\eta\bar{\alpha}\eta') - v(\mu\bar{\alpha}\mu')) = u(\mu\bar{\alpha}\mu') - W(\{\mu\bar{\alpha}\mu', \eta\bar{\alpha}\eta'\})$, that is, $W(\{\mu\bar{\alpha}\mu', \eta\bar{\alpha}\eta'\}) > u(\eta\bar{\alpha}\eta')$, which contradicts (14).

Since $v(\eta') - v(\mu') = v(\eta\alpha\eta') - v(\mu\alpha\mu')$ for all $\alpha \in L$, by Lemma 13 (i), $u(\mu') - W(\{\mu', \eta'\}) = u(\mu\alpha\mu') - W(\{\mu\alpha\mu', \eta\alpha\eta'\})$. Thus taking Continuity and the above claims together,

$$\begin{aligned} u(\mu') - W(\{\mu', \eta'\}) &= u(\mu\bar{\alpha}\mu') - W(\{\mu\bar{\alpha}\mu', \eta\bar{\alpha}\eta'\}) = u(\mu\bar{\alpha}\mu') - u(\eta\bar{\alpha}\eta') \\ &= \bar{\alpha}(u(\mu) - u(\eta)) + (1-\bar{\alpha})(u(\mu') - u(\eta')) \geq u(\mu) - u(\eta), \end{aligned}$$

as desired. ■

Let

$$B \equiv \{w \in [0,1] \,|\, w = v(\eta) - v(\mu) \text{ for some } \{\mu\} \succ \{\mu, \eta\}\}. \tag{16}$$

By Continuity, B is open in $[0,1]$. If $\{\mu\} \succ \{\mu, \eta\}$, then $\{\mu\} \succ \{\mu, \eta\alpha\mu\}$ for all $\alpha \in (0,1)$. Hence, B is an interval satisfying $\inf B = 0$. Moreover, by definition, $A \subset B$, or $\sup A \leq \sup B$.

Define $F : B \to \mathbb{R}_+$ by

$$F(w) \equiv \sup\{u(\mu) - u(\eta) \,|\, w = v(\eta) - v(\mu) \text{ for some } \{\mu\} \succ \{\mu, \eta\}\}. \tag{17}$$

Lemma 16 *F is weakly concave.*

Proof. Take $w_i \in B$, $i = 1, 2$, and $\alpha \in (0, 1)$. There exist $\mu_i^n, \eta_i^n \in \Delta$ such that $\{\mu_i^n\} \succ \{\mu_i^n, \eta_i^n\}$, $v(\eta_i^n) - v(\mu_i^n) = w_i$, and, $u(\mu_i^n) - u(\eta_i^n) \to F(w_i)$. Since $v(\eta_i^n) > v(\mu_i^n)$ and $u(\mu_i^n) > u(\eta_i^n)$, we have $v(\eta_1^n \alpha \eta_2^n) > v(\mu_1^n \alpha \mu_2^n)$ and $u(\mu_1^n \alpha \mu_2^n) > u(\eta_1^n \alpha \eta_2^n)$. Thus $\{\mu_1^n \alpha \mu_2^n\} \succ \{\mu_1^n \alpha \mu_2^n, \eta_1^n \alpha \eta_2^n\}$. Since

$$\alpha w_1 + (1-\alpha) w_2 = \alpha(v(\eta_1^n) - v(\mu_1^n)) + (1-\alpha)(v(\eta_2^n) - v(\mu_2^n)) = v(\eta_1^n \alpha \eta_2^n) - v(\mu_1^n \alpha \mu_2^n),$$

$$\begin{aligned} F(\alpha w_1 + (1-\alpha) w_2) &\geq \limsup u(\mu_1^n \alpha \mu_2^n) - u(\eta_1^n \alpha \eta_2^n) \\ &= \limsup \alpha(u(\mu_1^n) - u(\eta_1^n)) + (1-\alpha)(u(\mu_2^n) - u(\eta_2^n)) \\ &= \alpha F(w_1) + (1-\alpha) F(w_2). \end{aligned}$$

■

By Theorem 10.3 [16, p.85], F can be uniquely extended to the closure of B in a continuous and concave way.

Lemma 17 *(i) $F(w) > \varphi(w)$ for all $w \in A$. (ii) $F(\overline{w}) \geq \varphi(\overline{w})$. (iii) If $\overline{w} \notin A$, $F(\overline{w}) = \varphi(\overline{w})$.*

Proof. (i) There exist μ, η such that $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$ and $w = v(\eta) - v(\mu)$. By definition,

$$F(w) \geq u(\mu) - u(\eta) > u(\mu) - W(\{\mu, \eta\}) = \varphi(w).$$

(ii) By Lemma 13 (iii), φ is continuous. Moreover, since F is concave, F is continuous. For any sequence $w^n \to \overline{w}$, by part (i), $F(w^n) > \varphi(w^n)$. By continuity of F and φ, $F(\overline{w}) \geq \varphi(\overline{w})$.

(iii) Take any sequence $w^n \in A$ satisfying $w^n \to \overline{w}$. By part (ii), $F(\overline{w}) \geq \varphi(\overline{w})$. By contradiction, suppose $F(\overline{w}) > \varphi(\overline{w}) = \sup\{\varphi(w) | w \in A\}$. For all $w \in A$ and $\mu, \eta \in \Delta$ such that $w = v(\eta) - v(\mu)$ and $\{\mu\} \succ \{\mu, \eta\} \sim \{\eta\}$, by Lemma 15, we must have $\varphi(w) \geq u(\mu) - u(\eta)$. Thus there exist sequences $w^n \to \overline{w}$, $\{\mu^n\}_{n=1}^{\infty}$ and $\{\eta^n\}_{n=1}^{\infty}$ such that $w^n = v(\eta^n) - v(\mu^n) \in A$, $\{\mu^n\} \succ \{\mu^n, \eta^n\} \succ \{\eta^n\}$, and $u(\mu^n) - u(\eta^n) > c > \sup\{\varphi(w) | w \in A\}$, where $c > 0$ is a constant number. Since $\{\mu^n\}_{n=1}^{\infty}$ and $\{\eta^n\}_{n=1}^{\infty}$ are sequences in Δ, we can assume $\mu^n \to \mu^0$ and $\eta^n \to \eta^0$ without loss of generality. Since

$$u(\mu^n) - u(\eta^n) > c > \varphi(v(\eta^n) - v(\mu^n)) = u(\mu^n) - W(\{\mu^n, \eta^n\}),$$

continuity implies $u(\mu^0) - u(\eta^0) > u(\mu^0) - W(\{\mu^0, \eta^0\})$, that is, $W(\{\mu^0, \eta^0\}) > u(\eta^0)$. On the other hand, since $\overline{w} = v(\eta^0) - v(\mu^0) > 0$ and $u(\mu^0) > u(\eta^0)$, we have $\{\mu^0\} \succ \{\mu^0, \eta^0\}$. Hence $\{\mu^0\} \succ \{\mu^0, \eta^0\} \succ \{\eta^0\}$, which contradicts $\overline{w} \notin A$. ■

Since $v(\Delta) = [0, 1]$, $\max_x v - v(\mu) \in [0, 1]$ for all $x \in Z$ and $\mu \in x$. Now we define a function $\overline{\varphi} : [0, 1] \to \mathbb{R}_+$ as follows:

$$\overline{\varphi}(w) \equiv \begin{cases} \varphi(w) & \text{if } w \in [0, \overline{w}] \\ \frac{\varphi(\overline{w})}{\overline{w}} w & \text{if } w \in (\overline{w}, 1]. \end{cases} \tag{18}$$

Lemma 18 $\overline{\varphi}$ *is continuous, strictly increasing, and satisfies*

$$\overline{\varphi}(w)\begin{cases} = & \varphi(w) & \text{if } w\in\overline{A} \\ \geq & F(w) & \text{elsewhere.}\end{cases}$$

Proof. By Lemma 13, φ is continuous and strictly increasing on $[0,\overline{w}]$. Moreover, since $\frac{\varphi(\overline{w})}{\overline{w}}>0$, $\frac{\varphi(\overline{w})}{\overline{w}}w$ is continuous and increasing on $(\overline{w},1]$. Since $\overline{\varphi}(\overline{w})=\varphi(\overline{w})$, $\overline{\varphi}$ is continuous and strictly increasing on $[0,1]$.

If $\overline{w}\in A$, $\overline{w}=1$ by Lemma 7 (ii). Assume $\overline{w}\notin A$. Since F is concave, there exists a supporting affine function L at $(\overline{w},F(\overline{w}))$. That is, L satisfies that $L(w)\geq F(w)$ for all w and $L(\overline{w})=F(\overline{w})$. Since L is an affine function, $L(w)$ can be written as $aw+b$ for some $a,b\in\mathbb{R}$. If $b<0$, for small w, $L(w)<0$ and hence $\varphi(w)<F(w)\leq L(w)<0$, which is a contradiction. Thus, we must have $b\geq 0$. Since $F(\overline{w})=L(\overline{w})$,

$$\frac{F(\overline{w})}{\overline{w}}=a+\frac{b}{\overline{w}}\geq a.$$

Moreover, by Lemma, $\varphi(\overline{w})=F(\overline{w})$. Thus, we have $\frac{\varphi(\overline{w})}{\overline{w}}\geq a$. Therefore,

$$\frac{\varphi(\overline{w})}{\overline{w}}w-L(w)\begin{cases} = & 0 & \text{if } w=\overline{w} \\ \geq & 0 & \text{if } w>\overline{w} \\ \leq & 0 & \text{if } w<\overline{w}.\end{cases} \tag{19}$$

Now take any $w\in(\overline{w},1]$. By (19),

$$\overline{\varphi}(w)=\frac{\varphi(\overline{w})}{\overline{w}}w\geq L(w)\geq F(w),$$

as desired. ■

Lemma 19 *For all* $\mu,\eta\in\Delta$*,*

$$W(\{\mu,\eta\})=\max_{\nu\in\{\mu,\eta\}}\left\{u(\nu)-\overline{\varphi}\left(\max_{\{\mu,\eta\}}v-v(\nu)\right)\right\}.$$

Proof. Without loss of generality, assume $\{\mu\}\succsim\{\eta\}$. By Set Betweenness, $\{\mu\}\succsim\{\mu,\eta\}\succsim\{\eta\}$. There are four cases:

Case (i) $\{\mu\}\succ\{\mu,\eta\}\succ\{\eta\}$. In this case, $v(\eta)>v(\mu)$. By definition of φ, $W(\{\mu,\eta\})=u(\mu)-\varphi(v(\eta)-v(\mu))>u(\eta)$. Thus $W(\{\mu,\eta\})$ can be expressed as the desired form.

Case (ii) $\{\mu\}\succ\{\mu,\eta\}\sim\{\eta\}$: We have $v(\eta)>v(\mu)$. If $v(\eta)-v(\mu)\in A$, by Lemma 15, $W(\{\mu,\eta\})=u(\eta)\geq u(\mu)-\varphi(v(\eta)-v(\mu))$ as desired. If $v(\eta)-v(\mu)\notin A$, we have either $v(\eta)-v(\mu)=\sup A$ or $v(\eta)-v(\mu)\notin\overline{A}$. The former case implies $\varphi(v(\eta)-v(\mu))=F(v(\eta)-v(\mu))$ by Lemma 17 (iii). For the latter case, by Lemma 18, $\overline{\varphi}(v(\eta)-v(\mu))\geq F(v(\eta)-v(\mu))$. Thus, in each case,

$$\overline{\varphi}(v(\eta)-v(\mu))\geq F(v(\eta)-v(\mu))\geq u(\mu)-u(\eta).$$

Thus, $W(\{\mu,\eta\}) = u(\eta) \geq u(\mu) - \overline{\varphi}(v(\eta) - v(\mu))$.

Case (iii) $\{\mu\} \sim \{\mu,\eta\} \succ \{\eta\}$. By construction of v, $v(\mu) \geq v(\eta)$. Since $W(\{\mu,\eta\}) = u(\mu) > u(\eta) - \overline{\varphi}(v(\mu) - v(\eta))$, $W(\{\mu,\eta\})$ is represented by the desired form.

Case (iv) $\{\mu\} \sim \{\mu,\eta\} \sim \{\eta\}$. If $v(\eta) \geq v(\mu)$, $W(\{\mu,\eta\}) = u(\eta) \geq u(\mu) - \overline{\varphi}(v(\eta) - v(\mu))$. If $v(\mu) \geq v(\eta)$, we have $W(\{\mu,\eta\}) = u(\mu) \geq u(\eta) - \overline{\varphi}(v(\mu) - v(\eta))$. In either case, $W(\{\mu,\eta\})$ is represented by the desired form. ■

Finally, we can show that the representation extends to entire domain. The argument is similar that used in the conclusion of the proof [8, Thm 1]. Briefly, by GP [8, Lemma 2], Set-Betweenness implies that the representation extends to all finite menus. Then, given that the set of finite menus is dense in Z in the Hausdorff topology, the continuity of the representation permits the representation to extend to all menus. For a more detailed argument, see Lemmas 5 and 6 in the proof of Theorem 1.

C Appendix: Proof of Corollary 1

Let $(u, v, \overline{\varphi})$ be a representation constructed as in the proof of Theorem 3. If $\succsim$ satisfies Self-Control Linearity, we can show a counterpart of Lemma 14 (ii) as follows. A proof is omitted.

Lemma 20 *$\varphi : \overline{A} \to [0,1]$ satisfies $\varphi(\alpha w + (1-\alpha)w') = \alpha\varphi(w) + (1-\alpha)\varphi(w')$ for all $\alpha \in [0,1]$.*

Since $\varphi(0) = 0$, we have $\varphi(\lambda w) = \varphi(\lambda w + (1-\lambda)0) = \lambda\varphi(w)$ for all $w \in [0, \overline{w}]$ and $\lambda \in [0,1]$. Thus, φ is linear function on $[0, \overline{w}]$. Define $K \equiv \frac{\overline{\varphi}(\overline{w})}{\overline{w}}$. By Lemma 20, for all $w \in [0, \overline{w}]$,

$$\overline{\varphi}(w) = \overline{\varphi}\left(\frac{w}{\overline{w}}\overline{w}\right) = \overline{\varphi}(\overline{w})\frac{w}{\overline{w}} = Kw.$$

Moreover, by (18), $\overline{\varphi}(w) = Kw$ for all $w \in (\overline{w}, 1]$. That is, $\overline{\varphi}$ is written as a linear function with a positive slope K. Redefine v as Kv. Then, (u, v) is a GP representation.

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