Stochastic Approximation, Cooperative Dynamics and Supermodular Games*

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Abstract

This paper considers a stochastic approximation algorithm, with decreasing step size and martingale difference noise. Under very mild assumptions, we prove the non convergence of this process toward a certain class of repulsive sets for the associated ordinary differential equation (ODE). We then use this result to derive the convergence of the process when the ODE is *cooperative* in the sense of [Hirsch, 1985]. In particular, this allows us to extend significantly the main result of [Hofbauer and Sandholm, 2002] on the convergence of *stochastic fictitious play* in *supermodular games.*

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1 Introduction

Let $F : \mathbb{R}^d \to \mathbb{R}^d$ be a smooth vector field and $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space. We consider a $\mathbb{R}^d$-valued discrete time stochastic process $(x_n)_n$ whose general form can be written as the following recursive formula:

$$x_{n+1} - x_n = \frac{1}{n+1} \left(F(x_n) + U_{n+1} \right), \tag{1}$$

We assume that $(\Omega, \mathcal{F}, \mathbb{P})$ admits a filtration $(\mathcal{F}_n)_n$ such that x_0 is measurable with respect to $\mathcal{F}_0$, and $(U_n)_n$ is a $(\mathcal{F}_n)_n$-adapted sequence of random shocks (or perturbations). Throughout the paper, we make the following assumptions:

Hypothesis 1.1 *We assume that:*

(i) $(U_n)_n$ *is a martingale difference: for any* $n \in \mathbb{N}^*$,

$$\mathbb{E}\left(U_{n+1} \mid \mathcal{F}_n \right) = 0.$$

(ii) F *is Lipschitz continuous, with Lipschitz constant* L.

Such a stochastic approximation process is generally referred to as a Robbins-Monro algorithm (see [Robbins and Monro, 1951] or [Kiefer and Wolfowitz, 1952]). A natural approach to obtain information on the asymptotic behavior of the sample paths $(x_n(\omega))_n$ is to compare them to the trajectories of the ordinary differential equation

$$\dot{x} = F(x). \tag{2}$$

Indeed, one can interpret (1) as some kind of Cauchy-Euler approximation scheme for solving this ODE numerically, with a decreasing step size and an added noise. Since we assume that the noise has null expectation conditionally to the past, it is natural to expect that, for almost every $\omega \in \Omega$, the limit sets of the sample paths $(x_n(\omega))_n$ are related to the asymptotic behavior of the ODE solution curves. This approach was first introduced in [Ljung, 1977] and is usually referred to as the ODE method. Thereafter, the method has been studied and developed by many authors (including [Kushner and Clark, 1978], [Benveniste et al., 1990], [Duflo, 1996] or [Kushner and Yin, 2003]) for very simple dynamics (e.g. linear or gradient-like).

In a series of papers ([Benaïm and Hirsch, 1996] and [Benaïm, 1996] essentially), Benaïm and Hirsch proved that the asymptotic behavior of $(x_n)_n$ can be described with a great deal of generality through the study of the asymptotics of (2), regardless of the nature of F. In particular, under certain assumptions on the noise,

(a) the limit sets of $(x_n)_n$ are almost surely *internally chain recurrent* in the sense of Bowen and Conley (see [Bowen, 1975] and [Conley, 1978]). This result is detailled in section 2.1.

(b) the random process $(x_n)_n$ converges with positive probability to any given attractor of (2). See theorem 7.3 in [Benaïm, 1999] for a precise statement.

In addition, it was proved in [Pemantle, 1990] that, with probability one, $(x_n)_n$ does not converge to linearly unstable equilibria. Some additional references to non convergence results are given in section 3.

The motivation of this paper is threefold. First, under some additional assumptions on the noise, we prove the non convergence of $(x_n)_n$ toward a certain class of unstable sets (including linearly unstable equilibria, periodic orbits and normally hyperbolic sets), under less regularity assumptions than the existing results. This is detailled in section 3.

Secondly, in section 4, we use these results, combined with with the nature of limit sets (see point (a) above) and the structure of chain recurrent sets for cooperative dynamics (see [Hirsch, 1999]) to prove convergence of $(x_n)_n$ to the set of "stable" equilibria when F is cooperative and irreducible. This answers a question raised in [Benaïm, 2000].

Finally, these results are applied to prove the convergence of *stochastic fictitious play* in *supermodular games* in full generality. This proves a conjecture raised in [Hofbauer and Sandholm, 2002].

2 Background, Notation and Hypotheses

Let F denote a locally Lipschitz vector field on $\mathbb{R}^d$. By standard results, the Cauchy problem $\frac{dy}{dt} = F(y)$ with initial condition $y(0) = x$ admits a unique solution $t \to \Phi_t(x)$ defined on an open interval $J_x \subset \mathbb{R}$ containing the origin. For simplicity in the statement of our results we furthermore assume that F is *globally integrable*, meaning that $J_x = \mathbb{R}$ for all $x \in \mathbb{R}^d$. This holds in particular if F is sublinear; that is

$$\limsup_{||x||\to\infty} \frac{||F(x)||}{||x||} < \infty.$$

We let $\Phi = \{\Phi_t\}_{t\in\mathbb{R}}$ denote the flow induced by F.

A continuous map $\chi : \mathbb{R}_+ \to \mathbb{R}^d$ is called an *asymptotic pseudo trajectory* (APT) for Φ [Benaïm and Hirsch, 1996] if, for any $T > 0$,

$$\lim_{t\to+\infty} d_\chi(t, T) = 0,$$

where

$$d_\chi(t, T) = \sup_{h\in[0,T]} \|\chi(t + h) - \Phi_h(\chi(t))\|. \tag{3}$$

In other terms, for any $T > 0$, the curve joining $\chi(t)$ to $\chi(t+T)$ shadows the trajectory of the semiflow starting from $\chi(t)$ with arbitrary accuracy, provided t is large enough.

Remark 2.1 *Assume that Φ_1 restricted to $\chi(\mathbb{R}_+)$ is uniformly continuous. This holds in particular if χ or F are bounded maps. Then*

$$\lim_{t\to\infty} d_\chi(t, 1) = 0 \Leftrightarrow \forall\ T > 0, \lim_{t\to\infty} d_\chi(t, T) = 0.$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with some non decreasing sequence of σ-algebras $(\mathcal{F}_t)_{t\geq0}$. Throughout this paper we will consider an $(\mathcal{F}_t)_t$-adapted continuous time stochastic process $X = (X(t))_{t\geq0}$ verifying the following condition:

Hypothesis 2.1 There exists a map $\omega : \mathbb{R}_+^3 \to \mathbb{R}_+$ such that:

(i) For any $\delta > 0, T > 0$,

$$\mathbb{P}\left(\sup_{s\geq t} d_X(s, T) \geq \delta \mid \mathcal{F}_t\right) \leq \omega(t, \delta, T),$$

(ii) $\lim_{t\to\infty}\omega(t,\delta,T) = 0$.

A sufficient condition ensuring hypothesis 2.1 is that

$$\mathbb{P}\left(d_X(t,T) \geq \delta \mid \mathcal{F}_t\right) \leq \int_t^{t+T} r(s,\delta,T)ds \tag{4}$$

for some $r : \mathbb{R}^3 \mapsto \mathbb{R}_+$ such that

$$\int_0^{\infty} r(s,\delta,T)ds < \infty.$$

In this case

$$\omega(t,\delta,T) = \int_t^{\infty} r(s,\delta,T)ds.$$

The proof of the following proposition is obvious.

Proposition 2.2 *Under hypothesis 2.1, X is almost surely an asymptotic trajectory for Φ.*

Example 2.3 (Diffusion processes)
Let X be solution to the stochastic differential equation

$$dX(t) = F(X(t))dt + \sqrt{\gamma(t)}dB_t,$$

where F is a globally Lipschitz vector field, (B_t) a standard Brownian motion on $\mathbb{R}^d$ and $\gamma : \mathbb{R}_+ \mapsto \mathbb{R}_+$ a decreasing continuous function. Assume that

$$\int_0^{+\infty} \exp\left(\frac{-c}{\gamma(t)}\right) dt < +\infty$$

for all $c > 0$. Then (4) is satisfied with

$$r(t,\delta,T) = C\exp\left(-\frac{\delta^2 C(T)}{\gamma(t)}\right)$$

where C and $C(T)$ are positive constants. This is proved in ([Benaïm, 1999], Proposition 7.4)

Example 2.4 (Robbins-Monro algorithms) Let $(x_n)_n$ be a stochastic approximation algorithm governed by the recursive formula

$$x_{n+1} - x_n = \gamma_{n+1}\left(F(x_n) + U_{n+1}\right), \tag{5}$$

where $\gamma_n \geq 0$, $\sum_n \gamma_n = \infty$, and which satisfies Hypothesis 1.1. Assume furthermore that one of the two following conditions holds:

(i) There exists some $q \geq 2$ such that

$$\sum \gamma_n^{1+q/2} < +\infty \text{ and } \sup_n \mathbb{E}\left(\|U_n\|^q\right) < +\infty.$$

(ii) **(a)** The sequence $(U_n)_n$ is *subgaussian* (for instance bounded) meaning that

$$\mathbb{E}(\exp(\langle \theta, U_{n+1} \rangle))|\mathcal{F}_n) \leq \exp(\Gamma||\theta||^2)$$

for some $\Gamma > 0$; and

(b) for any $c > 0$,

$$\sum_n \exp\left(\frac{-c}{\gamma_n}\right) < +\infty.$$

Set $\tau_n := \sum_{i=1}^n \gamma_i$. We call X the continuous time affine interpolated process induced by $(x_n)_n$ and $\overline{\gamma}$ the piecewise constant deterministic process induced by $(\gamma_n)_n$:

$$X(\tau_i + s) := x_i + s\frac{x_{i+1} - x_i}{\gamma_{i+1}}, \text{ for } i \in \mathbb{N}, \ s \in [0, \gamma_{i+1}]$$

and

$$\overline{\gamma}(\tau_i + s) := \gamma_{i+1} \text{ for } s \in [0, \gamma_{i+1}[.$$

Under one of the above condition (i) or (ii), this continuous time process is an asymptotic pseudo trajectory of the flow induced by F (see [Benaïm, 1999]). Additionally, we have the following result (see [Benaïm, 1999] and more specifically [Benaïm, 2000]):

Proposition 2.5 *Let* $k_0 := \inf\left\{k \in \mathbb{N} \mid \ \gamma_k \leq \frac{B\delta^2}{2}\right\}$. *Then, for any* $s \geq \tau_{k_0}$, *condition (4) holds with*

$$r(s, \delta, T) = \frac{B\overline{\gamma}^{q/2}(s)}{\delta^q}$$

in the first case, and

$$r(s, \delta, T) = 2d \exp\left(\frac{-B\delta^2}{\overline{\gamma}(s)}\right)$$

in the second, where B is some positive constant depending only on the noise, the step size and the vector field.

2.1 The Limit set Theorem

A set $L \subset \mathbb{R}^d$ is said to be *invariant* (respectively *positively invariant*) for Φ provided $\Phi_t(L) \subset L$ for all $t \in \mathbb{R}$ (respectively $t \in \mathbb{R}_+$).

Let L be an invariant set for Φ. We let Φ^L denote the restriction of Φ to L. That is, $\Phi^L_t(x) = \Phi_t(x)$ for all $x \in L$ and $t \in \mathbb{R}$. Note that with such a notation $\Phi = \Phi^{\mathbb{R}^d}$.

An *attractor* for Φ^L is a nonempty compact invariant set $A \subset L$ having a neighborhood U in L such that

$$\lim_{t\to\infty} \operatorname{dist}(\Phi^L_t(x), A) = 0$$

uniformly in $x \in U$. Note that if L is compact, L is always an attractor for Φ^L. An attractor for Φ^L distinct from L is called a *proper attractor*.

The *basin of attraction* of A for Φ^L is the open set (in L) consisting of every $x \in L$ for which $\lim_{t\to\infty} \operatorname{dist}(\Phi_t(x), A) = 0$.

A *global attractor* for Φ is an attractor which basin is $\mathbb{R}^d$. If such an attractor exists, Φ (respectively F) is called a *dissipative* flow (respectively vector field).

A compact invariant set L is said to be *internally chain-transitive* or *attractor free* if Φ^L has no proper attractor (see e.g. [Conley, 1978]).

A fundamental property of asymptotic pseudo trajectories is given by the following result due to [Benaïm, 1996] for stochastic approximation processes and [Benaïm and Hirsch, 1996] for APT. We refer to [Benaïm, 1999] for a proof and more details; and also to [Pemantle, 2007] for a recent overview and some applications.

Theorem 2.6 *Let χ be a bounded APT, then its limit set*

$$\mathcal{L}(\chi) = \bigcap_{t\geq 0} \overline{\chi([t,\infty[)}$$

is internally chain transitive.

Corollary 2.7 *Under hypothesis 2.1, the limit set of X is almost surely internally chain transitive on the event $\{\sup_{t\geq 0} \|X(t)\| < \infty\}$.*

3 Non convergence toward normally hyperbolic repulsive sets

From Corollary 2.7 we know that the limit set of X is internally chain transitive (ICT). However not every ICT set can be such a limit set because the noise may push the process away from certain "unstable" sets. For equilibria this question has been tackled by several authors including [Pemantle, 1990], [Tarrès, 2001], [Brandiere and Duflo, 1996] and it was proved that, under natural conditions, X has zero probability to converge toward a linearly unstable equilibrium. This has been extended to linearly unstable periodic orbit by [Benaïm and Hirsch, 1995] and to more general normally hyperbolic sets by [Benaïm, 1999]). The proofs of all these results rely on the assumption that the unstable manifold of the set (to be defined below) is sufficiently smooth (at least $C^{1+\alpha}$ with $\alpha > 1/2$). While for linearly unstable equilibria or periodic orbit such a regularity assumption follows directly from the regularity of the vector field, the situation is much trickier for more general sets.

The purpose of this section is to extend the non convergence results mentioned above under less regularity assumptions. This will prove to be of fundamental importance in our analysis of cooperative dynamics and supermodular games in section 4.

Let S be a $\mathcal{C}^1$, $(d-k)$-dimensional ($k \in \{1, .., d\}$) submanifold of $\mathbb{R}^d$ and Γ a compact invariant set contained in S. We assume that S is *locally invariant* meaning that there exists a neighborhood U of Γ in $\mathbb{R}^d$ and a positive time t_0 such that

$$\Phi_t(U \cap S) \subset S$$

for all $|t| \leq t_0$. We let $\mathcal{G}(k, d)$ denote the Grassman manifold of k dimensional planes in $\mathbb{R}^d$. For $p \in S$, the tangent space of S in p is denoted T_pS.

Definition 3.1 Γ is called a *normally hyperbolic repulsive set* if there exists a continuous map

$$p \in \Gamma \mapsto E^u_p \in \mathcal{G}(k, d),$$

such that

(i) for any $p \in \Gamma$,

$$\mathbb{R}^d = T_p S \oplus E_p^u,$$

(ii) for any $t \in \mathbb{R}$ and any $p \in \Gamma$,

$$D\Phi_t(p) E_p^u = E_{\Phi_t(p)}^u,$$

(iii) there exists positive constants λ and C such that, for any $p \in \Gamma$, $w \in E_p^u$ and $t \geq 0$, we have

$$\|D\Phi_t(p) w\| \geq C e^{\lambda t} \|w\|.$$

The two basic examples of normally hyperbolic sets are the following. For more details, see ([Benaïm, 1999], Section 9).

Example 3.2 (Linearly unstable equilibrium): If $\Gamma = \{p\}$, where p is a linearly unstable equilibrium (not necessarily hyperbolic), then it is a normally hyperbolic repulsive set.

Example 3.3 (Hyperbolic linearly unstable periodic orbit): If Γ is a periodic orbit, the unity is always a Floquet multiplier. It is hyperbolic if the others multipliers all have moduli different from 1 and it is linearly unstable if at least one has modulus strictly greater than one. If both assumptions are checked then Γ is a normally hyperbolic repulsive set.

For further analysis, it is convenient to extend the map $p \to E_p^u$ to a neighborhood of Γ and to approximate it by a smooth map. More precisely it is shown in [Benaïm, 1999], Section 9.1 that there exists a neighborhood $\mathcal{N}_0 \subset U$ of Γ and a C^1 bundle

$$\tilde{E}^u = \{(p, v) \in S \cap \mathcal{N}_0 \times \mathbb{R}^d : \; v \in \tilde{E}_p^u\}$$

where $\tilde{E}_p^u \in \mathcal{G}(k, d)$ such that:

(i) For all $p \in S \cap \mathcal{N}_0$, $\mathbb{R}^d = T_p S \oplus \tilde{E}_p^u$;

(ii) the map $H : \tilde{E}^u \mapsto \mathbb{R}^d$ defined by $H(p,v) = p + v$ induces a C^1 diffeormorphism from a neighborhood of the zero section $\{(p,0) \in \tilde{E}^u\}$ onto $\mathcal{N}_0$.

Let now $V : \mathcal{N}_0 \mapsto \mathbb{R}_+$ be the map defined by $V(x) = ||v||$ for $H^{-1}(x) = (p,v)$. The form of V implies that there exits $L > 0$ such that

$$d(x,S) \leq V(x) \leq Ld(x,S) \tag{6}$$

for all $x \in \mathcal{N}_0$. Then according to Lemma 9.3 in [Benaïm, 1999] there exist a bounded neighborhood $\mathcal{N}_1 \subset \mathcal{N}_0$ of Γ, and numbers $T > 0$, $\rho > 1$ such that

$$\forall x \in \mathcal{N}_1,\ \ V(\Phi_T(x)) \geq \rho V(x). \tag{7}$$

Given a neighborhood $\mathcal{N} \subset U$ of Γ we let

$$\mathsf{Out}_\epsilon = \mathsf{Out}_\epsilon(\mathcal{N}, S) := \{x \in \mathcal{N} \mid d(x, S \cap \mathcal{N}) \geq \epsilon\}.$$

and

$$\mathsf{In}_\epsilon = \mathsf{In}_\epsilon(\mathcal{N}, S) := \mathcal{N} \setminus \mathsf{Out}_\epsilon.$$

Lemma 3.4 **(i)** *There exists a bounded neighborhood $\mathcal{N} \subset U$ of $\Gamma, T > 0$ and $\rho > 1$ such that for all $\epsilon > 0$,*

$$\Phi_T(\mathsf{Out}_\epsilon(\mathcal{N}, S)) \cap \mathcal{N} \subset \mathsf{Out}_{\rho\epsilon}(\mathcal{N}, S).$$

In particular, every compact invariant subset contained in $\mathcal{N}$ lies in S.

(ii) *For all $R > 0$ there exists a finite set $\{v_1, \ldots, v_n\} \subset \mathbb{R}^d$ and a Borel map $I : \Gamma \mapsto \{1, \ldots, n\}$ such that for all $p \in \Gamma$ and $v \in B(v_{I(p)}, 1)$,*

$$p + \epsilon v \in \mathsf{Out}_{R\epsilon}.$$

Proof. Choose $k \in \mathbb{N}$ such that $\rho^k > L$ and $\mathcal{N} \subset \mathcal{N}_1$ be small enough so that $\Phi_{kT}(\mathcal{N}) \subset \mathcal{N}_1$. Then, using (6) and (7) for all $x \in \mathcal{N}$,

$$d(\Phi_{kT}(x), S) \geq \frac{1}{L} V(\Phi_{kT}(x)) \geq \frac{\rho^k}{L} V(x) \geq \frac{\rho^k}{L} d(x,S).$$

Replacing T by kT and ρ by $\frac{\rho^k}{L}$ gives the result.

We now prove the second assertion. Given $R > 0$, let $f : \Gamma \mapsto \mathbb{R}^d$ be a measurable function such that for all $p \in \Gamma, f(p) \in \tilde{E}^u_p$ and $\|f(p)\| = L(R+2)$ where L is the constant appearing in (6). The bundle $\tilde{E}^u$ being locally trivial, it is not hard to construct such a function. By compactness of $\overline{f(\Gamma)}$, there exists a finite set $\{v_1, \ldots, v_n\} \subset f(\Gamma)$ such that $f(\Gamma) \subset \cup_{i=1}^n B(v_i, 1)$. For $p \in \Gamma$, set

$$I(p) = \min\{i = 1, \ldots, n : \|f(p) - v_i\| \leq 1\}.$$

Then, for $I(p) = i$ and $v \in B(v_i, 1)$,

$$d(p + \epsilon f(p), S) \leq d(p + \epsilon v, S) + \epsilon \|f(p) - v)\| \leq d(p + \epsilon v, S) + 2\epsilon.$$

On the other hand, by (6),

$$d(p + \epsilon f(p), S) \geq \frac{1}{L} V(p + \epsilon f(p)) = \frac{\epsilon \|f(p)\|}{L} = \epsilon(R+2).$$

Hence

$$d(p + \epsilon v, S) \geq R\epsilon.$$

■

Corollary 3.5 *Let $\mathcal{N}, T$ and ρ be like in Lemma 3.4, and set $\delta = (\rho - 1) > 0$. Let Y be an asymptotic pseudo-trajectory verifying*

(i) $\chi(0) \in \mathsf{Out}_\epsilon$,

(ii) *for all $t \geq 0, d_\chi(t, T) \leq \delta\epsilon$.*

Then χ eventually leaves $\mathcal{N}$.

Proof. Suppose that χ remains in $\mathcal{N}$. We claim that $\chi(kT) \in \mathsf{Out}_\epsilon$ for all $k \in \mathbb{N}$. If $\chi(kT) \in \mathsf{Out}_\epsilon$ then $\Phi_T(\chi(kT)) \in \mathsf{Out}_{\rho\epsilon}$ by Lemma 3.4. Hence $\chi(kT + T) \in \mathsf{Out}_\epsilon$ since $d_\chi(kT, T) \leq \delta\epsilon$. This proves the claim by induction on k. It follows that the limit set of χ meets In_ϵ but, by the limit set theorem 2.6 and Lemma 3.4, this limit set has to be in S. A contradiction. ■

3.1 Non convergence: sufficient conditions

Throughout this section we let $\mathcal{N}, T$ and ρ be like in Lemma 3.4, and $\delta = (\rho - 1) > 0$. We let X be a continuous time $(\mathcal{F}_t)$-adapted process verifying hypothesis 2.1 and E_t be the event

$$E_t = \{\forall s \geq t : \; X(s) \in \mathcal{N}\}.$$

Lemma 3.6 *On the event* $\{X(t) \in \mathsf{Out}_\epsilon\}$,

$$\mathbb{P}(E_t|\mathcal{F}_t) \leq \omega(t, \delta\epsilon, T)$$

and

$$\mathbb{P}(E_t|\mathcal{F}_t) \leq 1 - [1 - \omega(t+1, \delta\epsilon, T)]\mathbb{P}(X(t+1) \in \mathsf{Out}_\epsilon|\mathcal{F}_t).$$

Proof. The first inequality follows from Corollary 3.5. Now

$$\begin{aligned}
& \mathbb{P}(E_t|\mathcal{F}_t) \leq \mathbb{P}(E_{t+1}|\mathcal{F}_t) \\
= \; & \mathbb{P}(E_{t+1}; X(t+1) \in \mathsf{Out}_\epsilon|\mathcal{F}_t) + \mathbb{P}(E_{t+1}; X(t+1) \in \mathsf{In}_\epsilon|\mathcal{F}_t) \\
= \; & \mathbb{E}(\mathbb{P}(E_{t+1}|\mathcal{F}_{t+1})\mathbf{1}_{X(t+1)\in\mathsf{Out}_\epsilon}|\mathcal{F}_t) + \mathbb{E}(\mathbb{P}(E_{t+1}|\mathcal{F}_{t+1})\mathbf{1}_{X(t+1)\in\mathsf{In}_\epsilon}|\mathcal{F}_t) \\
\leq \; & \omega(t+1, \delta\epsilon, T)\mathbb{P}(X(t+1) \in \mathsf{Out}_\epsilon|\mathcal{F}_t) + \mathbb{P}(X(t+1) \in \mathsf{In}_\epsilon|\mathcal{F}_t).
\end{aligned}$$

■

Lemma 3.7 *Assume that there exists a maps* $\epsilon : \mathbb{R}_+ \mapsto \mathbb{R}_+$ *with* $\lim_{t\to\infty} \epsilon(t) = 0$ *and constants* $c > 0$ *and* $c' < 1$ *such that for* t *large enough*

(i) $\mathbb{P}(X(t+1) \in \mathsf{Out}_{\epsilon(t)}|\mathcal{F}_t) \geq c$ *on the event* $\{X(t) \in \mathsf{In}_{\epsilon(t)}\}$.

(ii)

$$\omega\left(t, \delta\epsilon(t), T\right) < c',$$

Then

$$\mathbb{P}(X(t) \to \Gamma) = 0.$$

Proof. One has

$$\{X(t) \to \Gamma\} \subset \bigcup_{n\in\mathbb{N}} E_n$$

and it suffices to prove that $\mathbb{P}(E_n) = 0$ for all $n \in \mathbb{N}$.

For all $t \geq n$, $E_n \subset E_t$. Thus

$$\mathbb{P}(E_n|\mathcal{F}_t) \leq \mathbb{P}(E_t|\mathcal{F}_t) \leq \max(c', 1-(1-c')c),$$

where the last inequality follows from the assumptions and Lemma 3.6. Now, by a classical Martingale result,

$$1 > \max(c', 1-(1-c')c) \geq \lim_{t\to\infty} \mathbb{P}(E_n|\mathcal{F}_t) \to \mathbf{1}_{E_n}$$

almost surely. Hence the result. ■

Hypothesis 3.8 *Assume that there exists a map* $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ *with* $\lim_{t\to\infty} \gamma(t) = 0$ *and an adapted process* $(Y(t))_{t\geq 0}$ *such that*

(i) *For all* $\epsilon > 0$,

$$\lim_{t\to\infty} \mathbb{P}\left(\left\| \frac{X(t+1) - \Phi_1(X(t))}{\sqrt{\gamma(t)}} - Y(t+1) \right\| \geq \epsilon | \mathcal{F}_t \right) = 0,$$

(ii) *For all open set* $O \subset \mathbb{R}^d$

$$\liminf_{t\to\infty} \mathbb{P}(Y(t+1) \in O | \mathcal{F}_t) > 0.$$

(iii) *There exists* $a > 0$ *such that*

$$\limsup_{t\to\infty} \omega(t, a\sqrt{\gamma(t)}, T) < 1.$$

Theorem 3.9 *Let* X *be a continuous* $(\mathcal{F}_t)$*-adapted process verifying hypotheses 2.1 and 3.8. Then*

$$\mathbb{P}(X(t) \to \Gamma) = 0.$$

Proof. We shall prove that the assumptions of Lemma 3.7 are fulfilled with $\epsilon(t) = \frac{\sqrt{\gamma(t)}}{\alpha}$; where $\alpha = \frac{\delta}{a}$ and a is given by hypothesis 3.8(iii). Condition (ii) of the lemma is clearly verified.

To check condition (i) we assume that $X(t) \in \mathsf{In}_{\epsilon(t)}$. Hence (for t large enough), $\Phi_1(X(t))$ lies in $\mathcal{N}_0 \subset \mathcal{N}$ and we can write

$$\Phi_1(X(t)) = p(t) + v(t)$$

with $(p(t), v(t)) \in \tilde{E}^u_{p(t)}$ (see the beginning of the section). Then, by the triangle inequality,

$$\begin{aligned} d(X(t+1), S) \geq\ & d(p(t) + \alpha\epsilon(t)Y(t+1), S) - \|v(t)\| \\ & -\alpha\epsilon(t)\|Y(t+1) - \tilde{Y}(t+1)\|. \end{aligned}$$

with

$$\tilde{Y}(t+1) = \frac{X(t+1) - \Phi_1(X(t))}{\alpha\epsilon(t)}.$$

Now

$$\|v(t)\| = V(\Phi_1(X(t))) \leq Ld(\Phi_1(X(t), S)) \leq M\epsilon(t)$$

where the first inequality follows from the Lipschitz continuity of the map V (see (6)), and the second from the Lipschitz continuity of Φ_1 and invariance of S. Thus

$$\frac{d(X(t+1), S)}{\epsilon(t)} \geq U_t - V_t - M$$

where

$$U_t = \frac{d(p(t) + \alpha\epsilon(t)Y(t+1), S)}{\epsilon(t)}$$

and

$$V_t = \alpha\|Y(t+1) - \tilde{Y}(t+1)\|.$$

Let $R = \frac{2+M}{\alpha}$. Then by lemma 3.4 (ii) and hypothesis 3.8 (ii), there exists $c > 0$ such that

$$\mathbb{P}(U_t \geq (1+M)|\mathcal{F}_t) = \mathbb{P}(p(t) + \alpha\epsilon(t)Y(t+1) \in \mathsf{Out}_{R\alpha\epsilon(t)}|\mathcal{F}_t) \geq 2c.$$

Furthermore, by Hypothesis 3.8,

$$\lim_{t\to\infty} \mathbb{P}(V_t \geq 1|\mathcal{F}_t) \leq c$$

for t large enough. It follows that

$$\mathbb{P}\left(\frac{d(X(t+1), S)}{\epsilon(t)} \geq 1|\mathcal{F}_t\right) \geq \mathbb{P}(U_t - V_t \geq M + 1|\mathcal{F}_t)$$

$$\geq \mathbb{P}(U_t \geq 2 + M|\mathcal{F}_t) - \mathbb{P}(V_t \geq 1|\mathcal{F}_t) \geq c.$$

This proves that condition (i) of the lemma is verified. ■

Proposition 3.10 *Let X be like in example 2.3. Set $l(t) = \log(\gamma(t))$. Assume that*

(i) *Function l is sub-additive: $l(t+s) \leq l(t) + l(s)$. This holds in particular if l is concave and $l(0) = 0$*

(ii) *There exist constants $a \geq b > 0$ such that $-a \leq \dot{l}(t) \leq -b$.*

Then hypothesis 3.8 holds. In particular, conclusions of Theorems 2.6 and 3.9 hold.

The proof is given in appendix.

We now apply these results to the specific case of Robbins-Monro algorithm. An additional assumption on the noise is needed:

Hypothesis 3.11 *There exists positive real values $0 < \Lambda^- < \Lambda^+ < +\infty$ and a continuous map*

$$Q : \mathbb{R}^d \to \mathcal{S}^+ \left(\mathbb{R}^d\right) \cap [\Lambda^- I_d, \Lambda^+ I_d],$$

such that $\mathbb{E}\left(U_{n+1}U_{n+1}^T \mid \mathcal{F}_n\right) = Q(x_n)$.

Proposition 3.12 *Let $(x_n)_n$ be a Robbins-Monro algorithm like in example 2.4 with $\gamma_n = 1/n$ and $\mathbb{E}\left(\|U_n\|^{2p} \mid \mathcal{F}_{n-1}\right)$ almost surely bounded for some $p > 1$, which noise also satisfies hypothesis 3.11. Then the associated interpolated process $X(t)_{t\geq 0}$ satisfies Hypothesis 3.8 and therefore,*

$$\mathbb{P}\left(X(t) \to \Gamma\right) = 0.$$

Proof. In appendix.

4 Application to cooperative dynamics

Throughout this section we assume that for all $x \in \mathbb{R}^d$ the Jacobian matrix $DF(x) = (\frac{\partial F_i}{\partial x_j}(x))$ has nonnegative off-diagonal entries and is irreducible. Such a vector field F is said to be *cooperative* and *irreducible* [Hirsch, 1985]. We refer the reader to [Hirsch and Smith, 2006] for a recent survey on the subject. We furthermore assume that F is *dissipative*, meaning that it admits a global attractor.

For $x, y \in \mathbb{R}^d$, $x \geq y$ means that $x_j \geq y_j$ for all j. If, additionally, $x \neq y$, we write $x > y$. If $x_j > y_j$ for all j, it is denoted $x \gg y$. Given two sets $A, B \subset \mathbb{R}^d$ we write $A \leq B$ provided $x \leq y$ for all $x \in A$ and $y \in B$. Set A is called *unordered* if for all $x, y \in A$, $x \leq y \Rightarrow x = y$.

The vector field F being cooperative and irreducible, its flow has positive derivatives [Hirsch, 1985], [Hirsch and Smith, 2006]. That is $D\Phi_t(x) \gg 0$ for $x \in \mathbb{R}^d$ and $t > 0$. This implies that it is *strongly monotonic* in the sense that $\phi_t(x) \gg \phi_t(y)$ for all $x > y$ and $t > 0$.

We let $\mathcal{E}$ denote the equilibria set of F. A Point $p \in \mathcal{E}$ is called *linearly unstable* if the Jacobian matrix $DF(p)$ has at least one eigenvalue with positive real part. We let $\mathcal{E}^+$ denote the set of such equilibria and $\mathcal{E}^- = \mathcal{E} \setminus \mathcal{E}^+$.

An equilibrium point $p \in \mathcal{E}$ is said to be *asymptotically stable from below* if there exists $x < p$ such that $\phi_t(x) \to p$. The subset of equilibria which satisfy this property is denoted $\mathcal{E}_{asb}$. Note that if $p \in \mathcal{E}_{asb}$, then there exists a non empty open set of initial conditions from which the solution trajectories converge to p. In particular $\mathcal{E}_{asb}$ is countable. Given $p \in \mathcal{E}_{asb}$, we introduce the set of points whose limit set dominates p:

$$V(p) := \{x \mid \omega(x) \geq p\}$$

and we let S_p denotes it boundary: $S_p := \partial V(p)$. The following proposition is basically due to ([Hirsch, 1988], Theorem 2.1) but for the C^1 regularity proved by [Terescak, 1996]. Our statement follows from Proposition 3.2 in [Benaïm, 2000], where more details can be found.

Proposition 4.1 *There exists a unique equilibrium $p^* \in \mathcal{E}_{asb}$ such that $V(p^*) = \mathbb{R}^d$. For any other $p \in \mathcal{E}_{asb} \setminus \{p^*\}$, S_p is a C^1 unordered invariant hypersurface diffeomorphic to $\mathbb{R}^{d-1}$.*

For $p \in \mathcal{E}_{asb} \setminus \{p^*\}$ we let $\mathcal{R}(\Phi^{S_p})$ denote the chain recurrent set of Φ restricted to S_p; or equivalently, the union of all internally chain transitive sets contained in S_p. We also set

$$\mathcal{R}'_p = \mathcal{R}(\Phi^{S_p}) \setminus \{\mathcal{E}^- \cap S_p\}.$$

The first part of the next Theorem is proved in [Benaïm, 2000] (see the proof of Theorem 2.1) and the second part restates Theorem 3.3 in the same paper (relying heavily on [Hirsch, 1999].

Theorem 4.2 *For any* $p \in \mathcal{E}_{asb} \setminus \{p^*\}$ *the set* $\mathcal{R}'_p$ *is a repulsive normally hyperbolic set (in the sense of section 3). Any internally chain transitive set is either an ordered arc included in* $\mathcal{E}^-$ *or is contained in* $\mathcal{R}'_p$ *for some* $p \in \mathcal{E}_{asb} \setminus \{p^*\}$.

Remark 4.3 *By a result of [Jiang, 1991], if* F *is real analytic, it cannot have a nondegenerate ordered arc of equilibria*

As a consequence of these results we get the following

Theorem 4.4 *Let* X *be a continuous* $(\mathcal{F}_t)$*-adapted stochastic process verifying hypotheses 2.1 and 3.8. Then the limit set of* X *is almost surely an ordered arc contained in* $\mathcal{E}^-$. *In case* F *is real analytic,* $X(t)$ *converges almost surely to an equilibrium* $p \in \mathcal{E}^-$.

Proof. Follows from Theorems 2.6, 4.2 and 3.9 ■

Corollary 4.5 *Let* X *be the process given in example 2.3 with* $-a \leq \frac{\dot{\gamma}(t)}{\gamma(t)} \leq -b$ *with* $a \geq b > 0$. *Then the conclusions of Theorem 4.4 hold.*

Corollary 4.6 *Let* (x_n) *be the Robbins Monro algorithm given in example 2.4 with* $\gamma_n = \frac{1}{n}$. *Assume that hypothesis 3.11 holds. Then the conclusions of Theorem 4.4 hold.*

5 Perturbed best response dynamic in supermodular games

5.1 General settings

Let us consider a N persons game in normal form. Player i's action set is finite and denoted A^i, Δ^i is the mixed strategies set:

$$\Delta^i := \left\{ x^i = (x^i(\alpha))_{\alpha \in A^i} \mid x^i(\alpha) \geq 0, \ \sum_{\alpha \in A^i} x^i(\alpha) = 1 \right\}$$

and $u^i : A^i \mapsto \mathbb{R}$ his utility function. The set of action profiles (respectively mixed strategy profiles) is denoted $A := \times_{i=1}^N A^i$ (resp.

$\Delta := \times_{i=1}^N \Delta^i$). The utility functions $(u^i)_{i=1,..,N}$ are defined on A but linearly extended to Δ:

$$x = (x^1, .., x^N) \in \Delta \mapsto u^i(x) := \sum_{a=(a^1,..,a^N)\in A} u^i(a) x^1(a^1)...x^N(a^N).$$

We call $G(N, A, u)$ the game induced by these parameters. Throughout our study, we assume that agents play repeatedly and independently. By this, we mean that, denoting $a_n = (a_n^1, .., a_n^N)$ the action profile realized at stage n and $(\mathcal{F}_n)_n$ an adapted filtration, we have

$$\mathbb{P}\left(a_{n+1} = (a^1, ..., a^N) \mid \mathcal{F}_n\right) = \prod_{i=1}^N \mathbb{P}\left(a_{n+1}^i = a^i \mid \mathcal{F}_n\right).$$

For $a = (a^1, .., a^N)$, δ_{a^i} denotes the vertex of Δ^i corresponding to the pure strategy profile a^i and δ_a is the extreme point of the polyhedron Δ relative to the pure strategy profile a. At last, $\overline{x}_n$ is the empirical distribution of moves up to time n :

$$\overline{x}_n := \frac{1}{n} \sum_{m=1}^n \delta_{a_m} = \left(\frac{1}{n} \sum_{m=1}^n \delta_{a_m^1}, .., \frac{1}{n} \sum_{m=1}^n \delta_{a_m^N} \right).$$

Standing Notation As usual in game theory we let $a^{-i} = (a^j)_{j\neq i}$, $x^{-i} = (x^j)_{j\neq i}, A^{-i} = \times_{j\neq i} A^j$ etc. We may write (a^i, a^{-i}) for $a = (a^1, \ldots, a^N)$ and so on.

5.2 Perturbed best response dynamic

To shorten notation let us take the point of view of player 1. A *choice function* for player 1 is a continuously differentiable map $C : \mathbb{R}^{A^1} \mapsto \Delta^1$.

Let $f : \mathbb{R}^{A^1} \mapsto \mathbb{R}^+$ be a strictly positive probability density and $\varepsilon \in \mathbb{R}^{A^1}$ a random variable having distribution $f(x)dx$. We say that C is a *good stochastic choice function* if it is induced by such a stochastic perturbation ε, in the following sense: for all $\Pi \in \mathbb{R}^{A^1}$ $C(\Pi)$ is the law of the random variable

$$\mathsf{argmax}_{\beta\in A^1} \left(\Pi(\beta) + \varepsilon(\beta)\right).$$

A classical example of good stochastic choice function is the Logit map:

$$L(\Pi)(\alpha) = \frac{\exp\left(\eta^{-1}\Pi(\alpha)\right)}{\sum_{\beta\in A^1}\exp\left(\eta^{-1}\Pi(\beta)\right)}.$$

It is induced by a stochastic perturbation with extreme value density (see [Fudenberg and Levine, 1998] and [Hofbauer and Sandholm, 2002]).

Given a choice function C, the *smooth* or *perturbed best response* associated to C is the map $\mathbf{br}^1 : \Delta^{-1} \mapsto \Delta^1$ defined by

$$\mathbf{br}^1(y) = C(u^1(\cdot\ , y)).$$

Definition 5.1 *Let* $\mathbf{br}^1$ *be a perturbed best response for player* 1. *A smooth fictitious play (SFP) strategy induced by* $\mathbf{br}^1$ *is a strategy such that, for any other opponent's strategy,*

$$\mathbb{P}(a^1_{n+1} = .\ |\ \mathcal{F}_n) = \mathbf{br}^1(\overline{x}^{-1}_n), \tag{8}$$

where $\overline{x}^{-1}_n$ *is the empirical moves of the opponents up to time* n.

Stochastic fictitious play was originally introduced by Fudenberg and Kreps (see [Fudenberg and Kreps, 1993]) and the concept behind is that players use fictitious play strategies in a game where payoff functions are perturbed by some random variables in the spirit of [Harsanyi, 1973]. To be more precise, suppose that at time $n+1$, the payoff function to player 1 is the map

$$u^1_{n+1} : A \mapsto \mathbb{R},$$
$$a \mapsto u^1(a) + \varepsilon_{n+1}(a^1),$$

where $\varepsilon_n \in \mathbb{R}^{A^1}$ is a random vector which conditional law, given $\mathcal{F}_n$ is $f(x)dx$. Suppose furthermore that u^1_{n+1} is known to player 1 as well as all the actions $a_1, \ldots, a_n$ played up to time n. Fictitious play assumes that player 1 chooses the best response to $\overline{x}^{-1}_n$. That is

$$a^1_{n+1} = \mathsf{argmax}_{\beta\in A^1} u^1_{n+1}(\beta, \overline{x}^{-1}_n).$$

Hence equation (8) holds where br^1 is the smooth best response associated to the good stochastic choice function induced by ε_{n+1}.[1]

[1] Another approach is to consider that the player chooses to randomize slightly its moves playing a best response relative to a payoff function perturbed by a deterministic map. Hofbauer and Sandholm (see [Hofbauer and Sandholm, 2002]) proved that any admissible stochastic perturbation can be represented in term of a deterministic perturbation.

On the subject, see also the papers [Fudenberg and Levine, 1995], [Fudenberg and Levine, 1998] or [Benaïm and Hirsch, 1999].

Let us get back to the settings described earlier with N players. We are interested in the asymptotic behavior of the sequence $\overline{x}_n$ when every player adopts a smooth fictitious play strategy. In the remaining of the section, an N-uple of perturbed best response maps is given and we let $\mathbf{br} : \Delta \mapsto \Delta$ denote the map defined by

$$\mathbf{br}(x) := \left(\mathbf{br^1}(x^{-1}), .., \mathbf{br^N}(x^{-N})\right).$$

The set of *perturbed Nash equilibria*, i.e. the set of $x \in \Delta$ such that $\mathbf{br}(x) = x$ (which can be viewed as the Nash equilibria in an auxiliary perturbed game) will be refered to as PNE. A simple computation gives

$$\overline{x}_{n+1} - \overline{x}_n = \frac{1}{n+1}\left(\delta_{a_n} - \overline{x}_n\right).$$

Hence, the expected increments satisfy:

$$\mathbb{E}\left(\overline{x}_{n+1} - \overline{x}_n \mid \mathcal{F}_n\right) = \frac{1}{n+1}\left(\mathbf{br}(\overline{x}_n) - \overline{x}_n\right).$$

The recursive formula describing the evolution of the random process $(\overline{x}_n)_n$ can then be written

$$\overline{x}_{n+1} = \overline{x}_n + \frac{1}{n+1}\left(F(\overline{x}_n) + U_{n+1}\right), \tag{9}$$

where

(i) the vector field F defined by $F(x) = \mathbf{br}(x) - x$ is smooth,

(ii) the noise U_{n+1} is a bounded martingale difference by construction and given by

$$U_{n+1} := \delta_{a_{n+1}} - \mathbf{br}(\overline{x}_n).$$

The associated ODE is the perturbed best response dynamic, given by

$$\dot{x} = \mathbf{br}(x) - x. \tag{10}$$

Note that the set of stationary points for this dynamic is exactly PNE, the set of perturbed equilibria. Since the vector field F is taking values in the tangent space relative to Δ, $T\Delta := \times T\Delta^i$ the trajectories remain in Δ. By an obvious abuse of language, we will say that a $m \times m$ matrix A is positive definite if, for any $\zeta \in T\Delta$, we have

$$\zeta \neq 0 \Rightarrow \zeta^T A \zeta > 0.$$

In the following, the set of matrices which are positive definite in this sense is denoted $\mathcal{S}^+(T\Delta)$.

Lemma 5.2 *Assume that for each i the choice function of player i takes values into the interior [2] of Δ^i. Then there exists positive values $0 < \Lambda^- < \Lambda^+ < +\infty$ and a continuous function $Q : \Delta \to \mathcal{S}^+(T\Delta) \cap [\Lambda^- I_d, \Lambda^+ I_d]$ such that*

$$\mathbb{E}\left(U_{n+1}U_{n+1}^T \mid \mathcal{F}_n\right) = Q(\overline{x}_n).$$

Proof. Let, for $x \in \Delta$ and $i \in \{1, \ldots, N\}$ $Q^i(x)$ denote the quadratic form on $T\Delta^i$ defined by

$$Q^i(x)(\zeta^i) = \sum_{\alpha \in A^i} \langle \delta_\alpha - br^i(x^{-i}), \zeta^i \rangle^2 br^i(x^{-i})_\alpha.$$

Equivalently, $Q^i(x)(\zeta^i)$ is the variance of $\alpha \mapsto \langle \delta_\alpha, \zeta^i \rangle$ under the law $br^i(x^{-i})$. Let $Q(x)$ denote the quadratic form on $T\Delta$ defined by

$$Q(x)(\zeta) = \sum_{i=1}^N Q^i(x)(\zeta^i).$$

Since $br^i(x^{-i})_\alpha > 0$ and $\{\delta_\alpha - br^i(x^{-i}) : \alpha \in A^i\}$ spans $T\Delta^i$, $Q^i(x)$ is non-degenerate for all i. Hence $Q(x)$ is nondegenerate, and by compactness and continuity, there exist $\Lambda^+ \geq \Lambda^- > 0$ such that

$$\Lambda^- \|\zeta\|^2 \leq Q(x)(\zeta) \leq \Lambda^+ \|\zeta\|^2, \ \forall \zeta \in T\Delta.$$

Now

$$\mathbb{E}\left(\langle U_{n+1}U_{n+1}^T \zeta, \zeta \rangle \mid \mathcal{F}_n\right) = Q(\overline{x}_n)(\zeta).$$

Hence the result. ■

Finally, the discrete stochastic approximation (9) is a first case Robbins Monro algorithm with $q = 2$, which satisfies hypothesis 3.11.

[2] Notice that this property is always satisfied for good stochastic choice functions

5.3 Properties of the best response dynamic for supermodular games

We assume here that for each $i = 1, \dots, N$ the action set A^i is equipped with a total ordering denoted $\leq$; and we focus our attention on games such that, for a given player, the reward he obtains by switching to a higher action increases when his opponents choose higher strategies. Such games are called *supermodular* and arise in many economic applications : see e.g. [Topkis, 1979] or [Milgrom and Roberts, 1990].

Definition 5.3 *We say that the game $G(N, A, u)$ is (strictly) supermodular if, for any pair of distinct players (i, j) and any action profiles $a = (a^1, .., a^N)$ and $b = (a^1, .., a^N)$ such that $a^i > b^i$ and $a^{-i} = b^{-i}$, the quantity $u^i(a) - u^i(b)$ is (strictly) increasing in $a^j = b^j$, for $j \neq i$.*

Remark 5.4 *In the particular case where each action set A^i is equal to the couple $\{0, 1\}$, the state space is the hypercube $[0, 1]^N$ and these games have been defined as* coordination games *in (Benaim and Hirch, 1999)*

In the remainder of this section we set $A^i = \{1, \dots, m^i\}$ and we assume that $\leq$ is the natural ordering on integers. For player i, we define the invertible linear operator T^i:

$$\Delta^i \to \mathbb{R}^{m^i-1}, \;\; (x^i_k)_{k=1,..,m^i} \mapsto ((T^i(x^i))_j)_{j=1,..,m^i-1}$$

with

$$(T^i(x^i))_j = \sum_{k=j+1}^{m^i} x^i_k.$$

Two mixed strategies can be compared via this operator and $T^i(x^i) \leq T^i(y^i)$ if and only if y^i stochastically dominates x^i. In the same spirit, two strategy profiles can be compared introducing the operator T :

$$\Delta \to \times_{i=1,..,N} \mathbb{R}^{m^i-1}, \;\; (x^1, .., x^N) \mapsto (T^1(x^1), .., T^N(x^N)).$$

Naturally, we say that $T(x) \leq T(y)$ if $T^i(x^i) \leq T^i(y^i)$ for $i = 1, .., N$ and the order relation relative to T denoted $\leq_T$ in the sequel. The following result is proved in [Hofbauer and Sandholm, 2002].

Theorem 5.5 (Hofbauer and Sandholm, 2002) *Assume that the game is strictly supermodular and that every agent plays a smooth fictitious play strategy induced by a good stochastic choice function. Then*

(i) for $i = 1, .., N$, $y^{-i} \geq_T x^{-i} \Rightarrow \mathbf{br}^i(y^{-i}) \geq_T \mathbf{br}^i(x^{-i})$.

(ii) The conjugate dynamic[3] *is cooperative and irreducible. Hence, it is strongly monotone. In particular, if* $(\mathbf{x}(t))_{t\geq 0}$ *and* $(\mathbf{y}(t))_{t\geq 0}$ *solve (10) with* $\mathbf{x}(0) \leq_T \mathbf{y}(0)$ *(and* $\mathbf{x}(0) \neq \mathbf{y}(0)$*) then, for any* $t \geq 0$, $\mathbf{x}(t) \leq_T \mathbf{y}(t)$,

(iii) There exists two perturbed equilibria $\underline{x} \leq_T \overline{x}$ *such that any chain recurrent set relative to the perturbed best response dynamic is included into the interval* $[\underline{x}, \overline{x}]$,

Hofbauer and Sandholm then used this theorem combined with results from [Benaïm, 2000] to describe the limit set of stochastic fictious plays for supermodular game. In view of the new results obtained in this paper and specifically in section 4 we are are now able to improve notably their results and to prove the convergence of stochastic fictious play for supermodular games in full generality.

Theorem 5.6 *Assume that the assumptions of previous theorem are satisfied. Then the limit set of* $(\overline{x}_n)_n$ *is almost surely an ordered arc of PNE that are not linearly unstable. If we furthermore assume that the choice function is real analytic (for instance in the logit case), then* $(\overline{x}_n)_n$ *almost surely converges toward a non linearly unstable PNE.*

Proof. By Lemma 5.2 and Theorem 5.5, the conditions to apply Corollary 4.6 are met. ■

[3] we refer to the dynamic induced by the conjugation relation T, defined on $\left\{(v^1, ..., v^N) \in \times_{i=1}^N \mathbb{R}^{m^i-1} \mid 1 \geq v_1^i \geq ... \geq v_{m^i-1}^i \geq 0 \ \forall i\right\}$ and given by $\dot{v} = T\left(br(T^{-1}(v))\right) - v$.

6 Appendix

6.1 Proof of Proposition 3.10

The assumptions on γ easily imply that

$$\frac{\gamma(t)}{\gamma(s+t)} \geq \frac{1}{\gamma(s)} \geq e^{bs}.$$

Thus

$$\omega(t, a\sqrt{\gamma(t)}, T) \leq C \int_0^\infty \exp\left(-a^2 e^{bs} C(T)\right)$$

and condition (iii) of hypothesis 3.8holds. Let

$$A_s^t = [DF(\Phi_s(X_t)) - \frac{1}{2}\frac{\dot{\gamma}(t+s)}{\gamma(t+s)}]$$

and let $\{Y_s^t, s \geq 0\}$ be solution to

$$dY_s^t = A_s^t Y_s^t + dB_{t+s}$$

with initial condition $Y_0^t = 0$. Condition (i) of Hypothesis 3.8 follows from the following lemma.

Lemma 6.1

$$\lim_{t\to\infty} \mathbb{P}(\sup_{0\leq s\leq 1} \|Y_s^t - \frac{X_{t+s} - \Phi_s(X_t)}{\sqrt{\gamma(t+s)}}\| \geq \epsilon | \mathcal{F}_t) = 0.$$

In particular, Hypothesis 3.8 (i) holds with $Y(t) = Y_1^{t-1}$ for all $t \geq 1$.

Proof. Set $\alpha(s) = 1/\sqrt{\gamma(s)}$, $Z_s^t = X_{t+s} - \Phi_s(X_t)$ and $\hat{Y}_s^t = \alpha(t+s)Z_s^t$. Then

$$dZ_s^t = (F(X_{t+s}) - F(\Phi_s(X_t)))ds + \sqrt{\gamma(t+s)}dB_{t+s}$$

$$= [DF(\Phi_s(X_t))Z_s^t + o(\|Z_s^t\|)]ds + \sqrt{\gamma(t+s)}dB_{t+s}.$$

Hence

$$d\hat{Y}_s^t = [DF(\Phi_s(X_t)) + \frac{\dot{\alpha}(t+s)}{\alpha(t+s)}]\hat{Y}_s^t + dB_{t+s} + \alpha(t+s)o(\|Z_s^t\|),$$

where $o(z) = z\eta(z)$ and $\lim_{z\to 0}\eta(z) = \eta(0) = 0$. Then

$$Y^t_s - \hat{Y}^t_s = \int_0^s A^t_u(Y^t_u - \hat{Y}^t_u)du + \int_0^s \alpha(t+u)o(\|Z^t_u\|)du.$$

Thus, by Gronwall's inequality,

$$\sup_{0\le s\le 1}\|Y^t_s - \hat{Y}^t_s\| \le e^K R_t$$

with

$$R_t = \sup_{0\le s\le 1}\alpha(t+s)o(\|Z^t_s\|)$$

and

$$K = \sup_{s,t}\|A^t_s\| \le \|DF\| + \frac{a}{2}. \qquad (11)$$

To conclude the proof it remains to show that

$$\mathbb{P}(R_t \ge \delta|\mathcal{F}_t) \to 0$$

as $t \to \infty$.

It follows from the estimate given in example 2.3 that

$$\mathbb{P}(\sup_{0\le s\le 1}\|Z^t_s\| \ge \delta|\mathcal{F}_t) \le \int_t^{t+1} C\exp(\frac{-\delta^2 C(1)}{\gamma(s)})ds \le C\exp(-\frac{\delta^2 C(1)}{\gamma(t+1)})$$

Thus

$$\begin{aligned}\mathbb{P}(\sup_{0\le s\le 1}\alpha(t+s)\|Z^t_s\| \ge R|\mathcal{F}_t) &\le \mathbb{P}(\|Z^t_s\| \ge \frac{R}{\alpha(t+1)}|\mathcal{F}_t)) \\ &\le C\exp(-R^2C(1)).\end{aligned}$$

Now,

$$\mathbb{P}(\sup_{0\le s\le 1}\alpha(t+s)\|Z^t_s\|\eta(\|Z^t_s\|) \ge \delta|\mathcal{F}_t)$$

$$\le \mathbb{P}(\sup_{0\le s\le 1}\alpha(t+s)\|Z^t_s\| \ge R|\mathcal{F}_t) + \mathbb{P}(\sup_{0\le s\le 1}\eta(\|Z^t_s\|) \ge \frac{\delta}{R}|\mathcal{F}_t).$$

$$\le C\exp(-R^2C(1)) + \mathbb{P}(\sup_{0\le s\le 1}\eta(\|Z^t_s\|) \ge \frac{\delta}{R}|\mathcal{F}_t).$$

Since $\lim_{z\to 0}\eta(z) = 0$,

$$\limsup_{t\to\infty}\mathbb{P}(\sup_{0\le s\le 1}\alpha(t+s)\|Z^t_s\|\eta(\|Z^t_s\|) \ge \delta|\mathcal{F}_t) \le C\exp(-R^2C(1))$$

and since R is arbitrary, this proves the result. ■

It remains to prove that condition (ii) of hypothesis 3.8 holds.

Lemma 6.2 *Let Σ be a $n \times n$ self-adjoint positive definite matrix and*

$$f_\Sigma(x) = \frac{\exp\left(-\frac{1}{2}\langle \Sigma^{-1}x, x\rangle\right)}{\sqrt{\det(\Sigma)(2\pi)^n}}$$

the density of a centered Gaussian vector with covariance Σ. Let $0 < \alpha \leq \beta$ respectively denote the smallest and largest eigenvalues of Σ. Then

$$f_\Sigma(x) \geq (\frac{\alpha}{\beta})^{n/2} f_{\alpha Id}(x).$$

Proof. Follows from the estimates $\det(\Sigma) \leq \beta^n$ and $\langle \Sigma^{-1}x, x\rangle \leq \frac{\|x\|^2}{\alpha}$. ■

Since Y_s^t is a linear function of $\{B_{t+u},\ 0 \leq u \leq s\}$, it is a Gaussian vector under the conditional probability $\mathbb{P}(\cdot|\mathcal{F}_t)$. By Ito's formulae, its covariance matrix is solution to

$$\frac{d\Sigma_s^t}{ds} = A_s^t \Sigma_s^t + \Sigma_s^t A_s^{t*} + Id$$

with initial condition $\Sigma_0^t = 0$; where A_s^{t*} stands for the transpose of A_s^t. It is then easy to check that

$$\Sigma_s^t = \int_0^s U^t(u) U^{t*}(u) du$$

where $U^t(s)$ is the solution to

$$\frac{dU}{ds} = A_s^t U,\ U(0) = Id. \tag{12}$$

Using (12) we see that $U^t(s)$ is invertible and that its inverse $(U^t(s))^{-1}$ solves

$$\frac{dV}{ds} = -VA_s^t,\ V(0) = Id.$$

Using again (12) combined with the estimate (11) and Gronwall's lemma, we get

$$\|U^t(s)\| \leq e^{Ks}.$$

Similarly

$$\|(U^t(s))^{-1}\| \leq e^{Ks}.$$

It follows that for all vector h,

$$e^{-Ks}\|h\| \leq \|U^t(s)h\| \leq e^{Ks}\|h\|.$$

Hence

$$a\|h\|^2 \leq \langle \Sigma_1^t h, h\rangle \leq b\|h\|^2,$$

where $a = \int_0^1 e^{-2Ku}du$ and $b = \int_0^1 e^{2Ku}du$. The result then follows from Lemma (6.2). ■

6.2 Proof of Proposition 3.12

Recall that $(\mathcal{F}_n)_n$ is a given filtration to which the stochastic process $(x_n)_n$ is adapted. Let $m_n := \sup\{k \in \mathbb{N} \mid \tau_k \leq n\}$ and call $(\mathcal{G}_n)_n$ the sigma algebra $(\mathcal{F}_{m_n})_n$. Let $n \geq 1$ and $k_n := m_{n+1} - m_n$. We denote by t_j^n the quantity $\tau_{m_n+j} - \tau_{m_n}$ $(j = 0, .., k_n)$ and $t_n := t_{k_n}^n$. Notice that $|t_n - 1| \leq \gamma_{m_n}$.
For the continuous time interpolated process induced by a discrete process $(x_n)_n$, hypothesis 3.8 is satisfied if there exists a vanishing positive sequence $(\gamma(n))_n$ and a $\mathcal{G}_n$-adapted random sequence $(Y_n)_n$ such that

(i) for any $\alpha > 0$,

$$\lim_{n\to+\infty} \mathbb{P}\left(\left\|\frac{x_{m_{n+1}} - \Phi_{t_n}(x_{m_n})}{\sqrt{\gamma(n)}} - Y_{n+1}\right\| > \alpha \mid \mathcal{G}_n\right) = 0,$$

(ii) for any open set $O \subset \mathbb{R}^d$, there exists a positive number δ such that

$$\liminf_{n\to+\infty} \mathbb{P}\left(Y_{n+1} \in O \mid \mathcal{G}_n\right) > \delta \quad \text{almost surely.}$$

(iii) there exists $a > 0$ such that

$$\limsup_{n\to+\infty} \omega(n, a\sqrt{\gamma(n)}, T) < 1.$$

Let $\gamma(n) := \sum_{k=1}^{k_n} \gamma_{m_n+k}^2$. First, by proposition 2.5, the map ω corresponding to the process $(x_n)_n$ is given by

$$\omega(n,\delta,T) = \frac{B\int_n^{+\infty}\overline{\gamma}(u)du}{\delta^2}.$$

Hence,

$$\omega(n, a\sqrt{\gamma(n)}, T) \le \frac{B}{a^2}\frac{\sum_{m_n}^{+\infty}\gamma_i^2}{\sum_{m_n+1}^{m_{n+1}}\gamma_i^2}.$$

Since

$$\limsup_n \frac{\sum_{m_n}^{+\infty}\gamma_i^2}{\sum_{m_n+1}^{m_{n+1}}\gamma_i^2} < +\infty,$$

the quantity $\omega(n, a\sqrt{\gamma(n)}, T)$ is smaller than 1, for a large enough. The next lemma corresponds to Lemma 6.1.

Lemma 6.3 *Point (i) is satisfied for this choice of $(\gamma(n))_n$ and the random sequence $(Y_n)_n$ given by*

$$\frac{1}{\sqrt{\gamma(n-1)}}\sum_{j=1}^{k_{n-1}}\gamma_{m_{n-1}+j}\left(\prod_{k=j+1}^{k_{n-1}}\left(I_d+\gamma_{m_{n-1}+k}DF(\phi_{t_{k-1}^{n-1}}(x_{m_{n-1}}))\right)\right)U_{m_{n-1}+j}.$$

Proof. Set $\hat{Y}_{n+1} := \frac{x_{m_{n+1}}-\phi_{t_n}(x_{m_n})}{\sqrt{\gamma(n)}}$. We have, for $j = 0, .., k_n - 1$,

$$\phi_{t_{j+1}^n}(x_{m_n}) - \phi_{t_j^n}(x_{m_n}) = \gamma_{m_n+j+1}F\left(\phi_{t_j^n}(x_{m_n})\right) + \mathcal{O}(\gamma_{m_n+j}^2).$$

Then, denoting

$$\hat{Y}_j^n := \frac{1}{\sqrt{\gamma(n)}}\left(x_{m_n+j} - \phi_{t_j^n}(x_{m_n})\right) (j = 0, .., k_n),$$

we have

$$\begin{aligned}\hat{Y}_{j+1}^n - \hat{Y}_j^n &= \frac{\gamma_{m_n+j+1}}{\sqrt{\gamma(n)}}\left[F(x_{m_n+j}) - F\left(\phi_{t_j^n}(x_{m_n})\right) + U_{m_n+j+1}\right] \\ &\quad +\mathcal{O}\left(\frac{\gamma_{m_n+j+1}^2}{\sqrt{\gamma(n)}}\right).\end{aligned}$$

Consequently,

$$\begin{aligned}\hat{Y}^n_{j+1} - \hat{Y}^n_j &= \gamma_{m_n+j+1}\left(DF\left(\phi_{t^n_j}(x_{m_n})\right)\hat{Y}^n_j + \frac{R^n(j)}{\sqrt{\gamma(n)}} + \frac{U_{m_n+j+1}}{\sqrt{\gamma(n)}}\right) \\ &\quad + \mathcal{O}\left(\frac{\gamma^2_{m_n+j+1}}{\sqrt{\gamma(n)}}\right) \quad j = 0,..,k_n-1,\end{aligned}$$

where

$$R^n(j) := F(x_{m_n+j}) - F(\phi_{t^n_j}(x_{m_n})) - DF\left(\phi_{t^n_j}(x_{m_n})\right)\cdot\left(x_{m_n+j} - \phi_{t^n_j}(x_{m_n})\right).$$

By a recursive argument,

$$\begin{aligned}&\hat{Y}_{n+1} - Y_{n+1} = \hat{Y}^n_{k_n} - Y_{n+1} = \\ &\frac{1}{\sqrt{\gamma(n)}}\sum_{j=1}^{k_n}\gamma_{m_n+j}\left(\prod_{k=j+1}^{k_n}\left(I_d + \gamma_{m_n+k}DF(\phi_{t^n_{k-1}}(x_{m_n}))\right)\right)R^n(j) \\ &+\mathcal{O}\left(e^{-n/2}\right).\end{aligned}$$

since $\hat{Y}^n_0 = 0$ and $\sum_{j=0}^{k_n-1}\frac{\gamma_{m_n+j+1}}{\sqrt{\gamma(n)}} = \sqrt{\gamma(n)} = \mathcal{O}(e^{-n/2})$.

Recall that $\sum_{j=1}^{k_n}\gamma_{m_n+j} \leq 1 + \gamma_{m_{n+1}}$ and DF is bounded. Consequently, there exists a real number K such that for n large enough,

$$\begin{aligned}&\frac{1}{\sqrt{\gamma(n)}}\left\|\sum_{j=1}^{k_n}\gamma_{m_n+j}\left(\prod_{k=j+1}^{k_n}\left(I_d + \gamma_{m_n+k}DF(\phi_{t^n_{k-1}}(x_{m_n}))\right)\right)R^n(j)\right\| \\ &\leq e^K\frac{1}{\sqrt{\gamma(n)}}\sup_{j=1,..,k_n}R^n(j) = e^K R_n,\end{aligned}$$

where $R_n := \frac{1}{\sqrt{\gamma(n)}}\sup_{j=1,..,k_n}R^n(j)$. By an application of results due to Benaim (see [Benaïm, 1999], proposition 4.1, formula (11) and identity (13) with $q = 2$), we have

$$\mathbb{E}\left(\sup_{j=0,..,k_n-1}\|x_{m_n+j} - \phi_{t^n_j}(x_{m_n})\|^2 \mid \mathcal{G}_n\right) \leq C\gamma(n).$$

where C is some positive constant. Additionally, by definition of DF,

$$R^n(j)^2 \leq h\left(\|x_{m_n+j} - \phi_{t^n_j}(x_{m_n})\|^2\right),$$

for some function $h : \mathbb{R}_+^* \to \mathbb{R}_+^*$, strictly increasing and such that $h(x)/x \to_{x\to 0^+} 0^+$. An immediate consequence is that

$$\begin{aligned}
& \mathbb{P}\left(R_n \geq \alpha \mid \mathcal{G}_n\right) \\
\leq\ & \mathbb{P}\left(\sup_{j=0,..,k_n-1} h\left(\|x_{m_n+j} - \phi_{t_j^n}(x_{m_n})\|^2\right) \geq \alpha^2\gamma(n) \mid \mathcal{G}_n\right) \\
\leq\ & \mathbb{P}\left(\sup_{j=0,..,k_n-1} \|x_{m_n+j} - \phi_{t_j^n}(x_{m_n})\|^2 \geq h^{-1}\left(\alpha^2\gamma(n)\right) \mid \mathcal{G}_n\right) \\
\leq\ & \frac{C\gamma(n)}{h^{-1}\left(\alpha^2\gamma(n)\right)} \to_{n\to+\infty} 0,
\end{aligned}$$

which proves the result. ■

To simplify notations, we call E the euclidian space $\mathbb{R}^d$. Given $n \in \mathbb{N}$, the random variable x_n can be written $h_n(U_1, .., U_n)$, where $h_n : (E^n, (\mathcal{B}_E)^n) \to (E, \mathcal{B}_E)$ is a measurable function. We denote by $\mathcal{P}_U$ the probability distribution induced by the measurable process $U = (U_n)_n : (\Omega, \mathcal{F}) \to (E^{\mathbb{N}}, (\mathcal{B}_E)^{\mathbb{N}})$. We keep the notation $\mathcal{F}_n$ for the sigma field $(\mathcal{B}_E)^n \times E^{\mathbb{N}}$ when it does not imply any ambiguity.

Proposition 6.4 *There exists a function $P_n : (\mathcal{B}_E)^{\mathbb{N}} \times E^{\mathbb{N}} \to [0,1]$ called a* regular conditional distribution of U given $\mathcal{F}_n$ *in the sense that, for any $u \in E^{\mathbb{N}}$, $P_n(\cdot, u)$ is a probability measure on $((\mathbb{R}^d)^{\mathbb{N}}, (\mathcal{B}_E)^{\mathbb{N}})$ and that, for any $B \in (\mathcal{B}_E)^{\mathbb{N}})$, the random variable $P_n(B, \cdot)$ is $\mathcal{F}_n$-measurable with*

$$\mathbb{P}_n(B, \cdot) = \mathbb{P}_U(B \mid \mathcal{F}_n)(\cdot)\ \mathbb{P}_U - \textit{almost surely.}$$

For convenience, given $u \in E^{\mathbb{N}}$, we denote by $\mathbb{P}_n^u$ the probability measure $\mathbb{P}_n(\cdot, u)$ and $\mathbb{E}_n^u$ the corresponding expectation. Given a measurable function $y : (E^{\mathbb{N}}, (\mathcal{B}_E)^{\mathbb{N}}) \to (E, \mathcal{B}_E)$, we have

$$\mathbb{E}_n^{\omega}(y) = \mathbb{E}_U(y \mid \mathcal{F}_n) = \mathbb{E}(y(U) \mid \mathcal{F}_n)\ \mathbb{P}_U - \text{a.s.}.$$

Lemma 6.5 *Let $k < i$ be two natural numbers and $y : (E^{\mathbb{N}}, (\mathcal{B}_E)^{\mathbb{N}}) \to (E, \mathcal{B}_E)$ be a measurable function. There exists a subset $\Omega_0(y) \subset E^{\mathbb{N}}$ such that $\mathbb{P}_U(\Omega_0(y)) = 1$ and, for any $u_0 \in \Omega_0(y)$, $\mathbb{E}_k^{u_0}(y \mid \mathcal{F}_i)$ and $\mathbb{E}_U(y \mid \mathcal{F}_i)$ are $\mathbb{P}_U$-almost surely equal.*

Proof. The random variable $z := \mathbb{E}_U(y \mid \mathcal{F}_i)$ is $\mathcal{F}_i$-measurable. Pick a countable π-class $\mathcal{D}$ such that $\sigma(\mathcal{D}) = \mathcal{F}_k$. Given $A \in \mathcal{D}$, there exists a set $\Omega_0(y, A)$ such that $\mathbb{P}_U(\Omega_0(y, A)) = 1$ and, for any $u_0 \in \Omega_0(y, A)$, we have

(1) $\mathbb{E}_k^{u_0}\left(\mathbb{E}(\mathbb{I}_A y \mid \mathcal{F}_i)\right) = \mathbb{E}_U\left(\mathbb{E}(\mathbb{I}_A y \mid \mathcal{F}_i) \mid \mathcal{F}_k\right)(u_0)$,

(2) $\mathbb{E}_k^{u_0}\left(\mathbb{I}_A y\right) = \mathbb{E}_U\left(\mathbb{I}_A y \mid \mathcal{F}_k\right)(u_0)$.

(3) $\mathbb{I}_A \mathbb{E}_U(y \mid \mathcal{F}_i) = \mathbb{E}_U(\mathbb{I}_A y \mid \mathcal{F}_i)$ $\mathbb{P}_k^{u_0}$-a.s.

Let us construct $\Omega_0(y, A)$. First, there exist two sets $\Omega_0^1(y, A)$ and $\Omega_0^2(y, A)$ on which respectively points (1) and (2) are satisfied and such that $\mathbb{P}_U(\Omega_0^j(y, A)) = 1$, $j = 1, 2$. Now for the last point, one must first consider a set $\Omega^3(y, A)$ such that $\mathbb{P}_U(\Omega^3(y, A)) = 1$ and, for any $u \in \Omega^3(y, A)$,

$$\mathbb{I}_A(u)\mathbb{E}_U(y \mid \mathcal{F}_i)(u) = \mathbb{E}_U(\mathbb{I}_A y \mid \mathcal{F}_i)(u)$$

Then, by definition of $\mathbb{P}_k^{u_0}$, there exists a set $\Omega_0^3(y, A)$ (which depends on $\Omega^3(y, A)$) such that, $\mathbb{P}_U(\Omega_0^3(y, A)) = 1$ and, for any $u_0 \in \Omega_0^3(y, A)$,

$$\mathbb{P}_k^{u_0}(\Omega^3(y, A)) = \mathbb{P}_U(\Omega^3(y, A) \mid \mathcal{F}_k)(u_0) = 1.$$

Finally, pick $\Omega_0(y, A) := \Omega_0^1(y, A) \cap \Omega_0^2(y, A) \cap \Omega_0^3(y, A)$.

Now take

$$\Omega_0(y) := \cap_{A \in \mathcal{D}} \Omega(y, A).$$

By countability of $\mathcal{D}$, we have $\mathbb{P}_U(\Omega_0(y)) = 1$. There remains to prove that, for any $u_0 \in \Omega_0(y)$,

$$\int_A z \, d\mathbb{P}_k^{u_0} = \int_A y \, d\mathbb{P}_k^{u_0}, \text{ for any } A \in \mathcal{D}.$$

$$\begin{aligned}
\mathbb{E}_k^{u_0}(\mathbb{I}_A z) &= \mathbb{E}_k^{u_0}\left(\mathbb{I}_A \mathbb{E}_U(y \mid \mathcal{F}_i)\right) \\
&= \mathbb{E}_k^{u_0}\left(\mathbb{E}_U(\mathbb{I}_A y \mid \mathcal{F}_i)\right) \\
&= \mathbb{E}_U\left(\mathbb{E}_U(\mathbb{I}_A y \mid \mathcal{F}_i) \mid \mathcal{F}_k\right)(u_0) \\
&= \mathbb{E}_U\left(\mathbb{I}_A y \mid \mathcal{F}_k\right)(u_0) \\
&= \mathbb{E}_k^{u_0}\left(\mathbb{I}_A y\right).
\end{aligned}$$

The second equality follows from point (3), the third from point (1) and the fifth from point (2). The lemma is proved. ■

The following result is due to [Hall and Heyde, 1980] (see Theorem 3.4 or Theorem 2 page 351 in [Chow and Teicher, 1998] for a version adapted to our situation). It is a central limit result for double arrays. We apply it to prove point (ii).

Theorem 6.6 (Hall and Heyde) *For any $n \geq 1$, let k_n be a positive integer and $(\Omega_n, \mathcal{F}^n, \mathbb{P}_n)$ a probability space. Consider $\mathcal{F}_1^n \subset \mathcal{F}_2^n \subset \ldots \subset \mathcal{F}_{k_n}^n \subset \mathcal{F}^n$ an increasing family of sigma fields and $(y_j^n)_{j=1,..,k_n}$ a $(\mathcal{F}_j^n)_{j=1,..,k_n}$-adapted family of random variables. Assume that*

* *for $j = 1, .., k_n$,*
$$\mathbb{E}_n\left(y_j^n \mid \mathcal{F}_{j-1}^n\right) = 0,$$
* *we have*
$$\sum_{j=1}^{k_n} \mathbb{E}_n\left(\|Y_j^n\|^2 \mathbb{I}_{\|Y_j^n\| > \varepsilon} \mid \mathcal{F}_{j-1}^n\right) \xrightarrow[n \to +\infty]{dist.} 0,$$
* *there exists a positive, $\mathcal{F}_1^n$-adapted random sequence $(w_n)_n$ such that*
$$\sum_{i=1}^{k_n} \mathbb{E}_n\left(y_j^n (y_j^n)^T \mid \mathcal{F}_{j-1}^n\right) - w_n \xrightarrow[n \to +\infty]{dist.} 0,$$
* *there exists a positive random matrix η, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which satisfies*
$$\sum_{j=1}^{k_n} \mathbb{E}_n\left(y_j^n (y_j^n)^T \mid \mathcal{F}_{j-1}^n\right) \xrightarrow[n \to +\infty]{dist.} \eta.$$

Then, denoting $y_{n+1} := \sum_{j=1}^{k_n} y_j^n$, the sequence $(y_n)_n$ converges in distribution to some random variable y defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and whose characteristic function is given by $\mathbb{E}\left(e^{-\frac{1}{2}t^T \eta t}\right)$. In particular,

$$\lim_{n \to +\infty} \mathbb{E}_n\left(e^{i<t, y_{n+1}>}\right) = \mathbb{E}\left(e^{-\frac{1}{2}t^T \eta t}\right).$$

Let us get back to our settings. Let $n \in \mathbb{N}$ and $j \in \{1, ..k_n\}$. Consider the measurable functions $y_j^n : (E^{\mathbb{N}}, (\mathcal{B}_E)^{\mathbb{N}}) \to (E, \mathcal{B}_E)$, given by

$$y_j^n(u) := \frac{\gamma_{m_n+j}}{\sqrt{\gamma(n)}} \left(\prod_{k=j+1}^{k_n} \left(I_d + \gamma_{m_n+k} DF(\phi_{t_{k-1}^n}(x_{m_n})) \right) \right) u_{m_n+j},$$

where $x_n = h_n(u_1, .., u_n)$. Finally, call $y_n := \sum_{j=1}^{k_n} y_j^n$

Corollary 6.7 *Given a nonempty open set O in E, there exist $\delta > 0$ and a set Ω_0 such that $\mathbb{P}_U(\Omega_0) = 1$ and, for any $u_0 \in \Omega_0$,*

$$\liminf_n \mathbb{P}_U \left(y_{n+1} \in O \mid \mathcal{G}_n \right)(u_0) > \delta.$$

Proof. Let Ω_0 be the set

$$\bigcap_{n\in\mathbb{N}, j=1,..,k_n, r\in\mathbb{Q}} \Omega_0 \left(y_j^n,\ \|y_j^n\|^2 \mathbb{I}_{\|y_j^n\|>r}),\ y_j^n (y_j^n)^T,\ \mathbb{I}_{\|x_{m_n+j} - \Phi_{t_j^n}(x_{m_n})\|>r}, \mathbb{I}_{y_n \in O} \right).$$

By countability, $\mathbb{P}(\Omega_0) = 1$. Pick $u_0 \in \Omega_0$. We apply Theorem 6.6 to $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n) := (E^{\mathbb{N}}, (\mathcal{B}_E)^{\mathbb{N}}, \mathbb{P}_{m_n}^{u_0})$, $\mathcal{F}_j^n = \mathcal{F}_{m_n+j}$ and the double array of random variables $(y_j^n)_{n,j}$.

We now verify that the assumptions required to apply Theorem 6.6 hold. First of all

$$\mathbb{E}_{m_n}^{u_0} \left(y_j^n \mid \mathcal{F}_{j-1}^n \right) = \mathbb{E}_U (y_j^n \mid \mathcal{F}_{j-1}^n) = 0 \text{ a.s.}$$

Secondly, let

$$\Pi_{n,j} := \prod_{k=j+1}^{k_n} \left(I_d + \gamma_{m_n+k} DF \left(\phi_{t_{k-1}^n}(x_{m_n}) \right) \right).$$

A simple computation gives

$$e^{-2\|DF\|_\infty} \leq \|\Pi_{n,j}\| \leq e^{\|DF\|_\infty}.$$

Recall that there exists $p > 1$ such that the sequence of random variables $(\mathbb{E}_U(\|u_n\|^{2p} \mid \mathcal{F}_{n-1}))_n$ is almost surely bounded. Hence, taking q such that $1/p + 1/q = 1$ and choosing $r \in \mathbb{Q}$,

$$\begin{aligned}
\mathbb{E}_U\left(\|y_j^n\|^2 \mathbb{I}_{\|y_j^n\|>r} \mid \mathcal{F}_{j-1}^n\right) &\leq \mathbb{E}_U\left(\|y_j^n\|^{2p} \mid \mathcal{F}_{j-1}^n\right)^{1/p} \mathbb{P}_U\left(\|y_j^n\|^{2p} > r^{2p} \mid \mathcal{F}_{j-1}^n\right)^{1/q} \\
&\leq \frac{1}{r^{2p/q}} \mathbb{E}_U\left(\|y_j^n\|^{2p} \mid \mathcal{F}_{j-1}^n\right) \\
&\leq \frac{1}{r^{2p/q}} \frac{\gamma_{m_n+j}^{2p}}{\gamma(n)} e^{\|DF\|_\infty} \mathbb{E}_U\left(\|u_{m_n+j}\|^{2p} \mid \mathcal{F}_{j-1}^n\right) \\
&\leq C(r) \frac{\gamma_{m_n+j}^{2p}}{\gamma(n)} \mathbb{E}_U\left(\|u_{m_n+j}\|^{2p} \mid \mathcal{F}_{j-1}^n\right).
\end{aligned}$$

Consequently,

$$\sum_{j=1}^{k_n} \mathbb{E}_U\left(\|y_j^n\|^2 \mathbb{I}_{\|y_j^n\|>r} \mid \mathcal{F}_{j-1}^n\right) \leq C(r) \sup_j \gamma_{m_n+j}^{2(p-1)} \sup_j \mathbb{E}_U\left(\|u_{m_n+j}\|^{2p} \mid \mathcal{F}_{j-1}^n\right),$$

which converges to 0 almost surely. Since u_0 belongs to the set $\Omega_0(\|y_j^n\|^2 \mathbb{I}_{\|y_j^n\|>r})$, for any $j = 1, ..k_n$,

$$\sum_{j=1}^{k_n} \mathbb{E}_{m_n}^{u_0}\left(\|y_j^n\|^2 \mathbb{I}_{\|y_j^n\|>r} \mid \mathcal{F}_{j-1}^n\right) = \sum_{j=1}^{k_n} \mathbb{E}_U\left(\|y_j^n\|^2 \mathbb{I}_{\|y_j^n\|>r} \mid \mathcal{F}_{j-1}^n\right) \quad \mathbb{P}_U\text{−a.s.}$$

and the second point holds.
From now on, we call

$$W_n := \sum_{j=1}^{k_n} \mathbb{E}_U\left(\left(y_j^n\right)\left(y_j^n\right)^T \mid \mathcal{F}_{n,j-1}\right).$$

We have

$$\begin{aligned}
&\mathbb{E}_U\left(\left(y_j^n\right)\left(y_j^n\right)^T \mid \mathcal{F}_{n,j-1}\right) \\
= \; & \frac{1}{\gamma(n)} \gamma_{m_n+j}^2 \Pi_{n,j} \mathbb{E}_U\left(u_{m_n+j} u_{m_n+j}^T \mid \mathcal{F}_{n,j-1}\right) \Pi_{n,j}^T \\
= \; & \frac{1}{\gamma(n)} \gamma_{m_n+j}^2 \Pi_{n,j} Q(x_{m_n+j-1}) \Pi_{n,j}^T.
\end{aligned}$$

Consequently,

$$W_n = \frac{1}{\gamma(n)} \sum_{j=1}^{k_n} \gamma^2_{m_n+j} \Pi_{n,j} Q(x_{m_n+j-1}) \Pi^T_{n,j}.$$

Let w_n be the $\mathcal{F}_{n,1}$-measurable random variable defined by

$$w_n := \frac{1}{\gamma(n)} \sum_{j=1}^{k_n} \gamma^2_{m_n+j} \Pi_{n,j} Q \left(\phi_{t^n_{j-1}}(x_{m_n}) \right) \Pi^T_{n,j}.$$

Pick $r \in \mathbb{Q}$. By definition of Ω_0 and assumption 2.1 (i),

$$\begin{aligned}
&\mathbb{P}^{u_0}_{m_n} \left(\sup_{j=1,..k_n} \left\| \phi_{t^n_{j-1}}(x_{m_n}) - x_{m_n+j-1} \right\| > r \right) \\
&= \mathbb{P} \left(\sup_{j=1,..k_n} \left\| \phi_{t^n_{j-1}}(x_{m_n}) - x_{m_n+j-1} \right\| > r \mid \mathcal{G}_n \right) \\
&\leq \omega(n, r, 1) \to 0,
\end{aligned}$$

which implies that

$$W_n - w_n \xrightarrow[n \to +\infty]{dist} 0$$

Since the application Q takes values in $[\Lambda^- I_d, \Lambda^+ I_d]$ and $\|\Pi_{n,j}\|$ is bounded above and away from zero, we have

$$0 < a^- \leq \Pi_{n,j} Q(x_{m_n+j-1}) \Pi^T_{n,j} \leq a^+ < +\infty.$$

W_n is a convex combination of such quantities, therefore is bounded. Pick some increasing sequence of integers $(n_k)_k$. $(W_{n_k})_k$ admits a subsequence $(W_{n'_k})_k$ which converges in distribution to some random variable η^{u_0}, defined on the probability space induced by U and which takes values in $\mathcal{S}^+(\mathbb{R}^d) \cap [a^- I_d, a^+ I_d]$.
Now by Theorem 6.6 ,

$$y_{n'_k} \xrightarrow[n \to +\infty]{\mathcal{L}} y^{u_0},$$

with $\mathbb{E}_U(e^{i<t,y^{u_0}>}) = \mathbb{E}\left(e^{-\frac{1}{2} t^T \eta^{u_0} t} \right)$. In particular, by definition of Ω_0,

$$\lim_k \mathbb{P}_U \left(y_{n'_k+1} \in O \mid \mathcal{G}_{n'_k}) \right) (u_0) = \lim_k \mathbb{P}^{u_0}_{n'_k}(y_{n'_k+1} \in O) = \mathbb{P}(y^{u_0} \in O) > \delta,$$

where δ depends on the parameters a and b but not on $u_0 \in \Omega_0$ and $(n'_k)_k$. The proof is complete. ■

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