Dynamic Legislative Policy Making

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Abstract

We prove existence of stationary Markov perfect equilibria in an infinite-horizon model of legislative policy making in which the policy outcome in one period determines the status quo in the next. We allow for a multidimensional policy space and arbitrary smooth stage utilities. We prove that all such equilibria are essentially in pure strategies and that proposal strategies are differentiable almost everywhere. We establish upper hemicontinuity of the equilibrium correspondence, and we derive conditions under which each equilibrium of our model determines a unique invariant distribution characterizing long run policy outcomes. We provide a long run core convergence theorem giving conditions under which the invariant distributions generated by stationary equilibria must be close to the core in the canonical spatial model.

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1 Introduction

Political interaction in modern democracies counts among the most complex phenomena subjected to scientific inquiry, and practical considerations dictate that we attempt to accommodate this complexity in formal political modeling. Doing so would appear essential, for example, for the detailed analysis of the effects of public policy and the design of constitutions. In this spirit, we study policy making within a legislative body or, more generally, a government in which policy initiatives are systematically subjected to review by political actors with authority to enact policy. Our goal is to develop a model of policy making that (i) accounts for the multidimensional aspects of public policy and idiosyncratic details of policy preferences, (ii) captures the ongoing nature of policy making, and (iii) allows for the kinds of random shocks (e.g., on preferences and the environment) to which political interaction is subjected over time. We provide a benchmark model that satisfies these desiderata and allows us to confront the central theoretical difficulties arising in applications. The model is intentionally austere, in that we do not incorporate the rich spectrum of political institutions observed in the real world, but our approach is very general: we conclude with a discussion of how our results extend to an institutionally detailed version of the model. And although we are motivated by the application to legislatures and democratic politics, the issues we address are fundamental and would arise in a host of dynamic bargaining contexts, such as wage negotiation in labor markets, or collusion among members of a cartel, or deliberations among a board of directors, or treaty talks among states.

We lay the theoretical groundwork for future applications by establishing the existence of equilibria satisfying a number of desirable regularity properties in a class of models satisfying the objectives (i)–(iii) identified above. We consider a fully dynamic model of legislative policy making in which each period begins with a status quo policy and the random draw of a legislator, who proposes any feasible policy, which is then subject to an up or down vote. The policy outcome in that period is the proposed policy if it receives the support of a “decisive” coalition of legislators, the status quo otherwise, and the status quo in the next period is determined by the outcome that prevails in the current period. This process continues ad infinitum. Thus, a path of play in our model is an infinite sequence of policies over time, and in equilibrium legislators must anticipate future policy consequences of their decisions. In particular, when voting on a proposal, a legislator must compare the distribution over policy streams generated by the proposed policy with the distribution generated by the status quo. And legislators must select their proposed policies optimally in light of the future, while factoring in whether a proposed policy will garner the support of a decisive coalition. We deduce the existence of stationary equilibria in pure strategies, and moreover, we show that all stationary equilibria are essentially pure. In fact, equilibria are strict in the sense that proposers almost always have unique optimal
proposals, an essential property for the computational tractability of the model.\footnote{The role of pure strategies is especially important for the computational tractability of dynamic games, as highlighted by Herings and Peeters (2004) and Nowak (2007) and emphasized by Doraszelski and Satterthwaite (2007) in the context of a dynamic oligopoly model. See Theorem 5 of Duggan and Kalandrakis (2007) for more on the computational tractability of the model, and see Duggan et al. (2008) and Duggan and Kalandrakis (2008) for applications to US politics.} In equilibrium, continuation values of the legislators are differentiable and proposal strategies are continuous almost everywhere. We prove a general result on upper hemi-continuity of the equilibrium correspondence with respect to the parameters of the model, including the policy space itself as a variable. We give conditions under which each equilibrium admits a unique invariant distribution with desirable ergodic properties, providing an unambiguous prediction of long run policy outcomes generated by the equilibrium. Finally, we specialize to the multidimensional spatial model in which legislative preferences are close to admitting a core policy, and we provide conditions under which an asymptotic core convergence theorem holds.

We do not impose specific assumptions about the policy space or functional forms for legislators’ utility functions. Instead, we allow the set of alternatives to be a very general subset of any finite-dimensional Euclidean space defined by arbitrary smooth feasibility constraints, and we assume smooth stage utility functions but do not impose any further restrictions on preferences. Thus, we capture standard models with resource and consumption constraints, such as the classical spatial model of politics, economic environments, and distributive models in which a fixed surplus is allocated across legislators, and we obtain even a finite policy space as a special case. We incorporate uncertainty about future policy preferences and future effects of policy with the assumption that at the end of each period: next period’s status quo is realized as the sum of the current period’s policy outcome and an arbitrarily small stochastic shock, and legislators’ preferences next period are subject to arbitrarily small publicly observed stochastic shocks. The first of these two assumptions captures the fact that policy instruments are often pegged to the realization of random variables. For example, legislators may care about the real minimum wage, which is determined through a nominal minimum wage (the policy instrument) and the realization of the current price level (a random variable for our purposes). In fact, we would argue that because no legal document can account for all possible contingencies, all policy is implicitly subject to unforeseeable shocks: the implementation of any policy codified in law will ultimately be subject to review by courts, interpretation by administrators, and the vagaries of the policy environment. Thus, a particular policy decision in the current period is likely to effectively result in a different, albeit correlated, policy implemented in future periods. The second assumption captures the possibility of mild idiosyncratic deviations from the underlying systematic preferences of legislators from period to period. These shocks can be viewed as a reduced form representation of uncertainty about the preferences of constituent voters or about other aspects of the legislators’ electoral environments.
The equilibrium behavior of legislators in our model is potentially highly complex, owing to the vast multiplicity of histories in the game. At a minimum, proposal strategies must depend on the current status quo and preference parameters, and voting strategies must depend additionally on the policy proposed. It is therefore natural to focus on stationary Markov perfect equilibria, a refinement that precludes more complicated forms of history-dependence. Due to their relative simplicity, such strategies minimize the difficulty of strategic calculations and may therefore possess a focal quality. From a practical point of view, moreover, stationarity significantly facilitates the identification of empirical models in applications.\(^2\) Existence of stationary equilibria in general dynamic games is a difficult issue, however, for discontinuities can arise due to expectations of future play of the game: if players use discontinuous strategies, then a small change in one player’s action in the current period may lead to a large response by other players in later periods, creating a jump in the discounted sum of payoffs even if the underlying stage payoffs are continuous. It is customary in the literature on stochastic games to gain traction on existence by adding noise to the transition from the current state to next period’s state and imposing continuity assumptions on transition probabilities—in fact, an existence counterexample due to Harris et al. (1995) demonstrates that some such measure must be taken to be assured of existence. In our model, uncertainty about next period’s status quo and future preferences of legislators plays a similar, though diminished, role. These two types of noise, the common shock to current policy and the idiosyncratic shock to each legislator’s preferences, confer distinct analytical benefits: the idiosyncratic component is critical for uniqueness of best responses, while the common component is used to obtain needed continuity and compactness conditions. A similar decomposition of noise can be found in the dynamic industrial organization literature, as in Aguirregabiria and Mira (2007) and Doraszelski and Satterthwaite (2007).\(^3\) As we discuss in the next section, however, our formulation of noise is not sufficient for the model to fulfill the standard continuity assumptions on transition probabilities in the stochastic games literature, and we cannot obtain existence “off the shelf.”

Our core convergence theorem characterizes long run equilibrium policies of the spatial model of politics when the stochastic shocks are small and there exists a “core” policy that cannot be overturned by a decisive coalition, approximating a setting in which a social-choice analysis yields an unambiguous prediction on the basis of individual preferences alone (abstracting from the institutional details of the bargaining process). We establish that the stage utility of the core legislator provides an arbitrarily tight lower bound on that legislator’s equilibrium continuation value. We then

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\(^3\)The addition of noise to dynamic models has also been fruitful in such applications as the study of intergenerational transfers in growth economies. See, e.g., Bernheim and Ray (1989) and Nowak (2006).
show that the invariant distributions over policy outcomes generated by stationary legislative equilibria must be close, in the sense of weak convergence, to the point mass on the core policy. In the one-dimensional special case, the equilibria of our model generate long run policy outcomes arbitrarily close to the median legislator’s ideal point with arbitrarily high probability, reconciling the myopic prediction of the median voter theorem with the strategic incentives of farsighted agents in the context of dynamic bargaining. In the multidimensional setting, our analysis actually allows for the possibility that the core is empty and legislative preferences approximate the canonical model in which a core policy exists. Thus, we generalize the results of Fer- ejohn et al. (1984), who show that myopic majority voting concentrates probability near the core policy of the canonical model, to a setting in which voting is farsighted and policy transitions are governed by the optimal proposals of strategic agents. Our results hold under the assumption that the core legislator is recognized to propose with positive, but perhaps small, probability. Thus, that legislator’s direct influence over policy may be insignificant, as she may propose very rarely, but long run equilibrium policies are driven to the median by the alignment of other legislators’ preferences.

In Section 2, we give a detailed review of the bargaining literatures in political economy and stochastic games, as well as the related literature in dynamic industrial organization. In Section 3, we present the model formally and describe our solution concept. In Section 4, we state our existence and characterization results. In Section 5, we analyze the long run equilibrium policies of our model as the stochastic shocks become arbitrarily small and the preferences of the legislators approach become close to canonical. We conclude in Section 6, and we collect all proofs in Appendix A.

2 Literature Review

Before turning to the analysis, we give a more in-depth review of the literature on bargaining, as it relates to legislative modeling, the literature on existence of stationary Markov perfect equilibrium in stochastic games, and the related literature in dynamic industrial organization.4

Bargaining Most of the existing work in political economy on bargaining considers an infinite-horizon game where in each period one agent proposes a division of surplus and that proposal is either accepted, in which case the game ends with the proposed outcome, or rejected, in which case bargaining continues for at least one more round. Baron and Ferejohn (1989) extend the models of Rubinstein (1982) and Binmore (1987) to cover legislative politics by allowing for an arbitrary number of legislators and requiring the support of a majority to pass a proposal. A substantial literature cutting across economics and political science has grown from these papers,

4See the working paper version, Duggan and Kalandrakis (2007), for an in-depth review of these literatures.
but most assume that bargaining terminates once a proposal passes. While these models can be used to examine policy choices across legislative sessions by simply repeating the bargaining game each session, this is appropriate only if policies remain in place for a single session with an exogenously fixed default outcome at the beginning of the next. This is often the case in budgetary negotiations, but the model is inadequate for the analysis of the enactment of legal statutes or continuing legislation, where policy remains in place for the indefinite future and endogenously determines the status quo in subsequent periods.

A growing literature considers the effects of endogenizing the status quo. In this framework, each period begins with a status quo, then one agent makes a proposal, and that proposal is either accepted, in which case it becomes the current policy and the status quo for the next period, or rejected, in which case the current status quo remains in place until next period. In any case, the process is repeated next period, and so on. Extant studies provide constructions of stationary equilibria in special cases of the model. Baron (1996) analyzes the one-dimensional version of the model with single-peaked stage utilities. Kalandrakis (2004, 2005) establishes existence and continuity properties of equilibrium strategies in the distributive model, obtains a fully strategic result, and studies the composition of equilibrium coalitions and the effect of risk-aversion on equilibrium. Baron and Herron (2003) give a numerical calculation of equilibrium in a three-legislator, finite-horizon model. Fong (2005) considers a three-legislator model in which policies consist of locations in a two-dimensional space and allocations of surplus. Cho (2005) analyzes policy outcomes in a similar environment but with a stage game emulating aspects of parliamentary government. Similar in spirit to the above, Battaglini and Coate (2007b) characterize stationary equilibria in a model of public good provision and taxation with identical legislators and a stock of public goods that evolves over time. Battaglini and Coate (2007a) consider a dynamic model of public spending and taxation in which the state variable is the amount of public debt. All of the above analyses of stationary equilibria consist of explicitly constructing equilibrium strategies, which, given the dependence of proposals on the status quo, can be extremely complex. Battaglini and Palfrey (2007) study a discrete version of a three-player distributive model and find that equilibrium predictions are roughly consistent with experimental data when the players’ stage utilities are strictly concave. Diermeier and Fong (2008) characterize the pure strategy stationary equilibria in a discretized distributive model in which one player has monopoly agenda setting power and players are patient.

A number of related papers diverge in various ways from the above literature and our model. Acemoglu et al. (2008) prove existence and characterize pure strategy stationary equilibria in a finite model with endogenous status quo, assuming a small transition cost and sufficiently patient players but allowing the voting rule to vary

5In contrast, Epple and Riordan (1987) allow for history dependent strategies and derive folk theorem results in the distributive model.
with the state. Bernheim et al. (2006) analyze a model of a single policy choice in which the proposal on the floor is subject to change over time, and after a fixed number of rounds, the implemented policy is determined by a final up or down vote between the proposal offered in the last round and the previous proposal on the floor. The authors assume a finite policy space and strict preferences over policies for all legislators, so that backward induction yields a unique equilibrium outcome. They then extend the model to a finite number of policy choices over time, with the finite horizon again permitting backward induction. Penn (2005) considers a dynamic voting game with randomly generated policy proposals and probabilistic voting on these proposals. Lagunoff (2005a,b) investigates a class of stochastic games that incorporate a social choice solution concept and analyzes endogenous political institutions. Finally, Gomez and Jehiel (2005) consider a class of stochastic games and characterize efficiency properties of equilibrium when players are patient. Unlike our model, they assume a finite number of states and transferable utility.

**Stochastic Games** Existence of stationary Markov perfect equilibrium is a central issue in the literature on stochastic games, but general results have been elusive. They have relied on strong assumptions on transition probabilities—adding noise to next period’s state as a function of the state and action profile in the current period—and on departures from the concept of stationary equilibrium. Chakrabarti (1999) proves existence of (possibly non-stationary) Markov perfect equilibria in games for which the transition probability is norm continuous and is absolutely continuous with respect to a fixed probability measure, and Mertens and Parthasarathy (1987, 1991) drop the absolute continuity and obtain existence of equilibria that are nearly Markovian. Adding non-atomiticity, Chakrabarti (1999) proves existence of a stationary equilibrium, but now in semi-Markov perfect strategies. Dutta and Sundaram (1998) give a simple proof of the existence of (possibly non-stationary) Markov perfect $\epsilon$-equilibria under norm continuity, and Nowak (1985) obtains a Markov perfect $\epsilon$-equilibrium in stationary strategies under stronger conditions. Nowak and Raghavan (1992) prove existence of stationary Markov perfect equilibria with public randomization under the same conditions, and Duffie et al. (1994) add mutual absolute continuity of transition probabilities and show that the equilibrium induces an ergodic process. A smaller subset of work considers pure strategy equilibria. Whitt (1980) and Escobar (2006) give sufficient conditions for existence of pure strategy equilibria by imposing restrictive conditions on payoffs that are satisfied when, for example, players are sufficiently impatient. Amir (1996, 2002), Curtat (1996), and Nowak (2007) derive pure strategy equilibria using lattice-theoretic methods, while the latter author also gives conditions based on concavity of the stage game and a decomposition of the transition probability. Horst (2005) gives a sufficient condition that limits the dependence of a player’s payoff on actions of others.

Our model can be formulated as a stochastic game, but because a legislator’s proposal precisely determines the policy matched against the status quo in the en-
suing vote, the transition probability of our model (specifically, the transition from “proposal states” to “voting states”) violates all of the above continuity assumptions. Furthermore, the conditions used in the literature to obtain pure strategies are not satisfied in our model. Thus, known results on general stochastic games do not establish equilibrium existence in our model, even if we allow for mixing and consider weaker equilibrium concepts, such as Markov equilibrium in semi-stationary or correlated strategies, \( \epsilon \)-equilibrium, or \( p \)-equilibrium. It appears that the structure of our application presents special technical difficulties but at the same time opens a solution to the existence problem.

**Industrial Organization** A strand of literature in industrial organization, beginning with Ericson and Pakes (1995), has studied the dynamics of entry, exit, and investment in a framework with some similarities to our legislative policy making model. Ericson and Pakes (1995) consider an industry with, essentially, a finite number of states characterizing the productivity of all firms. Each period begins with a state, which determines gross profits for that period, and then active firms must choose a level of investment and whether to remain in the industry, while inactive firms must decide whether to enter the industry. The period ends with the draw of next period’s productivity state from a distribution that depends on the current state and investment decisions. Doraszelski and Satterthwaite (2007) observe, however, that equilibria in the Ericson-Pakes model may require mixed strategies. The authors surmount this difficulty by assuming incomplete information about scrap values and setup costs of firms, which are distributed iid, and by imposing restrictions on the transition probability: the former yield pure entry/exit strategies, while the latter give pure investment strategies. Thus, their model involves a decomposition of noise into a common component (the shock to the productivity state) and an idiosyncratic component (scrap values and setup costs), as does ours. The nature of their idiosyncratic noise is quite distinct, however. In Doraszelski and Satterthwaite (2007), scrap values and setup costs are private information, and their purification of entry/exit decisions is in the spirit of Harsanyi (1973). In contrast, our model is characterized by symmetric information, as preference shocks are realized at the beginning of the period and are publicly observed, and our result on generic uniqueness of best responses relies on a different logic. As we note later, the presence of incomplete information in our model would actually facilitate our equilibrium analysis, but we prefer to maintain the assumption of symmetric information in the interest of parsimony.

In recent applied work, Aguirregabiria and Mira (2007) consider an empirical model in which there is a finite set of productivity states and firms make discrete choices that determine profits and the distribution of states in the next period. In addition, the current period’s profits are subject to idiosyncratic, iid shocks. Thus, the model involves a decomposition of noise into common and idiosyncratic components, as does ours. As with Doraszelski and Satterthwaite (2007), however, the idiosyncratic shocks are private information. Furthermore, the idiosyncratic shocks of
Aguirregabiria and Mira (2007) are shocks to the actions of firms, rather than to economic outcomes, and thus generic uniqueness of best responses is immediate. In our model, preference shocks are applied to the utility of the current policy outcome, and not to the actions (proposals and votes) leading to that outcome. Finally, although idiosyncratic shocks are continuously distributed in Doraszelski and Satterthwaite (2007) and Aguirregabiria and Mira (2007), the set of productivity states is finite. This greatly simplifies the problem of imbedding continuation values within a compact space, an important step in obtaining existence of stationary Markov perfect equilibrium. An additional role of the common component of noise in our model, not needed in the above work, is to facilitate that imbedding.

3 Legislative Model

Framework We posit a finite set $N$ of legislators, $i = 1, \ldots, n$, who determine policy over an infinite horizon, with periods indexed $t = 1, 2, \ldots$. Legislative interaction proceeds as follows in each period. A status quo policy $q \in \mathbb{R}^d$ and a vector $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^{nd}$ of preference parameters are realized and publicly observed. A legislator $i \in N$ is drawn at random, with probabilities $p_1, \ldots, p_n$, to propose a policy $y \in X \cup \{q\}$, where $X \subseteq \mathbb{R}^d$ represents the set of feasible policies. The legislators vote simultaneously to accept $y$ or reject it in favor of the status quo $q$. The proposal passes if a coalition $C \in \mathcal{D}$ of legislators vote to accept, and it fails otherwise, where $\mathcal{D}$ is a collection of coalitions described later. The policy outcome for period $t$, denoted $x_t$, is $y$ if the proposal passes and is $q$ otherwise. Each legislator $j$ receives utility $\hat{u}_j(x_t, \theta_j)$, where $\theta_j \in \mathbb{R}^d$ is the legislator’s utility shock. Finally, the status quo $q'$ for period $t + 1$ is drawn from the density $g(\cdot|x_t)$, a new vector $\theta' = (\theta'_1, \ldots, \theta'_n)$ of preference shocks is drawn from the density $f(\cdot)$ and publicly observed, and the above procedure is repeated in period $t + 1$. Payoffs in the dynamic game are given by the expected discounted sum of stage utilities, as is standard, and we denote the discount factor of legislator $i$ by $\delta_i \in [0, 1)$.

We represent a general voting rule by a nonempty collection $\mathcal{D} \subseteq 2^N \setminus \{\emptyset\}$ of decisive coalitions satisfying only the minimal monotonicity requirement that if one coalition is decisive and we add legislators to that coalition, then the larger coalition is also decisive. Formally, we assume that if $C \in \mathcal{D}$ and $C \subseteq C' \subseteq N$, then $C' \in \mathcal{D}$. This allows us to capture majority rule in the obvious way, as the collection $\mathcal{D} = \{C \subseteq N : |C| > n/2\}$, and we obtain many other common voting rules as special cases. To obtain unanimity rule, we set $\mathcal{D} = \{N\}$, and in general we obtain any quota rule by setting $\mathcal{D} = \{C \subseteq N : |C| \geq q\}$, where $q \in [0, N]$. Dictatorship is the special case $\mathcal{D} = \{C \subseteq N : i \in C\}$, where $i$ is the dictator, and we can give any legislator $i$ a veto by specifying $i \in C$ for all $C \in \mathcal{D}$. Our framework captures far more complex voting rules as well. See Banks and Duggan (2000, 2006) for examples of how a bicameral system with executive veto, as in the US system, is obtained, along with examples.
capturing more esoteric rules of the US Congress.

We impose a number of regularity conditions on the model. We assume that the set of feasible policies, $X$, is cut out by a finite number $k$ of functions $h_\ell : \mathbb{R}^d \to \mathbb{R}$ indexed by $K = \{n + 1, \ldots, n + k\}$. We partition $K$ into inequality constraints, $K^{in}$, and equality constraints, $K^{eq}$, and we assume that

$$X = \{x \in \mathbb{R}^d : h_\ell(x) \geq 0, \ell \in K^{in}, h_\ell(x) = 0, \ell \in K^{eq}\}.$$ 

We further assume that $X$ is compact, and that $h_\ell$ is $r$-times continuously differentiable for all $\ell \in K$, where we maintain the assumption that $r \geq \max\{2, d\}$. For technical reasons, we impose the weak condition that for all $x \in X$, $\{Dh_\ell(x) : \ell \in K(x)\}$ is linearly independent, where $K(x)$ is the subset of $\ell \in K$, including equality constraints, such that $h_\ell(x) = 0$. With these assumptions, we capture standard models with resource and consumption constraints, such as the classical spatial model of politics, public good economic environments, and distributive models in which an amount of surplus is to be allocated among the legislators’ districts. Moreover, equality constraints allow us to capture quite general manifolds, and in particular we obtain an arbitrary finite set $X \subseteq \mathbb{R}^d$ of policies as a special case.

The presence of preference shocks in the model captures uncertainty about the legislators’ future policy preferences. We assume that stage utilities are given by $\hat{u}_i(x, \theta_i) = u_i(x) + \theta_i \cdot x$, where $u_i : \mathbb{R}^d \to \mathbb{R}$ is $r$-times continuously differentiable. In the special case of negative quadratic utility, i.e., $u_i(x) = -||x - \hat{x}_i||^2$, where $\hat{x}_i \in \mathbb{R}^d$ is a fixed ideal point, the preference shock is equivalent to adding a noise term to the ideal point of legislator $i$, while in the case of linear utility, the shock is merely a perturbation of a legislator’s gradient. We choose the linear parameterization for the shock because it respects the standard convexity and continuity assumptions in the literature and is the simplest way of introducing well-behaved shifts of the indifference curves of legislators. It is a trivial matter to extend our results to the model with a more complex parameterization, as long as it contains a linear component. With regard to the distribution of preference shocks, we assume that the vector $\theta = (\theta_1, \ldots, \theta_n)$ is distributed according to a density $f$ with support contained in an open set $\Theta \subseteq \mathbb{R}^{nd}$, and we further assume a compact set $\tilde{X} \supseteq X$ and a bound $b_f$ such that $|u_i(x) + \theta_i \cdot x|f(\theta) \leq b_f$ for all $i \in N$, all $\theta \in \Theta$, and all $x \in \tilde{X}$.

The noise on the status quo captures the idea that legislators are uncertain about the way policy decisions today will be implemented in the future. We assume that

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6Of course, we allow $r = \infty$.

7To see this, note that $u_i(x) + \theta_i \cdot x = -(x - (\hat{x}_i + \frac{1}{4}\theta_i)) \cdot (x - (\hat{x}_i + \frac{1}{4}\theta_i)) + \theta_i \cdot \hat{x}_i + \frac{1}{4}\theta_i \cdot \theta_i$, which is just the sum of a term, $\theta_i \cdot \hat{x}_i + \frac{1}{4}\theta_i \cdot \theta_i$, constant in $x$ and the quadratic utility with ideal point $\hat{x}_i + \frac{1}{4}\theta_i$.

8We could, as well, elaborate the model by assuming an additional autocorrelated preference shock. What is needed is that there be some component of the preference shock that is independently distributed across periods.
the density \( g : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R} \), with values \( g(q|x) \), is jointly measurable in \((q, x)\), and that for all \( x \), the support of the density \( g(\cdot|x) \) lies in the compact set \( \tilde{X} \). We do not assume that the support of \( g(\cdot|x) \) lies in \( X \), though of course we allow it. Furthermore, we assume a bound \( b_g \) such that for all \( q \), we have: \( g(q|x) \) is \( r \)-times continuously differentiable in \( x \); if \( r < \infty \), then all derivatives of order 1, \ldots, \( r \) are bounded in norm by \( b_g \), and the \( r \)-th derivative of \( g(q|x) \) with respect to \( x \) is Lipschitz continuous with modulus \( b_g \); and if \( r = \infty \), then derivatives of all orders 1, 2 \ldots are bounded in norm by \( b_g \).\(^9\) For later reference, let \( b = (b_f, b_g) \) denote the vector of bounds described above. It is a trivial matter to extend the model to allow the density \( g \) of next period’s status quo to depend on the current status quo, in addition to the current policy outcome.

Our approach to existence involves the addition of noise to policy outcomes and legislator utilities, but we emphasize that the status quo and the utility shocks at the beginning of a period \( t \) are commonly known, so that a proposer knows whether any given policy will pass or fail if proposed. Furthermore, once a vote is taken, the policy outcome is pinned down for period \( t \): the legislators know, conditional on the outcome of voting, what the policy outcome in the current period will be, and a new status quo is drawn for period \( t + 1 \) only after legislators receive their period \( t \) utilities from outcome \( x_t \). Thus, our formulation of noise in the model is consistent with the view that while legislators are completely informed in the current period, there is at least some uncertainty about future policy preferences and the policy environment. We view these as natural modeling assumptions. In any case, the variance of the densities \( f \) and \( g(\cdot|x) \) may be arbitrarily low, with the sole caveat that the variance of \( g(\cdot|x) \) must be bounded above zero uniformly across \( x \). Thus, we allow for the selection of preference shocks and the status quo to be arbitrarily close to deterministic, so that the element of noise in the model can be made negligible from a substantive standpoint.

An alternative would be to model uncertainty in the current period as well, so that a legislator may not be able to precisely predict whether a proposal will pass, and legislators would not be able precisely predict the effects of a policy implemented in the current period. An argument can be made that such uncertainty exists in real-world legislative systems. We take it as evident, however, that uncertainty about the future is of a higher order of magnitude, and we bring out that contrast in the model by simply assuming complete information for the stage game in any given period. In any case, the addition of noise in the current period to the model would facilitate the equilibrium analysis. Put differently, we establish existence and various properties of equilibrium without recourse to the assumption of noise in the current period.

**Strategies and Payoffs** A strategy in the game consists of two components, one giving the proposals of legislators when recognized to propose and other giving

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\(^9\)This assumption precludes the possibility that \( g(\cdot|x) \) is uniform. See Duggan et al. (2008) for separate arguments for this case.
the votes of legislators after a proposal is made. While these choices can conceivably
depend on histories arbitrarily, we seek subgame perfect equilibria in which legislators
use stationary Markov strategies, which we denote $\sigma_i = (\pi_i, \alpha_i)$. Our main focus will
be on pure strategies, which, as we will show in the sequel, is without loss of generality.
Thus, legislator $i$'s proposal strategy is a measurable mapping $\pi_i: \mathbb{R}^d \times \Theta \to \mathbb{R}^d$,
where $\pi_i(q, \theta)$ is the policy proposed by $i$ given status quo $q$ and utility shocks $\theta$.
And legislator $i$'s voting strategy is a measurable mapping $\alpha_i: \mathbb{R}^d \times \mathbb{R}^d \times \Theta \to \{0, 1\}$,
where $\alpha_i(y, q, \theta) = 1$ if $i$ accepts proposal $y$ given status quo $q$ and utility shocks $\theta$
and $\alpha_i(y, q, \theta) = 0$ if $i$ rejects. We let $\sigma = (\sigma_1, \ldots, \sigma_n)$ denote a stationary strategy
profile. We may equivalently represent voting strategies by the set of feasible proposals
a legislator would vote to accept. We define this acceptance set for $i$ as $A_i(q, \theta; \sigma) = \{y \in X \cup \{q\}: \alpha_i(y, q, \theta_i) = 1\}$. Letting $C$ denote a coalition of legislators, we then define

$$A_C(q, \theta; \sigma) = \bigcap_{i \in C} A_i(q, \theta; \sigma) \quad \text{and} \quad A(q, \theta; \sigma) = \bigcup_{C \in \mathcal{C}} A_C(q, \theta; \sigma)$$

as the coalitional acceptance set for $C$ and the legislative acceptance set, respectively.
The latter consists of all policies that would receive the votes of all members of at
least one decisive coalition, and would therefore pass if proposed.

Given strategy profile $\sigma$, we define legislator $i$'s induced preferences in the game
by

$$U_i(y, \theta_i; \sigma) = (1 - \delta_i)(u_i(y) + \theta_i \cdot y) + \delta_i v_i(y; \sigma),$$

where $v_i(x; \sigma)$ is $i$'s continuation value at the beginning of period $t + 1$ from policy
outcome $x$ in period $t$.\footnote{Note that these continuation values are “ex ante,” in the sense that they are calculated at the
beginning of the period, before $q$ and $\theta$ are realized.} We initially assume that legislators use “deferential” voting
strategies, in the sense that when indifferent between a proposed policy and the status quo, they vote to accept. This assumption, which will turn out to be without loss of
generality, then allows us to focus on no-delay equilibria, in which no legislator ever
proposes a policy that is rejected. (In lieu of that, the legislator can just as well propose the status quo.) Our measurability assumptions on strategies imply that
continuation values are also measurable, and therefore for all $x$, we have

$$v_i(x; \sigma) = \int_q \int_\theta \sum_j p_j U_i(\pi_j(q, \theta), \theta_i; \sigma)f(\theta)g(q|x) \, d\theta dq.$$ (1)

To extend these ideas to allow for mixing and non-deferential voting, we let
$\pi_i: \mathbb{R}^d \times \Theta \to \mathcal{P}(\mathbb{R}^d)$ denote a mixed proposal strategy, where $\mathcal{P}(\mathbb{R}^d)$ is the set
of Borel probability measures on $\mathbb{R}^d$. We equip this space with the weak* topology,
and we assume $\pi_i$ is Borel measurable. Here, $\pi_i(q, \theta)$ represents the distribution of
i’s policy proposal given status quo $q$ and shocks $\theta$. We define voting strategies as measurable mappings $\alpha_i : \mathbb{R}^d \times \mathbb{R}^d \times \Theta \rightarrow [0, 1]$, where now $\alpha_i(q, \theta)$ is the probability, ranging between zero and one, that $i$ accepts proposal $y$ given $q$ and $\theta$. A mixed strategy for legislator $i$ is then $\sigma_i = (\pi_i, \alpha_i)$, and we let $\sigma = (\sigma_1, \ldots, \sigma_n)$ denote a mixed strategy profile. Given a profile $\sigma$ of mixed strategies, we define induced preferences $U_i(y, \theta_i; \sigma)$ as above, but the legislators’ continuation values now have the following more complicated form:

$$v_i(x; \sigma) = \int_q \int_\theta \sum_j p_j \int_y [\alpha(y, q, \theta; \sigma)] U_i(y, \theta_i; \sigma) + (1 - \alpha(y, q, \theta; \sigma)) U_i(q, \theta_i; \sigma)] \pi_j(q, \theta)(dy)f(\theta)g(q|x)d\theta dq,$$

where

$$\alpha(y, q, \theta; \sigma) = \sum_{C \in \mathcal{D}} \left( \prod_{j \in C} \alpha_j(y, q, \theta) \right) \left( \prod_{j \notin C} (1 - \alpha_j(y, q, \theta)) \right)$$

is the probability that a proposal $y$ is accepted by a decisive coalition of legislators. Note that, in (2), we now integrate over the policies proposed by each legislator $i$, and given a realization $y$ from the mixed proposal strategy we now account for the possibility that $y$ may pass with a probability intermediate between zero and one.

**Legislative Equilibrium** With this formalism established, we can now define classes of stationary Markov perfect equilibria of special interest. Intuitively, we require that legislators always propose optimally and that they always vote in their best interest. It is well-known that the latter requirement is unrestrictive in simultaneous voting games, however, as arbitrary outcomes can be supported by Nash equilibria in which no voter is pivotal. To address this difficulty, we follow the standard approach of refining the set of Nash equilibria in voting subgames by requiring that legislators delete votes that are dominated in the stage game. Thus, we say a strategy profile $\sigma$ is a pure stationary legislative equilibrium if the following conditions hold:

- for all shocks $\theta$, every status quo $q$, every proposal $y$, and every legislator $i$,
  $$\alpha_i(y, q, \theta_i) = \begin{cases} 1 & \text{if } U_i(y, \theta_i; \sigma) \geq U_i(q, \theta_i; \sigma) \\ 0 & \text{else.} \end{cases}$$

- for all shocks $\theta$, every status quo $q$, and every legislator $i$, $\pi_i(q, \theta)$ solves
  $$\max_{y \in A(q, \theta; \sigma)} U_i(y, \theta; \sigma)$$

This notion will be the main equilibrium concept of our analysis. Note that it not only imposes the requirement that legislators use pure strategies and eliminate stage-dominated voting strategies, but it also builds in the feature that voters defer to the
We then without loss of generality restrict proposers to the legislative acceptance set. Thus, this notion of equilibrium is relatively restrictive.

In contrast, we also define the following, conceptually less restrictive notion of equilibrium. We say a profile $\sigma$ of mixed strategies is a *mixed stationary legislative equilibrium* if

- for all shocks $\theta$, every status quo $q$, every proposal $y$, and every legislator $i$,

$$\alpha_i(y, q, \theta_i) = \begin{cases} 1 & \text{if } U_i(y, \theta_i; \sigma) > U_i(q, \theta_i; \sigma) \\ 0 & \text{if } U_i(y, \theta_i; \sigma) < U_i(q, \theta_i; \sigma) \end{cases}.$$ 

- for all shocks $\theta$, every status quo $q$, and legislator $i$, $\pi_i$ puts probability one on solutions to

$$\max_{y \in X \cup \{q\}} \alpha(y, q, \theta; \sigma) U_i(y, \theta_i; \sigma) + (1 - \alpha(y, q, \theta; \sigma)) U_i(q, \theta_i; \sigma),$$

One difference between this notion of equilibrium and that of pure stationary legislative equilibrium is that we now allow a legislator, in case there are multiple optimal proposals, to mix over those proposals. A second difference is that we allow a legislator to vote with arbitrary probabilities when indifferent between a proposed policy and the status quo. Consistent with stage-game weak dominance, however, we require that legislators with strict preferences vote deterministically. This complicates the optimization problem of a proposer $i$, for the utility maximization in the legislative acceptance set may no longer pass with probability one.

We say that a mixed strategy profile $\sigma$ is *equivalent* to a strategy profile $\sigma$ if for all $q$ and almost all $\theta$, the policy outcome determined by $(q, \theta)$ is $\pi_i(q, \theta)$ with probability one. Formally, for all $q$, there exists a measure zero set $\Theta(q)$ with probability one. Formally, for all $q$, there exists a measure zero set $\Theta(q)$ with probability one. Formally, for all $q$, there exists a measure zero set $\Theta(q)$ with probability one. Formally, for all $q$, there exists a measure zero set $\Theta(q)$ with probability one. Formally, for all $q$, there exists a measure zero set $\Theta(q)$ with probability one.

We consider two cases in the preceding definition because there are two, payoff equivalent ways the status quo can prevail during a given period—it can be proposed and pass or a proposal can be rejected—either of which suffices for the definition of equivalence. We will see that every mixed stationary legislative equilibrium is essentially pure, so that the added conceptual flexibility afforded by mixed strategies is not realized in equilibrium.

4 Principal Results

In this section, we take up the existence and characterization of pure and mixed stationary legislative equilibria, robustness of equilibria, and the long run distribution of equilibrium policies.
Pure Stationary Legislative Equilibria  The main result of this section is that there is a stationary legislative equilibrium satisfying a number of desirable technical properties.

**Theorem 1** There exists a pure stationary legislative equilibrium, \( \sigma \), possessing the following properties.

1. Continuation values are differentiable: for every legislator \( i \), \( v_i(x; \sigma) \) is \( r \)-times continuously differentiable as a function of \( x \).

2. Proposals are almost always strictly best: for every status quo \( q \), almost all shocks \( \theta \), every legislator \( i \), and every \( y \in A(q, \theta; \sigma) \) distinct from the proposal \( \pi_i(q, \theta) \), we have \( U_i(\pi_i(q, \theta), \theta; \sigma) > U_i(y, \theta; \sigma) \).

3. Proposal strategies are almost always continuous: for every status quo \( q \), almost all shocks \( \theta \), and every legislator \( i \) such that \( \pi_i(q, \theta) \neq q \), \( \pi_i(q, \theta) \) is continuously at \((q, \theta)\).

4. Binding voters, if any, are almost always not redundant: for every status quo \( q \), almost all shocks \( \theta \), and every legislator \( i \), if \( \pi_i(q, \theta) \neq q \) and there exists \( j \) such that \( U_j(\pi_i(q, \theta), \theta_j; \sigma) = U_j(q, \theta_j; \sigma) \), then

   \[ \{ \ell \in N : U_\ell(\pi_i(q, \theta), \theta_\ell; \sigma) \geq U_\ell(q, \theta_\ell; \sigma) \} \setminus \{ j \} \notin D. \]

Part 1 of Theorem 1 establishes that equilibrium continuation values inherit the differentiable structure of the components of the model, \( u_i \), \( h_\ell \), and \( g \). By part 2, the equilibrium exhibited in Theorem 1 is, in a sense, strict: for almost all realizations of noise on preferences and the status quo, a proposer has a unique optimal policy choice. Part 3 of the theorem establishes a potentially useful technical property of equilibrium policy proposals: although equilibrium policy strategies will generally be discontinuous, they are continuous on an open set of full measure.\(^{11}\) Part 4 of Theorem 1 establishes conditions under which a proposer will form minimal winning coalitions. We show that for all \( q \) and almost all \( \theta \), if the proposer is “constrained,” in the sense that the optimal policy proposal renders at least one other legislator indifferent between the proposal and the status quo, then all legislators who are indifferent between the proposal and the status quo are necessary coalition partners: the proposal fails if we remove any such legislator’s assent. An implication is that if the voting rule is a quota rule, then the assent of all legislators approving the proposal

\(^{11}\)In part 3 of Theorem 1, we state that equilibrium proposal strategies are almost everywhere continuous in order to conserve space. See the working paper version, Duggan and Kalandrakis (2007), for a more in-depth analysis where we prove almost everywhere differentiability. There, we actually demonstrate that for almost all realizations of noise, the optimal proposal problem of an agent can be written as a standard optimization problem with mixed constraints; furthermore, we show that the linear independence constraint qualification holds almost always, giving us a characterization of equilibrium proposals in terms of the Kuhn-Tucker first order conditions.
in these situations is necessary for the proposal to pass, and the winning coalition is of minimum size. This is reminiscent of Riker’s (1962) size principle, which maintains that winning coalitions are of minimal size necessary in order for a proposal to pass, and no larger. Part 4 can be viewed as a formalization of the size principle in a general, non-cooperative, dynamic model of policy making. In fact, the theorem suggests a caveat to the size principle: if a decisive coalition of legislators prefer the proposer’s ideal feasible point to the status quo, so the proposer is “unconstrained,” then there is nothing in the logic of Theorem 1 necessitating that the proposal receives only the support of a minimum winning coalition.

As expected, the proof of Theorem 1 proceeds by defining a suitable mapping, establishing the existence of a fixed point, and then verifying that it corresponds to a stationary legislative equilibrium with the claimed properties. Here, we give intuition for some key steps in the proof of existence, focusing for simplicity on the case \( r = \infty \). Let \( C^\infty(\mathbb{R}^d, \mathbb{R}^n) \) denote the space of smooth mappings from \( \mathbb{R}^d \) to \( \mathbb{R}^n \), endowed with the topology of \( C^\infty \)-uniform convergence on compacta.\(^{12}\) Given a vector \( v = (v_1, \ldots, v_n) \in C^\infty(\mathbb{R}^d, \mathbb{R}^n) \) of continuation value functions, define \( U_i(y, \theta; v) \) and \( A_i(q, \theta; v) \), in the obvious way, as the induced utilities and acceptance sets when continuation values are given by \( v \). We then consider a legislator \( i \)'s optimal proposal problem,

\[
\max_{y \in A_i(q, \theta; v)} U_i(y, \theta_i; v) \tag{3}
\]

and we let \( \pi_i(q, \theta; v) \) denote a selection from the solutions to this program. This selection then determines a new vector of continuation values, \( \hat{v} = (\hat{v}_1, \ldots, \hat{v}_n) \), for the legislators, and we define \( \psi \) as the mapping that takes the vector \( v \) to the new vector \( \hat{v} \), i.e., \( \psi(v) = \hat{v} \).

The existence proof consists in verifying that \( \psi \) satisfies the conditions of Glicksberg’s fixed point theorem. The technical roles of the noise on the status quo is to smooth out continuation values and to allow us to restrict the domain and range of \( \psi \) to a compact subset of \( C^\infty(\mathbb{R}^d, \mathbb{R}^n) \). To see how, note that the new continuation value \( \hat{v}_i \) of legislator \( i \) is defined by

\[
\hat{v}_i(x) = \int_q \int_\theta \sum_{j \in \mathbb{N}} p_j U_i(\pi_j(q, \theta; v), \theta_i; v) f(\theta) g(q|x) d\theta dq, \tag{4}
\]

and note further that the current period’s policy choice \( x \) enters this continuation value only through the density \( g(q|x) \). Thus, \( \psi \) is, in essence, the convolution of \( \int_\theta \sum_{j \in \mathbb{N}} p_j U_i(\pi_j(q, \theta; v), \theta_i; v) f(\theta) d\theta \), a generally discontinuous function of \( q \), with the function \( g(q|x) \). The result is a smooth function of the policy outcome \( x \). Furthermore, if we define \( \mathcal{V} \) as the compact space consisting of all functions \( v \in C^\infty(\mathbb{R}^d, \mathbb{R}^n) \)

\(^{12}\)See the appendix for the precise definition of this topology.
such that $v$ is appropriately bounded, then it is straightforward to verify that $ψ$ maps $\mathcal{V}$ into itself.

The preference shock plays a critical role in establishing continuity of the mapping $ψ$. The argument relies on the result, underlying part 2 of Theorem 1, that for any given $v$, for every status quo $q$, and almost all shocks $θ$, the proposer’s maximization problem has a unique solution. Thus, the selection $π_i(q, θ; v)$ is uniquely pinned down almost everywhere. The intuition behind this uniqueness result is straightforward: if legislator $i$ is indifferent between proposing two policies for one realization of $θ_i$, then, generically, a perturbation $θ'_i$ of $θ_i$ will break that indifference. This is depicted in Figure 1, where policies $x$ and $y$ maximize $U_i(\cdot, θ_i; v)$ over $A(q, θ_{−i}; v)$, the shaded region in the figure. Note that the constraint in $i$’s optimal proposal problem in (3) can be reformulated to exclude the constraint requiring that $i$ accept his own proposal, so we can write the constraint set as $A(q, θ_{−i}; v)$, which is independent of $θ_i$. Then a small perturbation to $θ'_i$ leads to a unique maximizer $z$. Key here is the fact that a perturbation of $θ_i$ does not affect the payoffs of other legislators or, therefore, the effective constraints of $i$’s maximization problem.

Having proved uniqueness of the selection $π_i(q, θ; v)$ almost everywhere, the preference shock delivers continuity of the mapping $ψ$ as follows. Using differentiability of $U_i(y, θ_i; v)$, we apply the transversality theorem to deduce that for any given $v ∈ C^∞(\mathbb{R}^d, \mathbb{R}^n)$, for every status quo $q$, almost all shocks $θ$, and every policy $y ∈ A(q, θ; v)$ distinct from $q$, the linear independence constraint qualification (LICQ) is satisfied at $y$.\textsuperscript{13} This is depicted in Figure 2. Here, for simplicity, we suppose the legislative acceptance set is the intersection of legislators 1’s and 2’s acceptance sets, which are shaded. Although the gradients of legislators 1 and 2 are linearly depen-

\textsuperscript{13}This means that the gradients of all binding feasibility constraints and all indifferent voters are linearly independent. See the appendix for the precise definition.
dent at \( y \), so LICQ is violated, a small shock to \( \theta_1 \) will lead to a perturbation of the acceptance set of legislator 1, given by the dashed curve in the figure. We then have the generic situation, in which LICQ is satisfied over the legislative acceptance set, save possibly the status quo. This in turn implies lower hemi-continuity of the legislative acceptance set correspondence \( A(q, \theta; v) \) for almost all \( \theta \), and by the theorem of the maximum, a legislator’s optimal proposal \( \pi_i(q, \theta; v) \) will be jointly continuous in \( (q, \theta; v) \) for almost all \( \theta \).\(^{14}\) This implies that the inner integral in (4), namely,

\[
\int_{\theta} \sum_{j \in N} p_j U_i(\pi_j(q, \theta; v), \theta_i; v) f(\theta) g(q|x) d\theta,
\]
is continuous as a function of \((x, v)\). Then continuity of \( \hat{v}_i(x) = \psi(v)(x) \) in \((x, v)\) follows by an application of Lebesgue’s dominated convergence theorem. It is straightforward to apply this argument to all higher derivatives, delivering continuity of the mapping \( \psi \) in the topology of \( C^\infty \)-uniform convergence on compacta, thereby permitting the application of Glicksberg’s theorem.

Mixed Stationary Legislative Equilibria The next result justifies our focus on pure stationary legislative equilibria. It establishes that every mixed equilibrium is equivalent to a pure one. Furthermore, because every pure equilibrium is a special case of mixed, it shows that every pure stationary legislative equilibrium satisfies the properties of Theorem 1.

**Theorem 2** Every mixed stationary legislative equilibrium is equivalent to a pure stationary legislative equilibrium satisfying the properties in parts 1–4 of Theorem 1.

Much of the intuition for this result has already been discussed. Given a mixed stationary legislative equilibrium, with continuation value \( v \), our earlier observation

\(^{14}\)In the appendix, we give a more powerful argument that the optimal proposal is in fact differentiable.
that the solution, \( \pi_i(q, \theta) \), to a legislator’s optimal proposal problem in (3) is almost always unique carries over without change. This does not immediately rule out the possibility of non-degenerate mixed strategies, however, because one or more legislators may be indifferent between \( \pi_i(q, \theta) \) and the status quo, and these legislators could conceivably vote to accept with probability less than one. But our subsequent claim that LICQ holds at every policy \( y \in A(q, \theta; v) \) distinct from \( q \) relied only on the differentiability of the equilibrium continuation values \( v \), and inspection of (2) reveals that even in a mixed equilibrium, continuation values will inherit the differentiability assumed in the model: the current period’s policy \( x \) enters the righthand side of (2) only through the function \( g(q|x) \), which is appropriately smooth in \( x \). An implication is that the proposer can find policies arbitrarily close to \( \pi_i(q, \theta) \) that are strictly better than the status quo for a decisive coalition of legislators. Such proposals will pass with probability one in equilibrium, and existence of an optimal proposal (a necessary condition for equilibrium) demands that \( \pi_i(q, \theta) \) will also pass with probability one. Since \( \pi_i(q, \theta) \) is the unique solution to (3), any optimal mixed proposal strategy must put probability one on that policy.

**Continuity of the Equilibrium Correspondence** A desirable property of equilibria is robustness, which we formalize in terms of upper hemicontinuity of the equilibrium correspondence with respect to the parameters of the model. In our framework, a model is specified by parameters \( \gamma = ((p^i_\gamma, u^i_\gamma, \delta^i_\gamma)_{i \in \mathbb{N}}, X^\gamma, f^\gamma, g^\gamma) \), where \( X^\gamma \) is a compact set defined by a finite number of equality and inequality constraints such that the gradients of binding constraints are linearly independent. Given the vector \( b = (b_f, b_g) \) of bounds, let \( \Gamma \) denote a metric space of possible parameterizations satisfying our maintained assumptions from Section 3. In particular, we assume that for all \( \gamma \in \Gamma \), there exist a compact set \( \tilde{X}^\gamma \) such that (i) the function \( |u^i_\gamma(x) + \theta_i x| f^\gamma(\theta) \) is bounded by \( b_f \) over \( i \in \mathbb{N}, \theta \in \Theta \), and \( x \in \tilde{X}^\gamma \), (ii) for all \( x \), the support of \( g^\gamma(\cdot|x) \) is contained in \( \tilde{X}^\gamma \), and (iii) for almost all \( q, g^\gamma(q|x) \) is \( r \)-times continuously differentiable in \( x \), (iv) if \( r < \infty \), then derivatives of order \( 1, \ldots, r \) are bounded in norm by \( b_g \), and the \( r \)-th derivative is Lipschitz continuous with modulus \( b_g \), and if \( r = \infty \), then derivatives of all orders \( 1, 2, \ldots \) are bounded in norm by \( b_g \). In addition, we assume that the parameterization is continuous: (v) \( p^i_\gamma \) and \( \delta^i_\gamma \) are continuous in \( \gamma \), (vi) \( u^i_\gamma(x) \) is jointly continuous in \( (x, \gamma) \), (vii) \( X^\gamma \) is continuous in \( \gamma \) with the Hausdorff metric on closed subsets of \( \mathbb{R}^d \), (viii) for all \( \theta, f^\gamma(\theta) \) is continuous in \( \gamma \), and (ix) for all \( q, g^\gamma(q|x) \) is continuous in \( (x, \gamma) \). Note that our parameterization is especially general with respect to the set of feasible policies, for we do not assume that policy spaces are generated by a common set of parameterized feasibility constraints. Moreover, we do not assume that the number of feasible constraints is bounded across \( \Gamma \), and so our notion of continuity allows us to approximate a general policy space with a sequence of finite approximations cut out by an increasing number of constraints.

We define the equilibrium correspondence \( E : \Gamma \rightrightarrows \mathcal{C}^r(\mathbb{R}^d, \mathbb{R}^n) \) so that \( E(\gamma) \) consists of the set of pure stationary legislative equilibrium continuation values \( v \in \mathcal{C}^r(\mathbb{R}^d, \mathbb{R}^n) \).
Theorem 1 shows that $E$ is nonempty-valued. The next result establishes that the equilibrium correspondence $E$ is upper hemicontinuous. Thus, equilibria are robust in the sense that a small perturbation of the parameters of our model cannot produce new equilibria far from the original equilibrium set.

**Theorem 3** The equilibrium correspondence $E : \Gamma \rightrightarrows C^{\infty}(\mathbb{R}^d, \mathbb{R}^n)$ is upper hemicontinuous.

The proof of upper hemicontinuity shares much of the structure of the argument sketched above for existence in Theorem 1. We note there that for all $v$, all $q$, and almost all $\theta$, the legislators’ optimal proposals, $\pi_i(q, \theta; v)$, are continuous at $(q, \theta, v)$. The same observation holds when we allow the parameters of the model to vary as described in (i)–(ix). That is, letting $\pi_i^\gamma(q, \theta; v)$ denote legislator $i$’s optimal proposal in model $\gamma$, our earlier argument establishes that for all $v$, all $q$, all $\gamma$, and almost all $\theta$, $\pi_i^\gamma(q, \theta; v)$ is jointly continuous at $(q, \theta, v, \gamma)$. Once this is proven, upper hemicontinuity of the equilibrium correspondence follows from the application of Lebesgue’s dominated convergence theorem.

**Ergodic Properties of Stationary Legislative Equilibria** A stationary legislative equilibrium, say $\sigma^*$, determines a stochastic process on policies, and we may then consider the equilibrium dynamics of policy outcomes in our model. Given Borel measurable $Y \subseteq \mathbb{R}^d$, let $I_Y$ denote the indicator function of $Y$. We define the transition probability on policy outcomes by

$$P(x, Y) = \int_q \int_\theta \sum_{i \in N} p_i I_Y(\pi_i^*(q, \theta)) f(\theta) g(q|x) d\theta dq,$$

which is the probability, conditional on policy outcome $x$ this period, that next period’s policy outcome will lie in the set $Y$. We define the associated Markov operator $T$ on the space of bounded, Borel measurable functions $\phi : \tilde{X} \to \mathbb{R}$ by $T\phi(x) = \int \phi(z) P(x, dz)$. The adjoint $T^*$ operates on the Borel measures on $\tilde{X}$, denoted $\xi$, and is defined by $T^*\xi(Y) = \int P(x, Y) \xi(dx)$. This describes the distribution of outcomes in the next period, given a distribution $\xi$ of policy outcomes in the current period. The iterates of $T^*$, denoted $T^m$, give the distribution of policy outcomes $m$ periods hence and are therefore key in describing the long run policy outcomes of the model.

Before proceeding, we provide definitions for some useful technical conditions on Markov chains. We say $P$ satisfies the **Feller property** if for all bounded, continuous $\phi : \tilde{X} \to \mathbb{R}$, the mapping $T\phi : \tilde{X} \to \mathbb{R}$ is also bounded and continuous. Since we assume $\tilde{X}$ is compact, continuity of course implies boundedness, so Feller reduces to the requirement that $T$ map continuous functions to continuous functions. We say $P$ satisfies **Doeblin’s condition** if there is a finite Borel measure $\varphi$, an integer $n$, and $\epsilon > 0$ such that $\varphi(Y) \leq \epsilon$ implies $P^m(x, Y) \leq 1 - \epsilon$, where $P^m$ is the $m$-period transition defined inductively by $P^m(x, Y) = \int P^{m-1}(y, Y) P(x, dy)$. Intuitively, this
means that if a set is small according to \( \varphi \), then \( P \) cannot assign a high probability to the set for any initial policy \( x \). Finally, we say \( P \) is aperiodic if there do not exist an integer \( \beta \geq 2 \) and pairwise disjoint, measurable subsets \( C_1, \ldots, C_\beta \) such that for all \( j = 1, \ldots, \beta \) and all \( x \in C_j \), we have \( P(x, C_{j+1 \mod \beta}) = 1 \). This condition is useful in deducing strong convergence properties for the Markov chain on policies.

In the next section, we provide a sharp characterization of long run equilibrium policies when the model is close to the canonical spatial model in which legislators’ stage utilities are such that a core policy exists. Here, we consider general properties of the long run distribution of equilibrium policies while imposing minimal structure on the model. It is straightforward to show that \( T \) maps continuous functions to continuous functions and, therefore, satisfies the Feller property. It is also tight, and it therefore immediately admits at least one invariant distribution \( \xi^* \), such that \( \xi^* = T^\ast \xi^* \). Thus, each stationary legislative equilibrium determines an “ergodic Markov equilibrium,” in the sense of Duffie et al. (1994). Furthermore, we can show that under the weak assumption that the density \( g \) is bounded, \( P \) satisfies Doeblin’s condition, with the implication that from any initial distribution \( \xi \) on \( X \), the sequence of long run average distributions, \( \frac{1}{m} \sum_{t=1}^{m} T^t \xi, m = 1, 2, \ldots \), converges to an invariant distribution in the total variation norm (Doob (1953)). This result is similar in spirit to that of Hellwig (1980), who uses Doeblin’s condition to establish ergodic properties of temporary equilibria. While it provides a minimal characterization of long run policy outcomes, however, the result is weak in several respects: it concerns the long run average distributions, rather than the distribution of policy outcomes in each period \( t \); the limiting invariant distribution can depend on the initial distribution; and the rate of convergence is only known to be arithmetic. In particular, we have not precluded the possibility that there are multiple invariant distributions.

Under further restrictions on the transition probability, standard results on Markov processes can be applied to address these shortcomings. Although the equilibrium transition probability is endogenous, we can obtain the desired properties by imposing restrictions on the exogenous density \( g(q|x) \). We first address the convergence issues raised above by assuming that every policy \( x \) lies in the support of \( g(\cdot|x) \), so that \( x \) itself is a possible status quo in the next period. This weak assumption is sufficient for aperiodicity of \( P \) and delivers fast convergence to an invariant distribution from any initial policy. With this assumption, we obtain convergence of per period policy distributions instead of the long run average distributions, and we obtain geometric convergence in the total variation norm. We then address the possibility of multiple invariant distributions by requiring overlapping supports of the status quo densities. Specifically, we assume that for every pair of policies \( x, x' \in \tilde{X} \), there is a status quo \( q \) that lies in the supports of \( g(\cdot|x) \) and \( g(\cdot|x') \). Though this assumption is restrictive from a theoretical perspective, we allow for the status quo densities to place arbitrarily low (but positive) probability at such a status quo, so it should not pose an impediment to applications of the model. This delivers uniqueness of the invariant
distribution corresponding to a given stationary legislative equilibrium and yields an unambiguous prediction of long run equilibrium policy outcomes.

**Theorem 4** Let \( \sigma^* \) be a stationary legislative equilibrium and \( T \) the associated Markov operator with adjoint \( T^* \).

1. The transition probability \( P \) satisfies the Feller property and admits at least one invariant distribution.

Assume \( g \) is bounded on \( \tilde{X} \times \tilde{X} \). Then \( P \) satisfies the following.

2. It satisfies Doeblin's condition, and given any initial distribution \( \xi \), the sequence of long run average distributions, \( \frac{1}{m} \sum_{t=1}^{m} T^{*t} \xi \), converges arithmetically to an invariant distribution \( \xi^* \) in the total variation norm: there is a constant \( M > 0 \) such that for all \( m \), we have

\[
\left\| \frac{1}{m} \sum_{t=1}^{m} T^{*t} \xi - \xi^* \right\| \leq \frac{M}{m}.
\]

3. If for every policy \( x \in \tilde{X} \), we have \( g(x|x) > 0 \), then \( P \) is aperiodic and given any initial distribution \( \xi \), the sequence of per period policy distributions converges geometrically to an invariant distribution \( \xi^* \) in total variation norm: there are constants \( M \) and \( \rho \), with \( \rho < 1 \), such that for all \( m \), we have \( \|T^{*m} \xi - \xi^*\| \leq M \rho^m \).

4. If for every pair of policies \( x, x' \in \tilde{X} \) there exists \( q \in \tilde{X} \) such that \( g(q|x)g(q|x') > 0 \), then \( P \) admits a unique invariant distribution, say \( \xi^* \). Given any initial distribution \( \xi \), the sequence of iterates, \( T^{*m} \xi \), converges geometrically to \( \xi^* \) in the total variation norm: there are constants \( M \) and \( \rho \), with \( \rho < 1 \), such that for all \( m \), we have \( \|T^{*m} \xi - \xi^*\| \leq M \rho^m \).

Note that parts 1 and 2 hold very generally. Furthermore, the arguments in Theorem 4 do not take advantage of the details of equilibrium strategies we have established in Theorem 1. We conjecture that in more structured environments, uniqueness of the ergodic distribution may follow under even weaker conditions on the status quo density.

## 5 Core Convergence Theorem

In this section, we explore long run equilibrium policies when the model becomes arbitrarily close to the canonical spatial model of politics, in which the stage utilities of the legislators are Euclidean (in fact, quadratic) and there is a “core” policy
that cannot be overturned by any decisive coalition. The latter precondition is automatically satisfied when the policy space is unidimensional, but our analysis also covers multidimensional environments that are “almost” canonical. In particular, we consider models in which the stochastic shocks are small and the profile of legislators’ stage utilities are close to a quadratic profile admitting a core policy, say \( x^* \).

We first establish that the core legislator’s stage utility provides an arbitrarily tight lower bound on his equilibrium continuation value. We then show that the invariant distribution over policy outcomes generated by stationary legislative equilibria must be close, in the sense of weak convergence, to the point mass on \( x^* \). Our results bear on an earlier literature in social choice that considers the location of policies relative to the core. It is well-known that the core is nonempty only for a negligible set of legislative preferences when the policy space is multidimensional. In the generic case, the predictions of the static social choice approach are vacuous. Ferejohn et al. (1984) take a dynamic approach by positing a transition probability on the space of policies generated by myopic majority voting: given a policy \( x_t \) in period \( t \), the distribution over policies in period \( t + 1 \) is determined by first randomizing over majority coalitions and then uniformly drawing \( x_{t+1} \) from the set of policies preferred (according to stage utilities) to \( x_t \) by all members of the coalition. The authors establish that this transition probability admits a unique invariant distribution, and that if individual preferences are Euclidean and such that \( x^* \) is almost a core policy, then the invariant distribution generated by myopic voting piles probability close to \( x^* \).

In contrast, we consider policy dynamics governed by equilibrium play in a non-cooperative game-theoretic framework when voters are farsighted. Thus, for example, even if the policy outcome \( x_t \) is close to being a core policy relative to the stage utilities of the legislators, the equilibrium policy in the following period is determined by the legislators’ strategic preferences, which incorporate expectations of future play; and it is not clear apriori that the stability properties of \( x_t \) with respect to stage utilities are maintained when legislators condition their votes on future expectations. Furthermore, in our model the transition on policies is intermediated by stochastic shocks to the status quo and to the stage utilities of legislators. Due to the noise on the status quo, the long run distribution of policies will not generally converge to a point mass, even when the core is nonempty, and the concept of the core cannot even be defined independently of preference shocks. It is not immediately clear how a game-theoretic convergence result in the spirit of Ferejohn et al. (1984) should be formulated. Our approach is to consider models in which legislator stage utilities either admit a core policy or are close to doing so, and in which the shocks to the status quo and stage utilities are small. This raises a technical challenge, in that the limiting canonical model lies outside the class of models we consider: letting the stochastic shocks diminish to zero, there can be no presumption that a limit (if one exists) of legislative bargaining equilibria from our model forms an equilibrium in the model with no noise. Thus, we are not able to conduct our analysis “in the limit” and work backward, as would normally be the case; rather we are forced to conduct
the analysis along the sequence of equilibria.

We will assume $\mathcal{D}$ is proper, i.e., for all $C \in \mathcal{D}$, we have $N \setminus C \notin \mathcal{C}$, and strong, i.e., for all $C \subseteq N$, either $C \in \mathcal{D}$ or $N \setminus C \in \mathcal{D}$. A well-known known example is majority rule with $n$ odd. When $n$ is even, it is trivial to modify majority rule by designating one legislator, say $v$, who breaks ties.\(^{15}\) The core, relative to the unperturbed stage utilities of the legislators, is the set of policies $x \in X$ such that for all $y \in \mathbb{R}^d$, we have $\{i \in N : u_i(y) > u_i(x)\} \notin \mathcal{D}$. Under further weak assumptions, if $x$ belongs to the core, then it is the unique core policy, it is the unique utility-maximizing policy for some legislator (the core legislator), and for all $y \in \mathbb{R}^d \setminus \{x\}$, we have $\{i \in N : u_i(x) > u_i(y)\} \in \mathcal{D}$. These implications hold in our framework if, for example, stage utilities are strictly quasi-concave.\(^{16}\) We simplify the analysis by considering models with policy spaces contained in a fixed, compact set $X \subseteq \mathbb{R}^d$. Because we consider models with arbitrarily small noise, we cannot maintain a bound on derivatives of the status quo density that is uniform across models. Therefore, given a vector $b = (b_f, b_g)$ of bounds, let $\Gamma^b_X$ be the class of models with policy space contained in $X$ and respecting the fixed vector of bounds $b$ as in Section 3, and let $\Gamma^\infty_X = \bigcup \{\Gamma^b_X : b \in \mathbb{R}^d_+\}$ consist of any model with $X^\gamma \subseteq X$ and satisfying our assumptions with respect to some vector of bounds.

We thus describe the canonical spatial model by a profile $(\hat{x}_i)_{i \in N}$ of ideal policies, a core legislator $k$, and a common discount factor $\delta$. We measure the distance of our model from the canonical one by a two-dimensional metric. Given $\epsilon > 0$, we say a model $\gamma = ((p_i,u_i,\delta)_{i \in N},X,f,g) \in \Gamma^\infty$ is $\epsilon$-canonical with respect to $(k, (\hat{x}_i)_{i \in N}, \delta) \in N \times \mathbb{R}^{nd} \times [0,1)$ if:

(i) for all $x \in \mathbb{R}^d$, $\{i \in N : u_i^c(x) > u_i^c(\hat{x}_k)\} \notin \mathcal{D}$, where $u_i^c$ is quadratic with ideal point $\hat{x}_i \in X$, i.e., $u_i^c(x) = -||\hat{x}_i - x||^2$,

(ii) for all $i \in N$, $\sup \{x \in B_\epsilon(X) : |u_i(x) - u_i^c(x)| < \epsilon$ and $|\delta_i - \delta| < \epsilon$.

(iii) supp$f \subseteq B_\epsilon(0)$,

(iv) for all $x \in X$, supp$g(\cdot|x) \subseteq B_\epsilon(x)$.

We say $\gamma$ is $\epsilon$-canonical if there exists $(k, (\hat{x}_i)_{i \in N}, \delta)$ satisfying (i)–(iv).\(^{17}\)

The above conditions describe a neighboring canonical spatial model in which the legislators’ utilities are quadratic and admit a non-empty core, the ideal policy of a

\(^{15}\)More generally, any voting rule $\mathcal{D}$ can be extended to a strong rule $\hat{\mathcal{D}}$ defined as follows: $C \in \hat{\mathcal{D}}$ if and only if either $C \in \mathcal{D}$ or $v \in C$ and $N \setminus C \notin \mathcal{D}$, where legislator $v$ breaks ties.

\(^{16}\)An even weaker assumption under which the implications hold is that each $u_i$ has a unique maximizer in $X$ and that this policy is the unique solution to the first order condition $Du_i(x) = 0$ for $i$.

\(^{17}\)If multiple canonical models satisfy these conditions, then we select an arbitrary one.
core legislator, \( k \), and in which legislators possess a common discount factor, \( \delta \), the distribution of the preference shock \( \theta \) is degenerate at zero, and the transition from current policy, \( x \), to next period’s status quo, \( q = x \), is deterministic. Distance to the neighboring model is modulated by the parameter \( \epsilon \). In the appendix, Lemma 7 establishes that when \( \epsilon \) is small, the core legislator is almost decisive: a proposed policy will pass only if it gives legislator \( k \) a dynamic utility almost equal to his dynamic utility from the status quo; and if a policy gives \( k \) a dynamic utility slightly higher than the status quo, then it will pass if proposed. This lemma is stated in terms of strategic preferences, and so it does not immediately yield restrictions on equilibrium policies relative to stage utilities, but Lemma 8 leverages this result to provide a lower bound for the equilibrium dynamic utility of the core legislator in which proposals of other legislators are absent and \( k \)'s proposals are not subject to voting constraints.

The next result establishes that the stage utility of the core legislator \( k \) provides an arbitrarily tight lower bound on the legislator’s equilibrium continuation value when the model is close to canonical. Thus, if a policy \( y \) is passed in the current period, the distribution of future policies cannot put appreciable probability on the set of policies worse than \( y \) for legislator \( k \).

**Theorem 5** Assume \( \mathcal{Q} \) is proper and strong. Fix \( \bar{\delta} < 1 \). For all \( \lambda > 0 \), there exist \( \bar{\epsilon} > 0 \) such that for all \( \epsilon < \bar{\epsilon} \), all \( \epsilon \)-canonical models \( \gamma \in \Gamma^\infty_X \) with \( \delta_k \leq \bar{\delta} \), all stationary legislative equilibria \( \sigma \in E(\gamma) \), and all \( y \in X \), we have \( v_k(y; \sigma) \geq u_k(y) - \lambda \).

Our characterization of long run policy outcomes considers a sequence of models becoming arbitrarily close to canonical. Call a sequence of models \( \{\gamma^m\} \) in \( \Gamma^\infty_X \) approximately canonical if there exists a sequence \( \{\epsilon^m\} \) such that:

(i) for all \( m \), \( \gamma^m \) is \( \epsilon^m \)-canonical with respect to \((k^m, (\hat{x}^m_i)_{i \in N}, \delta^m)\),

(ii) for all \( m \) and all \( x \), \( g^m(x|x) > 0 \),

(iii) \( \{(k^m, \hat{x}^m_{k^m}, \delta^m)\} \) is convergent with limit \((k^*, x^*, \delta^*)\) satisfying \( \delta^* < 1 \),

(iv) \( \lim \inf p^m_{k^*} > 0 \),

(v) \( \min\{||x - x^*|| : x \in X^m\} \to 0 \),

(vi) \( \epsilon^m \to 0 \).

Thus, we require that the models become arbitrary close to canonical, that in any given model, a policy \( x \) is a possible status quo following the choice of \( x \) in the current period,\(^{18}\) that the legislators do not become arbitrarily patient, and that the

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\(^{18}\)This assumption simplifies the proof of Theorem 6 by giving us more control over the invariant distributions as the noise in the model becomes small. See Duggan and Kalandrakis (2008) for a more general proof that drops this assumption.
core legislator has positive recognition probability in the limit. The above definition is general in the respect that it does not impose the restriction that the models $\gamma^m$ are $\epsilon^m$-canonical with respect to the same reference point, $(k, (\hat{x}_i)_{i \in N}, \delta)$.\footnote{In fact, the restriction of convergence of the sequence \{$(k^m, \hat{x}_{km}, \delta^m)$\} is only apparent, for given an arbitrary sequence with $\limsup \delta^m < 1$, Theorem 6 can always be applied to any convergent subsequence.} Furthermore, we let the policy space vary quite generally, requiring only that they are bounded by the compact space $\mathcal{X}$ and that the core alternative of the canonical model be close to feasible. Our final result shows, roughly, that in environments close to canonical, the long run equilibrium policies of our model are concentrated close to the core of the canonical model.

**Theorem 6** Assume that $\mathcal{D}$ is proper and strong and that \{${\gamma^m}$\} is approximately canonical. For each $m$, let $\sigma^m \in E(\gamma^m)$, and let $\xi^m$ be an invariant distribution corresponding to $\sigma^m$. Then \{${\xi^m}$\} converges weak$^*$ to the unit mass on $x^*$.

The proof is complicated by our inability to work in the limit. Rather, we work along the sequence and show that for all $\eta > 0$, the invariant probability of $B_\eta(x^*)$ goes to one. The proof proceeds by showing that as the model becomes close to canonical, the core legislator $k^*$ proposes policies close to $x^*$ with probability one, and in particular, there are alternatives $y^m$ in an ergodic set approaching $x^*$. Using Theorem 5, we show that the core legislator’s dynamic utility from $y^m$ approaches his ideal payoff in the neighboring canonical model, and therefore the long run probability that equilibrium policies lie in the ball, $P_t^m(y^m, B_\eta(x^*))$, converges to one as $m$ goes to infinity. At the same time, part 3 of Theorem 4, with assumption (ii) that $g(x|x) > 0$, imply that $P_t^m(y^m, B_\eta(x^*))$ converges to the invariant probability of the ball as $t$ goes to infinity. In the proof, we show that the latter convergence is geometric, allowing us to deduce the desired result.

6 Conclusion

We establish existence of stationary Markov perfect equilibria satisfying a number of desirable regularity properties in a general model of legislative policy making. Specializing to the unidimensional model, or the multidimensional model in which legislator preferences are close to the canonical form, we find that a median voter theorem holds: as the stochastic shocks in the model become small, the long run equilibrium policies are concentrated near the median, or the core policy in higher dimensions, with high probability. Our general analysis imposes no constraints on the dimensionality of the policy space, we do not assume convexity conditions on policy preferences, and we allow for any voting rule that can be expressed in terms of a collection of decisive coalitions. The main technical assumption we impose is differentiability, which, in combination with uncertainty about future policy preferences and noise in the imple-
mentation of future policies, allows us to bring methods of differentiable topology to bear on the existence problem. For reasons of space, we have limited the scope of our analysis to a benchmark model that is institutionally austere, in the sense that we abstract away from much of the detail of real-world political systems. It encapsulates all of the difficult technical issues we would encounter in more complex models, while offering advantages of efficiency in presentation. But our approach to existence and related issues is quite general and extends to a much larger class of models that can capture a substantial amount of institutional detail. It is trivial to augment the model with a finite set of states that control stage utilities, discount factors, the feasible policies, the voting rule, and the identity of the proposer and that evolve according to an exogenous Markov process, thereby capturing institutional features such as a legislative committee system and permitting the analysis of electoral incentives in policy-making.\textsuperscript{20} This richer version of the model opens the opportunity for the fine-tuned analysis of constitutional design issues and the development of applications, e.g., the computational analysis of the effect of the presidential veto (Duggan et al. (2008)) and the empirical analysis of the model using data from US presidential and congressional elections (Duggan and Kalandrakis (2008)).

A Proofs of Theorems

The appendix is organized as follows. We first derive a trio of lemmas that establish continuity properties and necessary conditions for solutions to the optimization problem of the proposer. We then state Lemma 4, which ensures that every feasible proposal under deferential voting can be approximated by a sequence of feasible proposals that are strictly preferred by the members of some decisive coalition. We proceed to define the mapping $\psi$, described in Section 4, and with Lemma 5 we establish that this mapping is continuous and that its domain and range can be restricted to a compact set. We then prove existence of legislative equilibrium in Theorem 1 by an application of Glicksberg’s theorem, and parts 1–4 of the theorem follow immediately from Lemmas 1–3. In Lemma 6, we show that all legislative equilibrium continuation values are fixed points $\psi$. The proof of Theorem 2, which reduces all mixed legislative equilibria to pure, relies mainly on Lemmas 1 and 4. Theorem 3, on upper hemicontinuity of the equilibrium correspondence, follows from Lemmas 5 and 6. Theorem 4 uses the continuity of optimal proposals along with known results on ergodicity of Markov chains. Theorems 5 and 6 follow with the help of Lemmas 7 and 8, which develops the near decisiveness of the core legislator.

Let $C^r(\mathbb{R}^d, \mathbb{R}^n)$ be the $r$-times continuously differentiable functions from $\mathbb{R}^d$ into

\textsuperscript{20}We capture a committee system by varying the voting rule and the set of feasible policies with the proposer in such a way that feasible policies are restricted to the policy jurisdiction of the committee to which the proposer belongs, and so that the assent of the committee, along with the that of the floor, is required for passage.
with the topology of $C^r$-uniform convergence on compacta. To describe this topology, let $\hat{r}$ be a non-negative integer and $Y \subseteq \mathbb{R}^d$, and define the norm $||\phi||_{\hat{r}, Y}$ on $C^r(\mathbb{R}^d, \mathbb{R}^n)$ as $\sup\{||\partial^r \phi(x)|| : x \in Y\}$, where $\partial \phi$ is the $\hat{r}$-th derivative of $\phi$. Then a sequence $\{\phi^m\}$ of functions converges to $\phi$ in $C^r(\mathbb{R}^d, \mathbb{R}^n)$ if and only if for every $\hat{r} = 0, 1, \ldots, r$ and every compact set $Y \subseteq \mathbb{R}^d$, we have $||\phi^m - \phi||_{\hat{r}, Y} \to 0$. We say $\phi^m \to \phi$ in $C^\infty(\mathbb{R}^d, \mathbb{R}^n)$ if and only if it converges in $C^r(\mathbb{R}^d, \mathbb{R}^n)$ for all $r = 0, 1, \ldots$.

Given $v = (v_1, \ldots, v_n) \in C^r(\mathbb{R}^d, \mathbb{R}^n)$, define the induced utility

$$U_i(y, \theta_i; v) = (1 - \delta_i)(u_i(y) + \theta_i \cdot y) + \delta_i v_i(y),$$

where future payoffs are assumed to be generated by $v$, and define the associated acceptance sets

$$A_i(q, \theta; v) = \{y \in X \cup \{q\} : U_i(y, \theta_i; v) \geq U_i(q, \theta_i; v)\}.$$ 

Let $C \subseteq N$ be any coalition and $\mathcal{C} \subseteq 2^N$ any nonempty collection of coalitions, and, following the conventions of Section 3, define

$$A_C(q, \theta; v) = \bigcap_{i \in C} A_i(q, \theta; v) \quad \text{and} \quad A_\mathcal{C}(q, \theta; v) = \bigcup_{C \in \mathcal{C}} \bigcap_{i \in C} A_i(q, \theta; v).$$

When $C = \emptyset$, we adopt the convention that $A_C(q, \theta; v) = X \cup \{q\}$. Lastly, let

$$\max_{y \in A_\mathcal{C}(q, \theta; v)} U_i(y, \theta; v)$$

be the optimal proposal problem of legislator $i$, given status quo $q$ and preference shocks $\theta$, if the collection of decisive coalitions were $\mathcal{C}$ and continuation values were $v$. When $\mathcal{C}$ consists of a single coalition, $C$, we use the obvious shorthand $\mathcal{P}_i(C, q, \theta; v)$, substituting $C$ for $\mathcal{C}$ in the notation defined above. Henceforth, the vector of functions $v$ will be assumed to range over $C^r(\mathbb{R}^d, \mathbb{R}^n)$, unless otherwise restricted.

Our first lemma establishes, among other things, that the legislators’ optimal proposals are essentially unique.

**Lemma 1**

1. For all $\mathcal{C}$, the correspondence $A_\mathcal{C} : \mathbb{R}^d \times \Theta \times C^r(\mathbb{R}^d, \mathbb{R}^n) \rightrightarrows \mathbb{R}^d$ has nonempty, compact values and closed graph in $(q, \theta, v)$,

2. Fix $v \in C^0(\mathbb{R}^d, \mathbb{R}^n)$. For all $i$ and all $\mathcal{C}$, there is a measurable function $\pi^c_i(\cdot; v) : \mathbb{R}^d \times \Theta \to \mathbb{R}^d$ such that for all $q$ and all $\theta$, $\pi^c_i(q, \theta; v)$ solves $\mathcal{P}_i(\mathcal{C}, q, \theta; v)$,

3. Fix $v \in C^0(\mathbb{R}^d, \mathbb{R}^n)$. For all $q$, there is a measure zero set $\Theta_1(q; v) \subseteq \Theta$ such that for all $\theta \notin \Theta_1(q; v)$, all $i$, and all $\mathcal{C}$, $\pi^c_i(q, \theta; v)$ is the unique solution to $\mathcal{P}_i(\mathcal{C}, q, \theta; v)$.

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21 We reserve the notation $D\phi$ for the two-dimensional Jacobian matrix of $\phi$. 
Proof We have $A_{i}(q, \theta; v) \neq \emptyset$ for all $(q, \theta, v)$, as the status quo $q$ belongs to $A_{i}(q, \theta; v)$ for all $i \in N$. By Mas-Colell's (1985) Theorem K.1.2, the function $U_{i}$ is jointly continuous in $(y, \theta_{i}, v)$, and it follows that the correspondence $A_{i}$ has closed graph in $(q, \theta, v)$. Compactness of $A_{i}(q, \theta; v)$ follows since it is a closed subset of $X \cup \{q\}$, a compact set. This completes the proof of part 1. To prove part 2, fix $v \in C^{0}(\mathbb{R}^{d}, \mathbb{R}^{n})$, and consider any $i$ and $C$. Then $U_{i}(\cdot; v)$ is a Carathéodory function, and Aliprantis and Border's (1999) Theorem 17.18 yields a measurable selection $\pi_{i}^{C}(\cdot; v) : \mathbb{R}^{d} \times \Theta \to \mathbb{R}^{d}$ from the correspondence of solutions to $\mathcal{P}_{i}(C, q, \theta; v)$. To prove part 3, fix $v \in C^{0}(\mathbb{R}^{d}, \mathbb{R}^{n})$, and consider any $q$, any $i$, and any $C$. Given preference shocks $\theta_{-i}$, let

$$A_{i}^{-1}(q, \theta_{-i}; v) = \bigcup_{C \in \mathcal{E}} A_{C \setminus \{i\}}(q, \theta; v)$$

denote the set of policies acceptable to all members, except possibly $i$, of some coalition in the collection $\mathcal{E}$. Note that if $y$ solves $\mathcal{P}_{i}(\mathcal{E}, q, \theta; v)$, then it also solves

$$\max_{y} U_{i}(y, \theta_{i}; v)$$

s.t. $y \in A_{i}^{-1}(q, \theta_{-i}; v)$.

Note that if $y \neq y'$, then $D_{\theta_{i}}[U_{i}(y, \theta_{i}; v) - U_{i}(y', \theta_{i}; v)] = y - y' \neq 0$. Thus, Mas-Colell's (1985) Theorem I.3.1 implies that there is a measure zero set $\Theta_{i}^{\mathcal{E}, \theta_{-i}}(q; v) \subseteq \mathbb{R}^{d}$ such that for all $\theta_{i} \notin \Theta_{i}^{\mathcal{E}, \theta_{-i}}(q; v)$, the program $\mathcal{P}_{i}(\mathcal{E}, q, \theta; v)$ admits a unique solution. Then

$$\Theta_{i}^{\mathcal{E}}(q; v) = \bigcup_{\theta_{-i} \in \mathbb{R}^{(n-1)d}} \left( \Theta_{i}^{\mathcal{E}, \theta_{-i}}(q; v) \times \{\theta_{-i}\} \right)$$

is measure zero. Finally, since $N$ is finite,

$$\Theta_{1}(q; v) = \bigcup_{i \in N} \bigcup_{\mathcal{E} \subseteq 2^{N}} \Theta_{i}^{\mathcal{E}}(q; v)$$

is measure zero, as desired. □

Before we state the next lemma, we develop necessary notation and recall some definitions. For the moment, fix continuation values $v$ and status quo $q$. For any subsets $C \subseteq N$ and $L \subseteq K$, define the functions $U^{C} : \mathbb{R}^{d} \times \Theta \to \mathbb{R}^{|C|}$ by $U^{C}(y, \theta; q, v) = (U_{j}(y, \theta_{j}; v) - U_{j}(q, \theta_{j}; v))_{j \in C}$ and $h^{L} : \mathbb{R}^{d} \to \mathbb{R}^{|L|}$ by $h^{L}(y) = (h_{\ell}(y))_{\ell \in L}$. Define the mapping $F^{C,L} : (\mathbb{R}^{d} \setminus \{q\}) \times \Theta \to \mathbb{R}^{|C|+|L|}$ by

$$F^{C,L}(y, \theta; q, v) = \begin{bmatrix} U^{C}(y, \theta; q, v) \\ h^{L}(y) \end{bmatrix},$$

where here (and whenever relevant) we view vectors as column matrices, making $F^{C,L}(y, \theta; q, v)$ a $(|C| + |L|) \times 1$ matrix. Derivatives are expanded via rows, e.g.,
\(D_u U^C(y, \theta; q, v)\) is a \(|C| \times d\) matrix. With regard to the program \(\mathcal{P}_i(C, q, \theta; v)\), consider \(y \in A_C(q, \theta; v)\) and let \(\overline{C} \subseteq C\) and \(\overline{K} \subseteq K\), with \(K^{eq} \subseteq \overline{K}\), represent the voting and feasibility constraints, respectively, that hold with equality at \(y\). We suppress the dependence of these sets on the pair \((q, \theta)\). Taking the coalition \(C\) as fixed, we say that \(y\) satisfies the linear independence constraint qualification (LICQ) at \((q, \theta)\) if \(D_y F_{q,v}^C(y, \theta; q, v)\) has full row rank. The next lemma establishes that for all \(q\) and almost all \(\theta\), LICQ holds at every policy other than the status quo.

**Lemma 2** Fix \(v\). For all \(q\), there exists a measure zero set \(\Theta_2(q; v) \subseteq \Theta\) such that for all \(\theta \notin \Theta_2(q; v)\), all \(i\), and all \(C\), and all \(y \in A_C(q, \theta; v) \setminus \{q\}\), \(y\) satisfies LICQ at \((q, \theta)\).

**Proof** Fix \(v\), and consider any \(q\) and any \(\overline{C} \subseteq N\) and \(\overline{K} \subseteq K\) such that \(\overline{C} \cup \overline{K} \neq \emptyset\). The derivative of the mapping \(F_{\overline{C}, \overline{K}}^C(\cdot; q, v)\) at \((y, \theta)\) \(\in \mathbb{R}^d \setminus \{q\} \times \Theta\) is the \((|\overline{C}| + |\overline{K}|) \times (d + |\overline{C}|d + (n - |\overline{C}|)d)\) matrix

\[
DF_{\overline{C}, \overline{K}}^C(y, \theta; q, v) = \begin{bmatrix}
D_y U^C(y, \theta; q, v) \\
\text{diag}[(1 - \delta_j)_{j \in \overline{C}}] \otimes (y - q)^T \\
Dh^K(y) \\
0 \\
0
\end{bmatrix},
\]

where \(\otimes\) denotes Kronecker product. Since \(y \neq q\) and \(\delta_i < 1\), the rows of \((1 - \delta_i)(y - q)^T \otimes I_{|\overline{C}|}\) are linearly independent. For all \((y, \theta)\) such that \(F_{\overline{C}, \overline{K}}^C(y, \theta; q, v) = 0\), \(\overline{K}\) is contained in the binding feasibility constraints at \(y\), and therefore the rows of \(Dh^K(y)\) are linearly independent by assumption. Thus, \(DF_{\overline{C}, \overline{K}}^C(y, \theta; q, v)\) has full row rank. We conclude that \(F_{\overline{C}, \overline{K}}^C\) is transversal to \(\{0\}\). For each \(\theta\), define \(F_{\overline{C}, \overline{K}}^C_\theta : \mathbb{R}^d \setminus \{q\} \to \mathbb{R}^{|\overline{C}| + |\overline{K}|}\) by \(F_{\overline{C}, \overline{K}}^C_\theta(y; q, v) = F_{\overline{C}, \overline{K}}^C(y, \theta; q, v)\). Note that \(F_{\overline{C}, \overline{K}}^C\) is \(r\)-times continuously differentiable, where \(r \geq d > \max \{0, d - (|\overline{C}| + |\overline{K}|)\}\). Thus, it follows by the transversality theorem that for almost all \(\theta\), \(F_{\overline{C}, \overline{K}}^C_\theta\) is transversal to \(\{0\}\). Let \(\Theta_{2, \overline{C}, \overline{K}}(q; v)\) be the measure zero set of \(\theta\)’s where this does not hold, and let \(\Theta_{2}(q; v)\) be the finite union of these sets over all \(\overline{C}\) and \(\overline{K}\) with \(\overline{C} \cup \overline{K} \neq \emptyset\), which also has measure zero.

The next lemma shows that for all \(q\) and almost all \(\theta\), if any legislator is indifferent between the optimal proposal and the status quo, then all such legislators are necessary in order for the proposal to be approved by a coalition in \(\mathcal{C}\): if one is removed, then the resulting coalition no longer belongs to \(\mathcal{C}\).

**Lemma 3** Fix \(v\). For all \(q\), there is a measure zero set \(\Theta_3(q; v) \subseteq \Theta\) such that for all \(\theta \notin \Theta_3(q; v)\), all \(i\), and all \(\mathcal{C}\), if \(\pi_i^C(q, \theta; v) \neq q\) and we define

\[C^* = \{j \in N : U_j(\pi_i^C(q, \theta; v), \theta_j; v) \geq U_j(q, \theta_j; v)\},\]

then there is an open set \(Z\) containing \((q, \theta)\) such that for all nonempty \(C\) satisfying \(U_j(\pi_i^C(q, \theta; v), \theta_j; v) = U_j(q, \theta_j; v)\) for all \(j \in C\), we have \(C^* \setminus C \notin \mathcal{C}\).

**Proof** Fix \(v\), and consider any \(q\). We claim that there is a measure zero set \(\tilde{\Theta}_3(q; v)\) such that for all \(\theta \notin \tilde{\Theta}_3(q; v)\), all \(i\), all \(C\), and all \(j \notin C \cup \{i\}\), if \(\pi_i^C(q, \theta; v) \neq q\),
implies that removes at least one $C_j$. Consider any coalition $C$. Note that for all $j \notin C \cup \{i\}$, $\pi_i^C(q, \theta_j; v)$ is independent of $\theta_j$. Thus, if $\theta_{-j}$ is such that $\pi_i^C(q, \theta_j; v) = q$, then the preference shocks $\theta_j'$ that solve the equality $U_j(\pi_i^C(q, (\theta_j, \theta_j')_j; v), \theta_j'; v) = U_j(q, \theta_j'; v)$ form a lower dimensional hyperplane in $\mathbb{R}^d$. We infer that there is a measure zero set $\tilde{\Theta}_\pi^{i,C,j,\theta_{-j}}(q; v) \subseteq \mathbb{R}^d$ such that for all $\theta_j \notin \tilde{\Theta}_\pi^{i,C,j,\theta_{-j}}(q; v)$, we have $U_j(\pi_i^C(q, (\theta_j, \theta_j')_j; v), \theta_j'; v) \neq U_j(q, \theta_j'; v)$. Then

$$
\tilde{\Theta}_\pi^{i,C,j}(q; v) = \bigcup \left\{ \theta \in \Theta : \theta_j \in \tilde{\Theta}_\pi^{i,C,j,\theta_{-j}}(q; v), \pi_i^C(q, \theta; v) \neq q \right\}
$$

is measure zero. Since $N$ is finite,

$$
\tilde{\Theta}_3(q; v) = \bigcup \left\{ \tilde{\Theta}_\pi^{i,C,j}(q; v) : i \in N, C \subseteq N, j \notin C \cup \{i\} \right\}
$$
is also measure zero, as desired. We now define $\Theta_3(q; v) = \Theta_1(q; v) \cup \Theta_2(q; v) \cup \tilde{\Theta}_3(q; v)$. Consider any $\theta \notin \Theta_3(q; v)$, any $i$, and any $C$, and suppose that $\pi_i^C(q, \theta; v) \neq q$. Define $C^*$ as in the statement of the lemma. Consider any nonempty $C$ satisfying $U_j(\pi_i^C(q, \theta; v), \theta_j; v) = U_j(q, \theta_j; v)$ for all $j \in C$. Since $\theta \notin \Theta_1(q; v)$, part 3 of Lemma 1 implies that $U_i(\pi_i^C(q, \theta; v), \theta_i; v) > U_i(q, \theta_i; v)$, so that $i \notin C$. Suppose, to obtain a contradiction, that $C^* \subseteq C \subseteq C^* \setminus C \subseteq C$ and take any $j \in C$. Note that $\pi_i^C(q, \theta; v)$ solves $\mathcal{P}_i(C^*, q, \theta; v)$, and since $\mathcal{P}_i(C', q, \theta; v)$ removes at least one constraint, we have $U_i(\pi_i^C(q, \theta; v), \theta_i; v) \geq U_i(\pi_i^C(q, \theta; v), \theta_i; v)$. Since $C^* \subseteq C$, we have $\pi_i^C(q, \theta; v) \in A_\mathcal{E}(q, \theta; v)$. Then, since $\theta \notin \Theta_1(q; v)$, part 3 of Lemma 1 implies $\pi_i^{C'}(q, \theta; v) = \pi_i^C(q, \theta; v) \neq q$. But then $j \notin C' \cup \{i\}$ and $U_j(\pi_i^{C'}(q, \theta; v), \theta_j; v) = U_j(\pi_i^C(q, \theta; v), \theta_j; v) = U_j(q, \theta_j; v)$ contradicts $\theta \notin \tilde{\Theta}_3(q; v)$. We conclude that $C^* \setminus C \notin C'$.

The next lemma shows that, generically, any feasible policy $x$ that is weakly preferred to the status quo by a decisive coalition of legislators can be approximated by feasible policies that are strictly preferred to the status quo by a decisive coalition.

**Lemma 4** Fix $v$. For all $q$, there is a measure zero set $\Theta_4(q; v) \subseteq \Theta$ such that for all $\theta \notin \Theta_4(q; v)$, all $i$, all $C$, and all $y \in A_\mathcal{E}(q, \theta; v) \setminus \{q\}$, there exists a sequence $\{y^m\}$ in $A_\mathcal{E}(q, \theta; v)$ such that $y^m \to y$ and for all $j$ and all $m$, $U_j(y^m, \theta_j; v) \neq U_j(q, \theta_j; v)$.

**Proof** Fix $v$, consider any $q$, and define $\Theta_4(q; v) = \Theta_2(q; v)$. Consider any $\theta \notin \Theta_4(q; v)$, any $i$, any $C$, and any $y \in A_\mathcal{E}(q, \theta; v) \setminus \{q\}$. Let $C^* = \{j \in N : U_j(y, \theta_j; v) \geq U_j(q, \theta_j; v)\}$. Since $y \in A_\mathcal{E}(q, \theta; v)$, we have $C^* \subseteq C$. Let $\overline{C}$ and $\overline{K}$ denote the voting and feasibility constraints, respectively, that bind at $y$ in program $\mathcal{P}_i(C^*, q, \theta; v)$. Since $y \neq q$ and $\theta \notin \Theta_2(q; v)$, $y$ satisfies LICQ at $(q, \theta)$. Define the mapping $F : \mathbb{R}^{d+1} \to \mathbb{R}^{|C| + |K|}$ by

$$
F(x, \epsilon) = \left[ \begin{array}{c}
(U_j(x, \theta_j; v) - \epsilon - U_j(q, \theta_j; v))_{j \in C} \\
(h_\epsilon(x))_{\epsilon \in K}
\end{array} \right],
$$

30
and note that \( F(y, 0) = 0 \). By LICQ, \( D_x F(y, 0) \) has full row rank, and the implicit function theorem (see Loomis and Sternberg (1968)) yields open sets \( P \subseteq \mathbb{R} \) around zero and \( Y \subseteq \mathbb{R}^d \) around \( y \) and a continuous mapping \( \phi: P \to Y \) such that \( \phi(0) = y \) and for all \( \epsilon \in P \), \( F(\phi(\epsilon), \epsilon) = 0 \). Defining the sequence \( \{y^m\} \) by \( y^m = \phi(1/m) \), continuity of \( \phi \) implies \( y^m \to y \). For all \( \ell \in K^m \setminus \overline{K} \), we have \( h_\ell(y) > 0 \), and continuity of \( h_\ell \) then implies that for sufficiently high \( m \), we have \( h_\ell(y^m) > 0 \). And \( F(y^m, 1/m) = 0 \) implies that for all \( \ell \in \overline{K} \), we have \( h_\ell(y^m) = 0 \). Thus, \( y^m \in X \) for sufficiently high \( m \). For all \( j \in C^* \setminus C \), so that \( U_j(y, \theta; v) > U_j(q, \theta; v) \), continuity of \( U_j \) implies that for sufficiently high \( m \), we have \( U_j(y^m, \theta; v) > U_j(q, \theta; v) \). And \( F(y^m, 1/m) = 0 \) implies that for all \( j \in \overline{C} \), we have \( U_j(y^m, \theta; v) - U_j(q, \theta; v) = 1/m > 0 \). Therefore, \( y^m \in A_{C^*}(q, \theta; v) \subseteq A_C(q, \theta; v) \). Furthermore, for all \( j \notin C^* \) such that \( U_j(y, \theta; v) < U_j(q, \theta; v) \), continuity implies that for sufficiently high \( m \), we have \( U_j(y^m, \theta; v) < U_j(q, \theta; v) \). Therefore, we have established the existence of a subsequence \( \{y^m\} \) in \( A_C(q, \theta; v) \), such that \( y^m \to y \) and \( U_j(y^m, \theta; v) \neq U_j(q, \theta; v) \) for all \( j \), as required.

As in Section 4, we now index models by \( \gamma = ((p_i, u_i, \delta_i)_{i \in N}, X, f, g) \), we let \( \Gamma \) denote the metric space of parameterizations satisfying the assumptions of the legislative model, and we continue to assume that the parameterization is continuous in the sense of that section. We define the induced utility \( U^\gamma_i(y, \theta; v) \) in model \( \gamma \) in the obvious way, and it is immediate that \( U_i \) is jointly continuous in \( (y, \theta, v, \gamma) \). Given model \( \gamma \in \Gamma \) and continuation value functions \( v \), Lemma 1 allows us to define measurable mappings \( \pi^\gamma_i(\cdot; v): \mathbb{R}^d \times \Theta \to \mathbb{R}^d \) such that for all \( q \) and almost all \( \theta \), \( \pi^\gamma_i(q, \theta; v) \) solves \( \mathcal{P}^\gamma_i(\Theta, q, \theta; v) \), i.e., it solves the proposer’s optimization problem at \( (q, \theta) \) in model \( \gamma \) when the voting rule is given by \( \Theta \) and continuation values are given by \( v \). We use these optimal proposal mappings to define a best response continuation value mapping \( \psi \) as follows: define \( \psi: C^0(\mathbb{R}^d, \mathbb{R}^n) \times \Gamma \to C^0(\mathbb{R}^d, \mathbb{R}^n) \) by

\[
\psi(v, \gamma)(x) = \int_q \int_{\theta} \sum_j p_j^\gamma U^\gamma_i(\pi^\gamma_i(q, \theta; v), \theta; v) f^\gamma(\theta) g^\gamma(q|x) d\theta dq.
\]

where \( \psi(v, \gamma) \in C^0(\mathbb{R}^d, \mathbb{R}^n) \) follows from the fact that \( \psi(v, \gamma) \) depends on \( x \) only through the density \( g^\gamma(q|x) \), which is continuous.\(^{22}\) When a model \( \gamma \) is fixed, we may write \( \psi^\gamma(v) \) for the value \( \psi(v, \gamma) \).

The next lemma establishes that the domain and range of \( \psi \) can be restricted to a compact space and that the mapping \( \psi \) is continuous on this space. Recall that \( b_f \) bounds \( |u_i(x) + x \cdot \theta_i| \), and \( b_g \) bounds the norms of derivatives of \( g(q|x) \) with respect to \( x \) of all orders, and let \( b_h \) be the Lebesgue measure of \( \bar{X} \). Without loss of generality, assume \( b_g, b_h \geq 1 \). Define \( \mathcal{V} \) to consist of functions \( v \in C^0(\mathbb{R}^d, \mathbb{R}^n) \) such that (i) if \( r < \infty \), then the derivatives of \( v \) of order \( 0, 1, \ldots, r \) are bounded in norm by \( \sqrt{nb_f b_g b_h} \), and the \( r \)-th derivative of \( v \) is Lipschitz continuous with modulus \( \sqrt{nb_f b_g b_h} \); and (ii)

\(^{22}\)This follows from a stronger result established in part 2 of Lemma 5.
if $r = \infty$, then the derivatives of $v$ of all orders $0, 1, 2, \ldots$ are bounded in norm by $\sqrt{m b_f b_g b_h}$. Denote by $M(\mathbb{R}^d, \mathbb{R}^n)$ the set of Borel measurable mappings from $\mathbb{R}^d$ to $\mathbb{R}^n$.

**Lemma 5**

1. The space $\mathcal{V}$ is nonempty, convex, and compact.

2. Consider $\gamma \in \Gamma$ and $\phi \in M(\mathbb{R}^d, \mathbb{R}^n)$ such that for all $i$, $\phi_i$ is bounded in absolute value by $b_f$ over $\bar{X}$. Define the mapping $\hat{\phi} \in M(\mathbb{R}^d, \mathbb{R}^n)$ by $\hat{\phi}(x) = \int_q \phi(q) g^{\gamma}(q|x) dq$ for all $x$. Then $\hat{\phi} \in \mathcal{V}$.

3. The mapping $\psi: \mathcal{V} \times \Gamma \to \mathcal{V}$ is continuous.

**Proof** The proof of part 1 is brief. Clearly, $\mathcal{V}$ is nonempty and convex. We claim that it is compact. In case $r < \infty$, Mas-Colell’s (1985) Theorem K.2.2 implies that $\mathcal{V}$ is compact in the topology of $C^r$-uniform convergence on compacta. In case $r = \infty$, compactness of $\mathcal{V}$ in the topology of $C^\infty$-uniform convergence on compacta follows from Mas-Colell’s Theorems K.2.2.1 and K.2.2.2.

For part 2, consider any $\gamma \in \Gamma$ and $\phi \in M(\mathbb{R}^d, \mathbb{R}^n)$ such that for all $i$, $\phi_i$ is bounded in absolute value by $b_f$. Define $\hat{\phi}$ as in the statement of the lemma. By Aliprantis and Burkinshaw’s (1990) Theorem 20.4, each function $\hat{\phi}_i$ is partially differentiable. Let $\partial^\alpha$ denote a partial derivative operator with respect to the coordinates of $x$ of any order $\hat{r} = 1, 2, \ldots, r$, with multi-index $\alpha$. Aliprantis and Burkinshaw’s result, with the expression in (1), implies that

$$\partial^\alpha \hat{\phi}(x) = \int_q \phi(q) \partial^\alpha g^{\gamma}(q|x) dq. \tag{5}$$

Since this depends on $x$ only through $\partial^\alpha g^{\gamma}(q|x)$, which is continuous, it follows that $\partial^\alpha \hat{\phi}_i$ is continuous. Indeed, consider a sequence $\{x^n\}$ in $\mathbb{R}^d$ converging to $x$. Then the integrand in $\partial^\alpha \hat{\phi}_i(x^n)$, as a function of $q$, converges pointwise to the integrand in $\partial^\alpha \hat{\phi}_i(x)$. Furthermore, we assume that the $\hat{r}$-th derivative of $g^{\gamma}(q|x)$ with respect to $x$ is bounded in norm by $b_g$, which implies $|\partial^\alpha g^{\gamma}(q|x^n)| \leq b_g$ for all $n$. Since the support of $g^{\gamma}(\cdot|x)$ lies in $\bar{X}$, a compact set, it follows that $\partial^\alpha g^{\gamma}(\cdot|x)$ is identically zero outside $\bar{X}$. Therefore, since $\phi$ is bounded in absolute value by $b_f$ on $\bar{X}$, we have $|\phi(q)\partial^\alpha g^{\gamma}(q|x)| \leq b_f b_g I_{\bar{X}}(q)$ for all $q$, and the claimed continuity follows from Lebesgue’s dominated convergence theorem. Therefore, $\hat{\phi}$ is $r$-times continuously differentiable. To prove that $\hat{\phi} \in \mathcal{V}$, first suppose $r < \infty$, and let $\partial$ be a derivative operator of order $\hat{r} = 0, 1, \ldots, r$, where we view $\partial \hat{\phi}(x)$ as a $n \times d^\hat{r}$ matrix and $\partial g^{\gamma}(q|x)$ as a $1 \times d^\hat{r}$ row vector. Then, viewing $\hat{\phi}(x)$ and $\phi(q)$ as $n \times 1$ column vectors, we have from (5) that $\partial \hat{\phi}(x) = \int_q \phi(q) \partial g^{\gamma}(q|x) dq$, and consequently,

$$||\partial \hat{\phi}(x)|| \leq \int_q ||\phi(q)\partial g^{\gamma}(q|x)|| dq \leq \int_q ||\phi(q)|| ||\partial g^{\gamma}(q|x)|| dq,$$
where the first inequality follows from Jensen's inequality and the second follows from Aliprantis and Border's (1999) Lemma 6.6. Note that \(|\phi(q)|| \leq \sqrt{nb_f}\). Again, \(\partial g^\gamma(x)\) is identically zero outside \(X\), and we therefore have

\[
||\partial \hat{\phi}(x)|| \leq \int_{\hat{X}} \sqrt{nb_f} ||\partial g^\gamma(q|x)|| dq \leq \int_{\hat{X}} \sqrt{nb_f} b_g dq = \sqrt{nb_f} b_g b_h.
\]

Let \(\partial\) be the \(r\)-th order derivative with respect to \(x\), and note that for all \(x\) and \(y\),

\[
||\partial \hat{\phi}(x) - \partial \hat{\phi}(y)|| = \left|\left| \int_{q} \phi(q)(\partial g^\gamma(q|x) - \partial g^\gamma(q|y)) dq \right|\right|
\leq \int_{q} ||\phi(q)|| ||\partial g^\gamma(q|x) - \partial g^\gamma(q|y)|| dq
\leq \int_{\hat{X}} \sqrt{nb_f} ||\partial g^\gamma(q|x) - \partial g^\gamma(q|y)|| dq
\leq \sqrt{nb_f} b_g b_h ||x - y||,
\]

where the last inequality follows from our assumption that the \(r\)-th derivative of \(g^\gamma(q|x)\) with respect to \(x\) is Lipschitz continuous with modulus \(b_g\) and the Lebesgue measure of \(\hat{X}\) is \(b_h\). Therefore, \(\partial \hat{\phi}\) is Lipschitz continuous with modulus \(\sqrt{nb_f} b_g b_h\), fulfilling (i). Now suppose \(r = \infty\), and consider any \(\hat{r} \geq 1\). As argued above, we have

\[
||\partial \hat{\phi}(x)|| \leq \sqrt{nb_f} b_g b_h,
\]

fulfilling (ii) and implying \(\hat{\phi} \in \mathcal{V}\).

For part 3, first consider any \((v, \gamma) \in \mathcal{V} \times \Gamma\), and define the measurable mapping \(w: \mathbb{R}^d \rightarrow \mathbb{R}^n\) by

\[
w_i(q) = \int_{\theta} \sum_j p_j \gamma U_i^\gamma(\pi_j^\gamma(q, \theta; v), \theta; v) f^\gamma(\theta) d\theta
\]

(6)

for all \(i\) and all \(q\). Recall that \(|u_i^\gamma(q + \theta_i \cdot x)| f^\gamma(\theta) \leq b_f\) for all \(i\), all \(\theta \in \Theta\), and all \(x \in \hat{X}\). Since \(\pi_j^\gamma(q, \theta; v) \in X \cup \{q\}\), we then have for all \(i\) and all \(q \in \hat{X}\),

\[
w_i(q) \leq \int_{\theta} \sum_j p_j \gamma \left[ (1 - \delta_j^\gamma)|u_i^\gamma(q, \theta; v)| \right]
\leq \int_{\theta} \sum_j p_j \gamma \left[ \theta_i \cdot \pi_j^\gamma(q, \theta; v) \right]
\leq \delta_j^\gamma |v_i(\pi_j^\gamma(q, \theta; v))| f^\gamma(\theta) d\theta \leq b_f.
\]

Noting that \(\psi(v, \gamma)(x) = \int_q w(q) g^\gamma(q|x) dq\), it follows from part 2 of the lemma that \(\psi(v, \gamma) \in \mathcal{V}\). We conclude that \(\psi: \mathcal{V} \times \Gamma \rightarrow \mathcal{V}\), as desired.

To prove continuity, consider sequences \(\{v^m\} \in \mathcal{V}\) and \(\{\gamma^m\} \in \Gamma\) with \(v^m \rightarrow v^* \in C^\infty(\mathbb{R}^d, \mathbb{R}^n)\) and \(\gamma^m \rightarrow \gamma^* \in \Gamma\). Write \(\gamma^m = ((p_i^m, u_i^m, \delta_i^m)_{i \in N}, X^m, f^m, g^m)\) and \(\gamma^* = ((p_i^*, u_i^*, \delta_i^*)_{i \in N}, X^*, f^*, g^*)\). We use superscript \(m\) for variables corresponding to model \(\gamma^m\), and we use a superscript asterisk for variables corresponding to \(\gamma^*\). We
first note that there exists a compact set $\hat{X} \subseteq \mathbb{R}^d$ such that for sufficiently high $m$, $X^m \subseteq \hat{X}$. Indeed, $X^*$ is compact by assumption. Letting $B$ be the closure of an open ball of finite, positive radius, it follows that $X^* + B$ is compact. Since $X^m \to X^*$ Hausdorff, we have $X^m \subseteq X^* + B$ for high enough $m$.

We claim that for all $q$, all $i$, and all $\theta \not\in \Theta_i^*(q; v^*) \cup \Theta_i^+(q; v^*)$, $\pi^m_i(q, \theta; v^m) \to \pi^*_i(q, \theta; v^*)$. If not, then because $\pi^m_i(q, \theta; v^m)$ lies in the compact set $\hat{X}$ for high enough $m$, we may go to a subsequence, still indexed by $m$, such that $\pi^m_i(q, \theta; v^m) \to x \neq \pi^*_i(q, \theta; v^*)$. Since $\pi^m_i(q, \theta; v^m) \in A_	heta^m(q, \theta; v^m)$ for all $m$, part 1 of Lemma 1 implies that $x \in A_	heta^m(q, \theta; v^*)$. And since $\theta \not\in \Theta_i^*(q; v^*)$, part 3 of Lemma 1 implies that $U^*_i(\pi^*_i(q, \theta; v^*), \theta; v^*) > U^*_i(x, \theta; v^*)$. We consider two cases. First, suppose $\pi^*_i(q, \theta; v^*) \neq q$, so by $\theta \not\in \Theta_i^*(q; v^*)$, Lemma 4 implies that there exists $y \in A^*(\pi^*_i(q, \theta; v^*))$ arbitrarily close to $\pi^*_i(q, \theta; v^*)$ such that $U^*_i(y, \theta; v^*) \neq U^*_i(x, \theta; v^*)$ for all $j$. Thus, there exists a decisive coalition $C \in \mathcal{D}$ such that $U^*_j(y, \theta; v^*) > U^*_j(q, \theta; v^*)$ for all $j \in C$. Furthermore, by $U^*_i(\pi^*_i(q, \theta; v^*), \theta; v^*) > U^*_i(x, \theta; v^*)$ and continuity of $U^*_i$, we may suppose $U^*_i(y, \theta; v^*) > U^*_i(x, \theta; v^*)$. Since $y \in X^*$, and since $X^m \to X^*$ Hausdorff, there exists a sequence $\{y^m\}$ in $\mathbb{R}^d$ such that $y^m \in X^m$ for all $m$ and $y^m \to y$.

By joint continuity, we then have for all $j \in C$ and for high enough $m$, $U^m_j(y^m, \theta; v^m) > U^m_j(q, \theta; v^m)$, implying $y^m \in A^m_j(q, \theta; v^m)$. But by joint continuity, we also have $U^m_i(y^m, \theta; v^m) > U^m_i(\pi^*_i(q, \theta; v^m), \theta; v^m)$ for high enough $m$, contradicting the fact that $\pi^*_i(q, \theta; v^*)$ solves $\mathcal{P}_m^i(D, q, \theta; v^m)$. For the second case, suppose $\pi^*_i(q, \theta; v^*) = q$. Then $\pi^*_i(q, \theta; v^*) = q \in A^m_i(q, \theta; v^m)$ for all $m$. By joint continuity, we have $U^m_i(q, \theta; v^m) > U^m_i(\pi^*_i(q, \theta; v^m), \theta; v^m)$ for high enough $m$, again contradicting the fact that $\pi^*_i(q, \theta; v^*)$ solves $\mathcal{P}_m^i(D, q, \theta; v^m)$. This establishes the claim.

We next claim that for all $i$ and all $\theta$, $\{U^m_i(\cdot, \theta; v^m)\}$ converges uniformly to $U^*_i(\cdot, \theta; v^*)$ on any compact set $Y \subseteq \mathbb{R}^d$. If not, then there exists $\epsilon > 0$ and a sequence $\{x^m\}$ in $Y$ such that

$$|(1 - \delta^m_i)(u^m_i(x^m) + \theta_i \cdot x^m) + \delta^m_i v^m_i(x^m) - (1 - \delta^*_i)(u^*_i(x^m) + \theta_i \cdot x^m) - \delta^*_i v^*_i(x^m)|$$

for all $m$. By compactness of $Y$, we may go to a convergent subsequence, still indexed by $m$, with $x^m \to x \in Y$. But $v^m \to v$ uniformly, and with continuity of our parameterization, we have

$$\lim_{m \to \infty} (1 - \delta^m_i)(u^m_i(x^m) + \theta_i \cdot x^m) + \delta^m_i v^m_i(x^m) = (1 - \delta^*_i)(u^*_i(x) + \theta_i \cdot x) + \delta^*_i v^*_i(x) = \lim_{m \to \infty} (1 - \delta^*_i)(u^*_i(x^m) + \theta_i \cdot x^m) + \delta^*_i v^*_i(x^m),$$

a contradiction. This establishes the claim.

Finally, let $\hat{v}^m = \psi(v^m, r^m)$ and $\hat{v}^* = \psi(v^*, r^*)$. Let $\partial$ denote a derivative operator with respect to the coordinates of $x$ of any order $\hat{r} = 0, 1, \ldots, r$. Consider any compact
The statement of Theorem 5 implicitly fixes a model $\sigma$ strategy profile $\psi$ for all $i$. By part 1 of Lemma 5, $U_i^m(\pi^m_i(q, \theta; v^m), \theta_i; v^m) f^m(\theta) \partial g^m(q|x^m) dq$. Consider the generic case of $(q, \theta)$ such that for all $j$, $\pi^m_j(q, \theta; v^m) \rightarrow \pi^*_j(q, \theta; v^*)$. By uniform convergence, from our preceding claim, we have $U_i^m(\pi^m_i(q, \theta; v^m), \theta_i; v^m) \rightarrow U^*_i(\pi^*_i(q, \theta; v^*), \theta_i; v^*)$. This gives us pointwise convergence of the integrand of $\partial v^m_i(x^m)$ for almost all $(q, \theta)$. Furthermore, since $\partial g^m(q|x^m)$ is zero outside $\bar{X}$ and since $v^m \in \mathcal{U}$, the terms in the above sequence are bounded in norm by the Lebesgue integrable function $b_i b_j b_k I_{\bar{X}}$. By Lebesgue’s dominated convergence theorem, and again using Aliprantis and Burkinshaw’s (1990) Theorem 20.4, we therefore have $\partial v^m_i(x^m) \rightarrow \int_q \int_\theta \sum_j p_j^m U_i^m(\pi^*_j(q, \theta; v^*), \theta_i; v^*) f^*(\theta) \partial g^*(q|x) d\theta dq = \partial \hat{v}_i^*(x)$. By continuity of $\partial \hat{v}_i^*$, we also have $\partial \hat{v}_i^*(x^m) \rightarrow \partial \hat{v}_i^*(x)$, but then $|\partial v^m_i(x^m) - \partial \hat{v}_i^*(x)| \rightarrow 0$. Since $i$ was arbitrary, we have $||\partial v^m_i(x^m) - \partial \hat{v}_i^*(x)|| \rightarrow 0$, a contradiction. We conclude that $\{\partial v^m_i\}$ converges to $\partial \hat{v}$ uniformly on $Y$, and therefore $\hat{v}^m \rightarrow \hat{v}$, as required.

We can at last turn to the proof of Theorem 1.

**Proof of Theorem 1** The statement of Theorem 1 implicitly fixes a model $\gamma \in \Gamma$. By part 1 of Lemma 5, $\mathcal{U}$ is nonempty, convex, and compact. By part 3 of Lemma 5, $\psi^\gamma$ maps $\mathcal{U}$ to $\mathcal{U}$ and the mapping $\psi^\gamma: \mathcal{U} \rightarrow \mathcal{U}$ is continuous. Therefore, Glicksberg’s theorem yields a fixed point $v^* \in \mathcal{U}$ such that $\psi^\gamma(v^*) = v^*$. We then construct equilibrium strategies as follows: for all $i$, we specify $\pi_i(q, \theta) = \pi^\gamma_i(q, \theta; v^*)$, and we specify $\alpha_i(y, q, \theta) = 1$ if $y \in A_i(q, \theta; v^*)$ and $\alpha_i(y, q, \theta) = 0$ otherwise. Evidently, the strategy profile $\sigma = (\pi_i, \alpha_i)_{i \in N}$ so defined is a pure stationary legislative equilibrium. Part 1 of Theorem 1 follows from $v^* \in \mathcal{U}$, and parts 2, 3, and 4 follow from part 3 of Lemma 1, part 3 of Lemma 3, and part 1 of Lemma 3, respectively.
The proof of existence in Theorem 1 relied on the fact that every fixed point of \( \psi^\gamma \) corresponds to a stationary legislative equilibrium in model \( \gamma \). Our final lemma establishes the converse: every pure strategy equilibrium continuation value \( v = \psi(v) \). Furthermore, all equilibrium continuation values of \( \gamma \) lie in \( \Psi \).

**Lemma 6** For all \((v, \gamma) \in M(\mathbb{R}^d, \mathbb{R}^n) \times \Gamma\), if \( v \in E(\gamma) \), then \( v \in \Psi \) and \( v = \psi^\gamma(v) \).

**Proof** Let \((v, \gamma) \in M(\mathbb{R}^d, \mathbb{R}^n) \times \Gamma\) such that \( v \in E(\gamma) \), and let \( \sigma \) be the stationary legislative equilibrium generating \( v \), so that \( v = v(\cdot; \sigma). \) As in the proof of part 3 of Lemma 5, define the measurable mapping \( w: \mathbb{R}^d \to \mathbb{R}^n \) by (6) for all \( i \) and all \( q \), so that \( v(x) = \int w(q)g^\gamma(q|x)dq \) for all \( x \). As argued in the proof of part 3 of Lemma 5, part 2 of that lemma then implies that \( v \in \Psi \). Part 3 of Lemma 1 therefore implies that for all \( i \) and almost all \((q, \theta)\), we have \( \pi_i(q, \theta) = \pi_i^\gamma(q, \theta; v) \). This in turn implies that \( v = \psi(v, \gamma) \).

We now complete the proofs of the remaining results of the paper.

**Proof of Theorem 2** The statement of Theorem 2 implicitly fixes a model \( \gamma \in \Gamma \), which we suppress notationally. Consider an arbitrary mixed stationary legislative equilibrium \( \pi \), and let the measurable mapping \( v: \mathbb{R}^d \to \mathbb{R}^n \) be defined by the equilibrium continuation values as \( v(x) = (v_1(x; \pi), \ldots, v_n(x; \pi)) \). To facilitate the proof, define

\[
W_i(y, q, \theta; \pi) = \pi(y, q, \theta; \pi) U_i(y, \theta; \pi) + (1 - \pi(y, q, \theta; \pi)) U_i(q, \theta; \pi)
\]

as the objective function of the proposer given strategy profile \( \pi \).

Now consider any \( q \), and set \( \Theta(q) = \Theta_1(q; v) \cup \Theta_4(q; v) \). Consider any \( \theta \notin \Theta(q) \). Since \( \theta \notin \Theta_1(q; v) \), part 3 of Lemma 1 implies that \( \pi^\theta_i(q, \theta; v) \) is the unique solution to \( \mathcal{P}_i(\mathcal{P}, q, \theta; v) \). We consider two cases. First, suppose that \( \pi^\theta_i(q, \theta; v) = q \). If we have \( \int_{X \setminus \{q\}} \pi(y, q, \theta; \pi) \pi_i(q, \theta)(dy) > 0 \), then there is a set \( Y \subseteq X \setminus \{q\} \) such that \( \pi_i(q, \theta)(Y) > 0 \) and for all \( y \in Y \), \( \pi(y, q, \theta; \pi) > 0 \). By definition of equilibrium, the latter implies \( Y \subseteq A^\theta(q, \theta; v) \). Then \( U_i(q, \theta; \pi) > U_i(q, \theta; \pi) \) for all \( y \in Y \), which implies \( W_i(y, q, \theta; \pi) < U_i(q, \theta; \pi) = W_i(q, q, \theta; \pi) \) for all \( y \in Y \), contradicting the fact that \( \pi_i \) places probability one on maximizers of \( W_i(\cdot, q, \theta; \pi) \). Therefore, \( \int_{X \setminus \{q\}} \pi(y, q, \theta; \pi) \pi_i(q, \theta)(dy) = 0 \). Second, suppose \( \pi^\theta_i(q, \theta; v) \neq q \). We claim that

\[
\sup_{y \in X} W_i(y, q, \theta; \pi) \geq U_i(\pi^\theta_i(q, \theta; v), \theta_i; v).
\]

To see this, note that since \( \theta \notin \Theta_4(q; v) \), Lemma 4 yields a sequence \( \{y^m\} \) in \( X \) such that \( y^m \to \pi^\theta_i(q, \theta; v) \) and for all \( m \), there is a decisive coalition \( C^m \) satisfying \( U_j(y^m, \theta_j; v) > U_j(q, \theta_j; v) \) for all \( j \in C^m \). By definition of equilibrium, it then follows that \( \pi_j(y^m, q, \theta_j) = 1 \) for all \( j \in C^m \), which implies \( \pi(y^m, q, \theta; \pi) = 1 \). By continuity, we then have \( W_i(y^m, q, \theta; \pi) = U_i(y^m, \theta_i; v) \to U_i(\pi^\theta_i(q, \theta; v), \theta_i; v) \), as claimed. Thus, by definition of equilibrium, the mixed proposal strategy \( \pi_i \).
must achieve an expected payoff of at least $U_i(\pi^\varphi_i(q, \theta; v), \theta; v)$. Next, we claim that $\pi_i(q, \theta)(\{\pi^\varphi_i(q, \theta; v)\}) = 1$ and $\overline{\alpha}(\pi^\varphi_i(q, \theta; v), q, \theta; \overline{\sigma}) = 1$. Consider any $y \neq \pi^\varphi_i(q, \theta; v)$, and note that if $y \notin A(q, \theta; v)$, then $\pi_i(y, q, \theta; \overline{\sigma}) = 0$, which implies $W_i(y, q, \theta; \overline{\sigma}) = U_i(q, \theta; v) < U_i(\pi^\varphi_i(q, \theta; v), \theta; v)$. And if $y \in A(q, \theta; v) \setminus \{q\}$, then $U_i(\pi^\varphi_i(q, \theta; v), \theta; v) > \max\{U_i(y, \theta; v), U_i(q, \theta; v)\}$, which implies $W_i(y, q, \theta; \overline{\sigma}) < U_i(\pi^\varphi_i(q, \theta; v), \theta; v)$. Therefore, we conclude that $\pi_i(q, \theta)$ indeed puts probability one on $\pi^\varphi_i(q, \theta; v)$. If we had $\overline{\alpha}(\pi^\varphi_i(q, \theta; v), q, \theta; \overline{\sigma}) \neq 1$, then $U_i(\pi^\varphi_i(q, \theta; v), \theta; v) > U_i(q, \theta; v)$ would imply $W_i(\pi^\varphi_i(q, \theta; v), q, \theta; \overline{\sigma}) < U_i(\pi^\varphi_i(q, \theta; v), \theta; v)$, contradicting our previous claim. Thus, conclude $\overline{\alpha}(\pi^\varphi_i(q, \theta; v), q, \theta; \overline{\sigma}) = 1$, as desired.

Finally, we specify pure proposal strategies by $\pi_i(q, \theta) = \pi^\varphi_i(q, \theta; v)$, and we specify pure voting strategies by $\alpha_i(y, q, \theta) = 1$ if $y \notin A(q, \theta; v)$ and $\alpha_i(y, q, \theta) = 0$ otherwise. The pure stationary strategy profile $\sigma = (\pi_i, \alpha_i)_{i \in N}$ generates the same policy outcomes as $\overline{\sigma}$ for almost all $(q, \theta)$ and, therefore, the same continuation values. By construction, proposal and voting strategies satisfy the equilibrium conditions of Section 3, and therefore $\sigma$ is a pure stationary legislative equilibrium. Evidently, $\overline{\sigma}$ is equivalent to $\sigma$, and by Lemma 6, the equilibrium continuation value function $v$ lies in $\mathcal{V}$ and is a fixed point of $\psi^\varphi$. Then the property of part 1 of Theorem 1 follows immediately, and the properties of parts 2, 3, and 4 follow from part 3 of Lemma 1, part 3 of Lemma 3, and part 1 of Lemma 3, respectively. \hfill \Box

**Proof of Theorem 3** Consider sequences $\{\gamma^m\}$ in $\Gamma$ and $\{v^m\}$ in $C^\varphi(\mathbb{R}^d, \mathbb{R}^n)$ such that $\gamma^m \rightarrow \gamma \in \Gamma$, $v^m \rightarrow v \in C^\varphi(\mathbb{R}^d, \mathbb{R}^n)$, and for all $m$, $v^m \in E(\gamma^m)$. By Lemma 6, we have $v^m = \psi(v^m, \gamma^m)$ for all $m$. Taking limits, we have $v^m \rightarrow v$ and, by part 3 of Lemma 5, $\psi(v^m, \gamma^m) \rightarrow \psi(v, \gamma)$. Thus, $v = \psi(v, \gamma)$, which implies $v \in E(\gamma)$, establishing closed graph of $E$. By Lemma 6, the range of $\psi$ lies in $\mathcal{V}$, a compact space, and therefore closed graph of $E$ implies upper hemicontinuity. \hfill \Box

**Proof of Theorem 4** Let $\sigma^*$ be a stationary legislative equilibrium. To establish that the transition probability $P$ satisfies the Feller property, consider any $\text{Borel}$ measurable set $Y \subseteq \mathbb{R}^d$, and let $I_Y$ be the indicator function of $Y$. Then by arguments in the proof of part 2 of Lemma 5, it follows that the function $TI_Y$, defined by

$$TI_Y(x) = P(x, Y) = \int_q \int_\theta \sum_i p_i I_Y(\pi^\varphi_i(q, \theta)) f(\theta) g(q|x) d\theta dq,$$

is continuous. If $\phi: \mathbb{R}^d \rightarrow \mathbb{R}$ is a simple function, i.e., $\phi(x) = \sum_{l=1}^L a_l I_{Y_l}(x)$ for a measurable partition $\{Y_1, \ldots, Y_m\}$ of $\mathbb{R}^d$ and coefficients $\{a_1, \ldots, a_m\}$, then $T\phi$ is likewise continuous. Now suppose $\phi: \mathbb{R}^d \rightarrow \mathbb{R}$ is an arbitrary bounded, continuous function. Consider any sequence $\{x^m\}$ in $\mathbb{R}^d$ with limit $x$. By Aliprantis and Border’s (1999) Theorem 11.6, for all $\epsilon > 0$, there exist simple functions $\phi^1, \phi^2: \mathbb{R}^d \rightarrow \mathbb{R}$ such that $\phi^1 \leq \phi \leq \phi^2$ and $\int (\phi^2(z) - \phi^1(z)) P(x, dz) < \epsilon$, which implies $T\phi^1 \leq T\phi \leq T\phi^2$ and $T\phi^2(x) - T\phi^1(x) < \epsilon$. By continuity, we have $|T\phi^1(x^m) - T\phi^1(x)| < \epsilon$ and
\[ |T\phi^2(x^m) - T\phi^2(x)| < \epsilon \text{ for } m \text{ high enough. Therefore, we have} \]
\[
T\phi(x) - T\phi(x^m) \leq T\phi^2(x) - T\phi^1(x^m) \\
= T\phi^2(x) - T\phi^1(x) + T\phi^1(x) - T\phi^1(x^m) \\
< 2\epsilon,
\]
with an analogous derivation establishing that \( |T\phi(x) - T\phi(x^m)| < 2\epsilon \) for high enough \( m \). Since \( \epsilon \) is arbitrarily small, it follows that \( T\phi \) is continuous. Since \( P(x, X) = 1 \) for all \( x \in \mathbb{R}^d \), it follows that \( P \) is tight. By Futia’s (1982) Theorem 2.9, therefore, \( T \) admits an invariant distribution, delivering part 1.

Assume for the remainder of the proof that \( g \) is bounded on \( \bar{X} \times \bar{X} \) by some \( M \in \mathbb{R} \). For all \( x \in X \) and all measurable \( Z \subseteq X \times \Theta \), let \( Q(x, Z) = \int_Z g(q|x)f(\theta)d(q, \theta) \) denote the probability that next period’s \( (q, \theta) \) lies in \( Z \), conditional on policy choice \( x \) this period. To verify Doeblin’s condition, define the finite Borel measure \( \eta \) on \( \mathbb{R}^d \) by
\[
\eta(Y) = \int_{\pi^{-1}(Y) \cap (\bar{X} \times \Theta)} f(\theta)d\theta dq.
\]
(Here, we integrate the status quo \( q \) with respect to Lebesgue measure.) Set \( \epsilon = \frac{1}{1+M} \), and consider any \( x \in \mathbb{R}^d \) and any measurable \( Y \subseteq \mathbb{R}^d \). Note that \( \eta(Y) \leq \epsilon \) implies \( M\eta(Y) \leq \frac{M}{1+M} \), and furthermore, we have
\[
P(x, Y) = \sum_{j \in N} p_j Q(x, \pi_j^{-1}(Y)) = \sum_{j \in N} p_j Q(x, \pi_j^{-1}(Y) \cap (\bar{X} \times \Theta)) \leq M\eta(Y) \leq 1 - \epsilon,
\]
where we use the assumption that the support of \( g(\cdot|x) \) lies in \( \bar{X} \). Therefore, \( P(x, Y) \leq 1 - \epsilon \), establishing Doeblin. By Futia’s (1982) Theorem 4.9, the Markov operator \( T \) is quasi-compact, and it follows that the adjoint \( T^* \) is also quasi-compact. (See Futia (1982), proof of Theorem 3.3.) For an arbitrary initial distribution \( \mu \), Futia’s (1982) Theorems 3.2 and 3.4 then yield convergence to an invariant distribution \( \mu^* \) at the rate claimed in part 2.

Suppose that for all \( x \in \bar{X} \), we have \( g(x|x) > 0 \). Let \( C_1, \ldots, C_\beta \) be pairwise disjoint, measurable sets such that for all \( j = 1, \ldots, \beta \) and all \( x \in C_k \), we have \( P(x, C_{j+1 \mod \beta}) = 1 \), and suppose \( \beta > 2 \). Let \( C_j \) denote the closure of \( C_j \), which then inherits compactness of \( \bar{X} \). We first claim that for \( j \neq \ell \), we have \( C_j \cap C_\ell = \emptyset \). Otherwise, consider \( x \in C_j \cap C_\ell \). Since \( g(x|x) > 0 \) and \( g \) is continuous, we can choose \( x_j \in C_j \), \( x_\ell \in C_\ell \), and an open set \( G \) containing \( x \) such that for all \( q \in X \), we have \( g(q|x_j) > 0 \) and \( g(q|x_\ell) > 0 \). Let \( i \) satisfy \( p_i > 0 \). Because \( P(x_j, C_{j+1 \mod \beta}) = 1 \), it follows that \( G \times \Theta \) contains a measure zero set \( Z_j \) such that for all \( (q, \theta) \in (G \times \Theta) \setminus Z_j \), we have \( \pi_i(q, \theta) \in C_{j+1 \mod \beta} \). Similarly, \( G \times \Theta \) contains a measure zero set \( Z_\ell \) such that for all \( (q, \theta) \in (G \times \Theta) \setminus Z_\ell \), we have \( \pi_i(q, \theta) \in C_{\ell+1 \mod \beta} \). But \( G \times \Theta \) has positive measure, so there exists \( (q, \theta) \in G \times \Theta \) such that \( \pi_i(q, \theta) \in C_{j+1 \mod \beta} \cap C_{\ell+1 \mod \beta} \).
a contradiction. By Mas-Colell’s (1985) Theorem I.3.1, there exists $\theta_i$ with positive marginal density such that $U_i(y, \theta_i; \sigma^*)$ has a unique maximizer on the set $\bigcup_{j=1}^{\beta} \overline{C_j}$, say $x^* \in \overline{C_j}$. In particular, we have shown

$$U_i(x^*, \theta_i; \sigma^*) > \max \left\{ U_i(y, \theta_i; \sigma) : y \in \bigcup_{\ell \neq j} C_\ell \right\}.$$  \hfill (7)

and by continuity of $U_i(\cdot; \sigma^*)$, there is an open set $G$ containing $\theta_i$ such that the strict inequality in (7) continues to hold for all $\theta_i \in G$. Furthermore, since $g(x^*|x^*) > 0$ and $g$ is continuous, there is an open set $H$ containing $x^*$ such that for all $\theta_i \in G$ and all $q \in H$, we have $g(q|x^*) > 0$ and

$$U_i(q, \theta_i; \sigma^*) > \max \left\{ U_i(y, \theta_i; \sigma) : y \in \bigcup_{\ell \neq j} C_\ell \right\}.$$  \hfill (8)

We claim that for all $(q, \theta')$ such that $\theta_i \in G$ and $q \in H$, we have $\pi_i(q, \theta') \notin C_{j+1} \bmod \beta$. Indeed, optimality of $\pi_i(q, \theta')$ implies $U_i(\pi_i(q, \theta'), \theta_i; \sigma^*) \geq U_i(q, \theta_i; \sigma^*)$, and then (8) implies $\pi_i(q, \theta') \notin C_{j+1} \bmod \beta$. Using continuity of $g$ and $x^* \in \overline{C_j}$, there exists $\bar{x} \in C_j$ such that for all $q \in H$, we have $g(q|\bar{x}) > 0$. But then with probability $Q(\bar{x}, H \times G \times \mathbb{R}^{(n-1)d}) > 0$, we have $\pi_i(q, \theta') \notin C_{j+1} \bmod \beta$, contradicting $P(\bar{x}, C_{j+1} \bmod \beta) = 1$. Thus, $P$ is aperiodic, and Doob’s (1953) Case (f) obtains. Since Doeblin’s condition holds, we can partition $\bar{X}$ into a finite number of ergodic sets $E_1, \ldots, E_\alpha$ and a transient set $(\bar{X} \setminus \bigcup_{j=1}^{\alpha} E_j)$, and for every ergodic set $E_j$ there is a unique invariant probability measure $\xi_j$ such that $\xi_j(E_j) = 1$ (Doob (1953), pp. 210–211). By Doob’s (1953) equation (5.13), we have for all measurable $Y \subseteq \bar{X}$,

$$\lim_{t \to \infty} P^t(x, Y) = \sum_{j=1}^{\alpha} \lim_{t \to \infty} P^t(x, E_j) \xi_j(Y),$$

where $\xi_x(\cdot) = \sum_{j=1}^{\alpha} \lim_{t \to \infty} P^t(x, E_j) \xi_j(\cdot)$ is an invariant probability measure. In fact, the limit is uniform and approached exponentially fast (see his explanation below equation (5.15)), and therefore there exist $M'$ and $\rho < 1$ such that for all $x \in \bar{X}$ and all $t$,

$$\sup_{Y} |P^t(x, Y) - \xi_x(Y)| \leq M' \rho^t,$$

where the supremum is over measurable subsets of $\bar{X}$. Given an initial probability measure $\xi$ on $\bar{X}$, define $\xi^*$ by $\xi^*(Y) = \int \xi_x(Y) \xi(dx)$. Then Jensen’s inequality yields

$$|T^{*t} \xi(Y) - \xi(Y)| \leq \int |P^t(x, Y) - \xi_x(Y)| \xi(dx) \leq M' \rho^t.$$

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Defining $M = 2M'$, we therefore have
\[
\|T^{st}\xi(\cdot) - \xi_x(\cdot)\| = 2 \sup_Y |T^{st}\xi(Y) - \xi_x(Y)| \leq M\rho',
\]
as required for part 3.

Since Doeblin's condition holds, we again partition $\bar{X}$ into a finite number of ergodic sets $E_1, \ldots, E_\alpha$ and a transient set. We will first show that there is just one ergodic set $E$, i.e., $\alpha = 1$. Suppose there are distinct ergodic sets, $E$ and $E'$, and consider any $x \in E$ and $x' \in E'$. By assumption, there exists $(\tilde{q}, \tilde{\theta})$ such that $g(\tilde{q}|x)g(\tilde{q}|x')f(\tilde{\theta}) > 0$, and continuity then implies $Z = \{(q, \theta) : g(q|x)g(q|x')f(\theta) > 0\}$ is a nonempty, open subset of $\bar{X} \times \Theta$. Let $i$ satisfy $p_i > 0$. Because $P(x, E) = 1$, there is a measure zero set $W \subseteq Z$ such that for all $(q, \theta) \in Z \setminus W$, we have $\pi_i(q, \theta) \in E$. But similarly, there is a measure zero set $W' \subseteq Z$ such that for all $(q, \theta) \in Z \setminus W'$, we have $\pi_i(q, \theta) \in E'$. But $Z$ has positive measure, so there exists $(q, \theta) \in Z \setminus (W \cup W')$, and we have $\pi_i(q, \theta) \in E \cap E'$, a contradiction. Thus, there is a unique ergodic set $E \subseteq \bar{X}$. An identical argument establishes that $E$ cannot be partitioned into cyclically moving subclasses, that is, pairwise disjoint, measurable subsets $C_1, \ldots, C_\beta \subseteq E$, with $\beta \geq 2$, such that for all $j = 1, \ldots, \beta$ and all $x \in C_j$, we have $P(x, C_{j+1 \text{ mod } \beta}) = 1$. If such a partition were possible, then for each $x \in C_1$ and $x' \in C_2$, there would exist $(q, \theta)$ such that $\pi_i(q, \theta) \in C_1 \cap C_2$, a contradiction. Thus, a strong version of Doeblin’s condition holds, delivering part 4 (see Doob (1953), page 221, on condition (D0)).

We now prove Theorems 5 and 6 relating equilibrium policies to the core of a nearby canonical model. Given model $\gamma \in \Gamma_X$, strategies $\sigma$, and a policy $y \in \bar{X}$ in any period, we can write the continuation value $v_i(y; \sigma)$ as the integral of $u_i(x) + \theta_i \cdot x$ with respect to a Borel probability measure $\mu_i$ on $\bar{X} \times \Theta$ as follows. We define $\mu^1$ by
\[
\mu^1(Y \times H) = \sum_j p_j \int_q \int_\theta I_Y(\pi_j(q, \theta)) I_H(\theta) f(\theta) g(q|y) d\theta dq
\]
for all open $Y \subseteq \bar{X}$ and open $H \subseteq \Theta$, and for $m \geq 2$, we define $\mu^m$ by
\[
\mu^m(Y \times H) = \sum_j p_j \int_x \int_\theta \left[ \int_q \int_{\theta'} I_Y(\pi_j(q, \theta')) I_H(\theta') f(\theta') g(q|x) d\theta' dq \right] \mu^{m-1}(d(x, \theta))
\]
Thus, given policy outcome $y$ in the current period, $\mu^m$ is the joint distribution on policies and preference shocks $m$ periods hence. We notationally suppress the dependence of $\mu^m$ on $y$, but we make this dependence clear from context. We then define $\mu_i = (1 - \delta_i) \sum_{m=1}^{\infty} \delta_i^{m-1} \mu^m$. As is well-known, the probability measures $\mu^m$ and $\mu_i$ extend uniquely to the Borel sigma-algebra on $\bar{X} \times \Theta$. We refer to $\mu_i$ as the continuation distribution of $y$ at $\sigma$ in $\gamma$ for legislator $i$. It is important to note that the definition of $\mu^m$ is independent of $i$, while $\mu_i$ depends on $i$ through the discount factor.
δ_i. When discount factors are common, the continuation distribution is common to all legislators. Letting \( I^m_i = \int_{x,\theta} [u_i(x) + \theta_i \cdot x] \mu^m(d(x, \theta)) \), we then have

\[
v_i(y; \sigma) = \int_{x,\theta} [u_i(x) + \theta_i \cdot x] \mu_i(d(x, \theta)) = (1 - \delta_i) \sum_{m=1}^{\infty} \delta_i^{m-1} I^m_i.
\]

Note that the marginal probability measure \( \mu^m_i \) on \( \Theta \) is given by \( f \) and that the supports of the conditionals \( \mu^m(\cdot | \theta) \) lie in \( \tilde{X} \).

When the model \( \gamma \) is \( \epsilon \)-canonical, we define

\[
\beta = \sup \{| |x - y| |^2 + | \theta_i \cdot y | : i \in N, x, y \in B_\epsilon(X), \theta \in B_\epsilon(0) \}
\]

and we therefore have \( |I^m_i| \leq \beta \). We write \( \mu^c \) for \((1 - \delta) \sum_{m=1}^{\infty} \delta_i^{m-1} \mu^m_i \), where because \( \delta \) is a common discount factor in the neighboring canonical model, the continuation distribution \( \mu^c \) is independent of \( i \). It is then straightforward to verify the following inequality:\(^{23}\)

\[
\left| \int_{x,\theta} [u_i(x) + \theta_i \cdot x] \mu_i(d(x, \theta)) - \int_{x,\theta} [u_i(x) + \theta_i \cdot x] \mu^c(d(x, \theta)) \right|
\]

\[
= \left| (1 - \delta) \left[ \sum_{m=2}^{\infty} (\delta_i^{m-1} - \delta^{m-1}) I^m_i \right] - (\delta_i - \delta) \left[ \sum_{m=1}^{\infty} \delta_i^{m-1} I^m_i \right] \right|
\]

\[
= |\delta_i - \delta| \left[ (1 - \delta) \sum_{m=2}^{\infty} \left( \frac{\delta_i^{m-1} - \delta^{m-1}}{\delta_i - \delta} \right) I^m_i \right] - \left[ \sum_{m=1}^{\infty} \delta_i^{m-1} I^m_i \right]
\]

\[
\leq |\delta_i - \delta| \left[ \frac{\beta}{1 - \delta_i} + \frac{\beta}{1 - \delta_i} \right]
\]

\[
\leq 2\epsilon \beta.
\]

We define continuation values and dynamic utilities for the neighboring canonical model, generated by quadratic stage utilities and the common discount factor, by

\[
v^c_i(y; \sigma) = \int_{x,\theta} [u^c_i(x) + \theta_i \cdot x] \mu^c(d(x, \theta))
\]

\[
U^c_i(y, \theta_i; \sigma) = (1 - \delta)(u^c_i(y) + \theta_i \cdot y) + \delta v^c_i(y; \sigma).
\]

\(^{23}\)The first equality in what follows uses the identity

\[
(1 - \delta) \sum_{m=2}^{\infty} \left( \frac{\delta_i^{m-1} - \delta^{m-1}}{\delta_i - \delta} \right) = \frac{1}{1 - \delta_i}.
\]
It follows from the foregoing that
\[
|v_i(y; \sigma) - v_i^c(y; \sigma)|
\]
\[
= \left| \int_{x,\theta} [u_i(x) + \theta_i \cdot x] \mu(d(x, \theta)) - \int_{x,\theta} [u_i^c(x) + \theta_i \cdot x] \mu^c(d(x, \theta)) \right|
\]
\[
\leq 2\epsilon \beta + \left| \int_{x,\theta} (u_i(x) - u_i^c(x)) \mu^c(d(x, \theta)) \right|
\]
\[
\leq 2\epsilon \beta + \epsilon.
\]
Thus, given any \( \hat{\theta} \in \text{supp}_f \), we have
\[
|U_i(y, \theta_i; \sigma) - U_i^c(y, \theta_i; \sigma)|
\]
\[
\leq (1 - \delta)|u_i(y) - u_i^c(y)| + |(\delta - \delta_i)(u_i(y) + \theta_i \cdot y)| + \delta|v_i(y; \sigma) - v_i^c(y; \sigma)|
\]
\[
+ |(\delta_i - \delta)\hat{v}_i(y; \sigma)|
\]
\[
\leq \epsilon + \epsilon \beta + [2\epsilon \beta + \epsilon] + \epsilon \beta
\]
\[
= \epsilon \bar{\sigma},
\]
where \( \bar{\sigma} = 2 + 4\beta \). Of course, \( \epsilon \bar{\sigma} \) can be made arbitrarily small by choice of \( \epsilon \).

The next result gives us a partial characterization of legislative equilibria of a model \( \gamma \in \Gamma_\infty^\infty \) in terms of the proximity of \( \gamma \) to a canonical model. The result gives a global bound that holds even if \( \gamma \) is not particularly close to canonical (i.e., \( \epsilon \) is not vanishingly small). It has greatest bite, however, when \( \epsilon \) is small and the core legislator \( k \) is close to pivotal.

**Lemma 7** Assume \( \mathcal{D} \) is proper and strong. For all \( \lambda > 0 \), there exists \( \bar{\sigma}(\lambda) > 0 \) such that for all \( \epsilon \) with \( \epsilon < \bar{\sigma}(\lambda) \) and all \( \epsilon \)-canonical models \( \gamma \in \Gamma_\infty^\infty \), all stationary strategy profiles \( \sigma \), all \( \hat{\theta} \in \text{supp}_f \), and all \( y, z \in \hat{X} \), the following hold: (i) if \( U_k(y, \hat{\theta}_k; \sigma) > U_k(z, \hat{\theta}_k; \sigma) + \lambda \), then \( \{ i \in N : U_i(y, \hat{\theta}_i; \sigma) > U_i(z, \hat{\theta}_i; \sigma) \} \in \mathcal{D} \), and (ii) if \( \{ i \in N : U_i(y, \hat{\theta}_i; \sigma) > U_i(z, \hat{\theta}_i; \sigma) \} \notin \mathcal{D} \), then \( U_k(y, \hat{\theta}_k; \sigma) > U_k(z, \hat{\theta}_k; \sigma) - \lambda \).

**Proof** Consider any \( \lambda > 0 \), and choose \( \bar{\sigma}(\lambda) > 0 \) so that \( \bar{\sigma}(\lambda) \bar{\sigma} < \lambda/6 \) and so that
\[
\sup \{ \theta_j \cdot (r - s) : \theta \in B_{\bar{\sigma}(\lambda)}(0), j \in N, r, s \in \text{conv}B_{\bar{\sigma}(\lambda)}(\mathcal{X}) \} < \frac{\lambda}{6}.
\]
We prove only (i), as (ii) follows by similar calculations. Consider any \( \epsilon \)-canonical model with \( \epsilon < \bar{\sigma}(\lambda) \), any stationary \( \sigma \), any \( \hat{\theta} \in \text{supp}_f \), and any \( y, z \in \hat{X} \). For notational simplicity, translate the set of alternatives so that \( \hat{x}_k = 0 \). Let \( \mu^c \) be the (common) continuation distribution of \( y \) generated by \( \sigma \) in the canonical model, and let \( \nu^c \) be the (common) continuation distribution corresponding to \( z \). By construction of \( \bar{\sigma} \), the inequality \( |U_i(x, \theta; \sigma) - U_i^c(x, \theta; \sigma)| < \lambda/6 \) holds for all \( i \). Suppose we have \( U_k(y, \hat{\theta}_k; \sigma) > U_k(z, \hat{\theta}_k; \sigma) + \lambda \), but \( \{ i \in N : U_i(y, \hat{\theta}_i; \sigma) > U_i(z, \hat{\theta}_i; \sigma) \} \notin \mathcal{D} \). Since \( \mathcal{D} \) is
strong, this implies that $C = \{i \in N : U_i(z, \hat{\theta}_i; \sigma) \geq U_i(y, \hat{\theta}_i; \sigma)\} \in \mathcal{D}$. Furthermore, our choice of $\epsilon$ implies that in the canonical model, we have

$$U_k^c(y, \hat{\theta}_k; \sigma) > U_k^c(z, \hat{\theta}_k; \sigma) + \frac{2\lambda}{3}$$

and

$$U_i^c(z, \hat{\theta}_i; \sigma) \geq U_i^c(y, \hat{\theta}_i; \sigma) - \frac{\lambda}{3}$$

for all $i \in C$. Note that for every legislator $i$, $u_i^c(y) + \hat{\theta}_i \cdot y$ is equivalent to

$$-||\hat{x}_i + \frac{1}{2}\hat{\theta}_i - y||^2 = -||\hat{x}_i + \frac{1}{2}\hat{\theta}_i||^2 + 2(\hat{x}_i + \frac{1}{2}\hat{\theta}_i) \cdot y - ||y||^2$$

up to a constant, and so, neglecting the constant term, we may write

$$v_i^c(y; \sigma) = \int_\theta \int_x -||\hat{x}_i + \frac{1}{2}\theta_i - x||^2 \mu_c(dx|\theta)\mu_\theta(d\theta)$$

$$= \int_\theta -||\hat{x}_i + \frac{1}{2}\theta_i - E_{\mu_c}[x|\theta]||^2 \mu_\theta(d\theta) - V_{\mu_c}[x]$$

$$= \int_\theta \left[ -||\hat{x}_i + \frac{1}{2}\theta_i||^2 + 2(\hat{x}_i + \frac{1}{2}\theta_i) \cdot E_{\mu_c}[x|\theta] - ||E_{\mu_c}[x|\theta]||^2 \right] \mu_\theta(d\theta) - V_{\mu_c}[x],$$

where the second equality uses mean-variance analysis. Of course, a similar decomposition holds for $U_i^c(z, \hat{\theta}_i; \sigma)$. Recall that the marginals of $\mu^c$ and $\nu^c$ on $\theta$ are equal: $\mu_{\theta}^c = \nu_{\theta}^c$.

By assumption, we have $U_k^c(z, \hat{\theta}_k; \sigma) < U_k^c(y, \hat{\theta}_k; \sigma) - 2\lambda/3$. Using the above decomposition (and recalling the normalization $\hat{x}_k = 0$), this implies

$$(1 - \delta)(||y||^2 - ||z||^2) + \delta \int_\theta (||E_{\mu_c}[x|\theta]||^2 - ||E_{\nu^c}[x|\theta]||^2) \mu_\theta(d\theta) + V_{\mu_c}[x] - V_{\nu^c}[x]$$

$$< (1 - \delta)\hat{\theta}_k \cdot (y - z) + \delta \int_\theta \hat{\theta}_k \cdot (E_{\mu_c}[x|\theta]) - E_{\nu^c}[x|\theta] \mu_\theta(d\theta) - \frac{2\lambda}{3}$$

$$\leq \sup \{\theta_j \cdot (r - s) : \theta \in B_{r, \lambda}(0), j \in N, r, s \in \text{conv}B_{r, \lambda}(X)\} - \frac{2\epsilon}{3}$$

$$< -\frac{\lambda}{2},$$

where we use equivalence of the marginals on $\theta$. For each $i \in C$, we have $U_i^c(y, \hat{\theta}_i) \leq U_i^c(z, \hat{\theta}_i) + \lambda/3$, which implies

$$(1 - \delta)2(\hat{x}_i + \frac{1}{2}\hat{\theta}_i) \cdot (y - z) + \delta \int_\theta 2(\hat{x}_i + \frac{1}{2}\theta_i) \cdot (E_{\mu_c}[x|\theta]) - E_{\nu^c}[x|\theta] \mu_\theta(d\theta) - \frac{\lambda}{3}$$

$$< (1 - \delta)(||y||^2 - ||z||^2) + \delta \int_\theta (||E_{\mu_c}[x|\theta]||^2 - ||E_{\nu^c}[x|\theta]||^2) \mu_\theta(d\theta) + V_{\mu_c}[x] - V_{\nu^c}[x]$$

$$< -\frac{\lambda}{2}.$$
where we again use $\mu_0^* = \nu_0^*$. The latter inequality then implies that

$$
2x_i \cdot \left[ (1 - \delta)(y - z) + \delta \int_\theta (E_{\mu^c}^\prime[x,\theta] - E_{\nu^c}^\prime[x,\theta]) \mu_0^*(d\theta) \right]
< -(1 - \delta)\hat{\theta}_i \cdot (y - z) - \delta \int_\theta \hat{\theta}_i \cdot (E_{\mu^c}^\prime[x,\theta] - E_{\nu^c}^\prime[x,\theta]) \mu_0^*(d\theta) + \frac{\lambda}{3} - \frac{\lambda}{2}
< \sup \{ \theta_j \cdot (r - s) : \theta \in B_\gamma(x)(0), j \in N, r, s \in \text{conv} B_\gamma(x)(X) \} - \frac{\lambda}{6}
< 0.
$$

Now define the vector $s = (1 - \delta)(y - z) + \delta \int_\theta (E_{\mu^c}^\prime[x,\theta] - E_{\nu^c}^\prime[x,\theta]) \mu_0^*(d\theta)$. We have shown that $C \subseteq \{ i \in N : \hat{x}_i \cdot s < 0 \}$. But (again recalling $\hat{x}_k = 0$) note that the derivatives at zero in direction $s$ for legislators in this set are negative: for all $i \in C$, $D_s u_i^c(\hat{x}_k) = 2\hat{x}_i \cdot s < 0$. But then for small enough $\xi > 0$, we have

$$\{ i \in N : u_i^c(\hat{x}_k - \xi s) > u_i^c(\hat{x}_k) \} \supseteq \{ i \in N : \hat{x}_i \cdot s < 0 \} \supseteq C \in \mathcal{F},$$

and $\hat{x}_k$ is not a core policy of the neighboring canonical model, a contradiction. 

We next use Lemma 7 to deduce a lower bound for the equilibrium dynamic utility of the core legislator in which proposals of other legislators are absent and $k$’s proposals are not subject to voting constraints. Now fix $\epsilon > 0$, choose $\epsilon < \gamma$, assume $\gamma$ is $\epsilon$-canonical, and consider a stationary legislative equilibrium $\sigma$. We claim that for all $y \in X \cup \{ q \}$, legislator $k$ can obtain dynamic utility at least equal to $U_k(y, \theta_k; \sigma) - \lambda$ by proposing $y$ when recognized to propose. To see this, note that if $U_k(y, \theta_k) - \lambda > U_k(q, \theta_k; \sigma)$, then by Lemma 7, we have $y \in A(q, \theta; \sigma)$, i.e., it will pass if proposed, generating a dynamic utility of $U_k(y, \theta_k; \sigma)$. Otherwise, if $U_k(y, \theta_k; \sigma) - \lambda \leq U_k(q, \theta_k; \sigma)$, then $k$ can propose the status quo and obtain a dynamic utility of at least $U_k(y, \theta_k; \sigma) - \lambda$, as claimed. Likewise, if another legislator is recognized to propose, then the core legislator obtains at least $U_k(q, \theta_k; \sigma) - \lambda$. These observations allow us to derive a simple lower bound on $k$’s equilibrium dynamic utility when recognized as proposer.

For every measurable mapping $\tilde{x}_k : \Theta \rightarrow \tilde{X}$ representing possible proposals of legislator $k$, the previous discussion implies that

$$
U_k(\tilde{x}_k(\hat{\theta}), \hat{\theta}_k; \sigma) \geq (1 - \delta_k)u_k(\tilde{x}_k(\hat{\theta})) + \tilde{x}_k(\hat{\theta}) \cdot \hat{\theta}_k + \delta_k p_k \int_\theta [U_k(\tilde{x}_k(\hat{\theta}), \theta_k; \sigma) - \lambda] f(\theta) d\theta + \delta_k (1 - p_k) \int_{q, \theta} [U_k(q, \theta_k; \sigma) - \lambda] g(q) \tilde{x}_k(\hat{\theta}) f(\theta) dq d\theta, 
$$

where we make use of the fact that after the current period, if $k$ is recognized, which occurs with probability $p_k$, then $U_k(\tilde{x}_k(\hat{\theta}), \theta_k; \sigma) - \lambda$ gives a lower bound on the legislator’s equilibrium dynamic utility: optimality of $k$’s proposal strategy implies
that \( U_k(\pi_k(\hat{q}, \hat{\theta}), \hat{\theta}_k; \sigma) \geq U_k(\hat{x}_k(\hat{\theta}), \hat{\theta}_k; \sigma) - \lambda \). Similarly, if a legislator other than \( k \) is recognized, which occurs with probability \( 1 - p_k \), then no policy generating a dynamic utility less than \( U_k(q, \theta_k; \sigma) - \lambda \) for \( k \) can pass in equilibrium. Building on these observations, we can expand and substitute iteratively in (9) to obtain a lower bound in which dynamic utilities are eliminated altogether. Now define \( g^1 = g \), and for \( s = 2, 3, \ldots \), define the iterated density \( g^s(q|x) \) by

\[
g^s(q|x) = \int_{q_1} \cdots \int_{q_{s-1}} g(q|q_{s-1})g(q_{s-1}|q_{s-2}) \cdots g(q_1|x) dq_{s-1} \cdots dq_1.
\]

The next lemma is a technical result that gives a pointwise bound on legislator \( k \)'s dynamic utility. Note that given \( \lambda \) in Lemma 8, \( \mathcal{E}(\lambda) \) is chosen as in Lemma 7.

**Lemma 8** Assume \( \mathcal{D} \) is proper and strong. For all \( \lambda > 0 \), all \( \epsilon \in \mathcal{E}(\lambda) \), all \( \epsilon \)-canonical models \( \gamma \in \Gamma_X \), and all stationary legislative equilibria \( \sigma \in E(\gamma) \), the following holds: for all measurable mappings \( \tilde{x}: \Theta \rightarrow \tilde{X} \) and all \( \hat{\theta} \in \Theta \), we have

\[
U_k(\tilde{x}(\hat{\theta}), \hat{\theta}_k; \sigma) \geq (1 - \delta_k) \left[ u_k(\tilde{x}(\hat{\theta})) + \tilde{x}(\hat{\theta}) \cdot \hat{\theta}_k \right] + \sum_{t=2}^{\infty} \delta_k^{t-1} p_k \int_{\hat{\theta}} (1 - \delta_k) \left[ u_k(\tilde{x}(\theta)) + \tilde{x}(\theta) \cdot \theta_k \right] \int_{q_{t-1}} \cdots \int_{q_1} [u_k(q_t) + q_t \cdot \theta_k] g^s(q_t|\tilde{x}(\theta)) f(\theta) dq'_t d\theta_t - \frac{\delta_k \lambda}{1 - \delta_k}. 
\]

**Proof** To simplify the explanation of the expression in the lemma, we suppose that \( \tilde{x}(\hat{\theta}) \) is the policy outcome in period \( t = 1 \). Iterating the inequality in (9), we generate an expansion over all sequences of realized proposers in periods \( t = 2, 3, \ldots \). In the first period, legislator \( k \)'s stage utility is \( u_k(\tilde{x}(\hat{\theta})) + \tilde{x}(\hat{\theta}) \cdot \hat{\theta}_k \), normalized by \( (1 - \delta_k) \). Whenever \( k \) is not recognized for \( s \) consecutive periods following the initial policy \( \tilde{x}(\hat{\theta}) \) in period \( t = 1 \), we insert \( \int_{q, \theta} [u_k(q) + q \cdot \theta] g^s(q|\tilde{x}(\theta)) f(\theta) dq d\theta \); this occurs with probability \( (1 - p_k)^s \) and is additionally weighted by \( (1 - \delta_k) \delta_k^{s-1} \). We insert \( u_k(\tilde{x}(\theta)) + \tilde{x}(\theta) \cdot \theta_k \) in periods \( t \geq 2 \) whenever \( k \) is recognized as proposer with shock \( \theta \); this occurs with probability \( p_k \) and is additionally weighted by \( (1 - \delta_k) \delta_k^{t-1} \). Whenever \( k \) is not recognized for \( s \) consecutive periods after being recognized to propose in period \( t \geq 2 \) with shock \( \theta \), we insert \( \int_{q', \theta'} [u_k(q') + q' \cdot \theta'_k] g^s(q'|\tilde{x}(\theta)) dq' d\theta' \) in the expansion; this occurs with probability \( (1 - p_k)^s \) and is weighted additionally by \( \delta_k^s \) times \( (1 - \delta_k) \delta_k^{t-1} p_k \). At each step in the iteration, a slackness term \( -\lambda \), appropriately discounted, is introduced, and these are collected in the term \( -\frac{\delta_k \lambda}{1 - \delta_k} \). Thus, we collect.
stage payoffs from sequences of realizations in which $k$ is not recognized for a finite, consecutive number of periods subsequent to the initial policy $\tilde{x}(\hat{\theta})$ in the expression

$$(1 - \delta_k) \sum_{s=1}^{\infty} \delta_s^k (1 - p_k) \int_{q,\theta} [u_k(q) + q \cdot \theta] g^s(q|\tilde{x}(\hat{\theta})) f(\theta) dq d\theta.$$ 

We then collect stage payoffs from sequences of realizations in which $k$ is recognized in a period $t \geq 2$ in the term

$$\sum_{t=2}^{\infty} \delta_{t-1} p_k \int_{q,\theta} (1 - \delta_k) [u_k(q) + \tilde{x}(\theta) \cdot \theta_k].$$

Finally, we collect stage payoffs from sequences in which $k$ is not recognized for a finite, consecutive number of periods subsequent to being recognized in period $t \geq 2$ with shock $\theta$ in the term

$$\delta_{t-1} p_k \sum_{s=1}^{\infty} \delta_s^k (1 - p_k) \int_{q',\theta'} [u_k(q') + q' \cdot \theta'_k] g^s(q'|\tilde{x}(\theta)) f(\theta') dq' d\theta'.$$

This yields the expression in the lemma.

**Proof of Theorem 5** Fix $\tilde{\delta} < 1$. We first claim that for all $\eta > 0$, there exists $\tilde{\epsilon}(\eta) > 0$ such that for all $\epsilon < \tilde{\epsilon}(\eta)$ and $\delta_k \leq \tilde{\delta}$, all stationary legislative equilibria $\sigma \in E(\gamma)$, all $y \in \tilde{X}$, and all $\theta \in \text{supp } f$, we have

$$U_k(y, \theta_k; \sigma) \geq u_k(y) - \eta.$$  

(10)

Given $\epsilon > 0$, we define

$$u(\epsilon) = \sup \{ ||x - y||^2 + ||x - z||^2 : x, y, z \in B_\epsilon(X) \}$$

$$\rho(\epsilon) = \sup \{ ||x \cdot \theta_k|| : x \in B_\epsilon(X), \theta \in B_\epsilon(0) \}$$

and we choose $\epsilon'$ so that $\epsilon' < \tilde{\delta}$ and $\rho(\epsilon') < \eta/4$. Let $s'$ satisfy $\frac{s' + u(\epsilon')}{1 - \delta} < \frac{\eta}{4}$, and choose $\epsilon'' > 0$ small enough that

$$\sum_{s=1}^{s'} \delta_s^k \left[ \epsilon'' + ||\hat{x}_k - y - (y - \hat{x}_k) \frac{s\epsilon''}{||y - \hat{x}_k||}|| \right] < \frac{\eta}{4}.$$ 

To apply Lemma 8, let $\lambda > 0$ be small enough that $\frac{(\delta + \epsilon')\lambda}{1 - \delta - \epsilon} < \frac{\eta}{4}$. Set $\tilde{\epsilon}(\eta) = \min \{ \epsilon', \epsilon'', \tilde{\epsilon}(\lambda) \}$, and consider any $\epsilon < \tilde{\epsilon}(\eta)$ and any $\epsilon$-canonical model $\gamma \in \Gamma^\infty_X$ with $\delta_k \leq \tilde{\delta}$, any stationary legislative equilibrium $\sigma$, any $y \in \tilde{X}$, and any $\theta \in \text{supp } f$. Using the
inequality in Lemma 8 with the constant mapping \( \tilde{x} \equiv y \), note that \( \tilde{x}(\hat{\theta}) \cdot \hat{\theta}_k \geq \rho(\epsilon) \) and \( q \cdot \hat{\theta}_k \geq \rho(\epsilon) \); we gather these terms to obtain

\[
U_k(y, \theta_k; \sigma) - u_k(y) \\
\geq (1 - \delta_k) \sum_{s=1}^{\infty} \delta^s_k (1 - p_k)^s \int_q [u_k(q) - u_k(y)] g^s(q|y) dq \\
+ (1 - \delta_k) \sum_{t=2}^{\infty} \delta^{t-1}_k p_k \sum_{s=1}^{\infty} \delta^s_k (1 - p_k)^s \int_{q'} [u_k(q') - u_k(y)] g^s(q'|y) dq' - \rho(\epsilon) - \frac{\delta_k \lambda}{1 - \delta_k}.
\]

We decompose the terms \( u_k(q) - u_k(y) \) as \((u_k(q) - u^c_k(q)) + (u^c_k(q) - u^c_k(y))\), and similarly for \( u_k(q') - u_k(y) \). Since \( \gamma \) is \( \epsilon \)-canonical, the support of \( g^s(\cdot|y) \) is contained in the ball of radius \( s \epsilon \) around \( y \), and therefore \( u^c_k(q) \) is bounded below over \( q \in \text{supp} g^s(\cdot|y) \) by \( y + (y - \hat{x}_k) \frac{s \epsilon}{||y - \hat{x}_k||} \). Thus, using \( \epsilon < 1 \), we have

\[
\sum_{s=1}^{\infty} \delta^s_k (1 - p_k)^s \int_q [u_k(q) - u_k(y)] g^s(q|y) dq \\
\geq \sum_{s=1}^{s'} \delta^s \left[ -\epsilon - ||\hat{x}_k - y - (y - \hat{x}_k) \frac{s \epsilon}{||y - \hat{x}_k||} || \right] - \sum_{s=s'+1}^{\infty} \delta^s \left[ -\epsilon - u(\epsilon) \right] \\
\geq -\frac{\eta}{2},
\]

with a similar expression for the terms involving \( u_k(q') - u_k(y) \). We conclude that

\[
U_k(y, \theta_k; \sigma) - u_k(y) \geq -\frac{\eta}{2} - \frac{\eta}{4} - \frac{\eta}{4} = \eta,
\]

as claimed.

To prove the theorem, consider any \( \lambda > 0 \) and choose \( \bar{\tau} \) so that (i) \( \bar{\tau} < \tau(\lambda/3) \), (ii) \( \bar{\tau} < \tilde{\epsilon}(\lambda/3) \), and (iii) for all \( y \in B_\tau(X) \) and all \( q \in B_\tau(y) \), we have \( |u_k(y) - u_k(q)| < \lambda/3 \).

It follows that for all \( \epsilon \)-canonical models \( \gamma \in \Gamma^\infty_X \) with \( \epsilon < \bar{\tau} \) and \( \delta_k \leq \bar{\tau} \), all \( \sigma \in E(\gamma) \), and all \( y \in X \), we have

\[
v_k(y; \sigma) = \sum_j p_j \int_{q, \theta} U_k(\pi_j(q, \theta), \theta_k; \sigma) f(\theta) g(q|y) d\theta dq \\
\geq \sum_j p_j \int_{q, \theta} \left[ U_k(q, \theta_k; \sigma) - \frac{\lambda}{3} \right] f(\theta) g(q|y) d\theta dq \\
\geq \int_q \left[ u_k(q) - \frac{2\lambda}{3} \right] g(q|y) dq \\
\geq u_k(y) - \lambda,
\]
where the first inequality follows from (i), Lemma 7, and our observation that legislator \( k \) is guaranteed a dynamic utility of at least \( U_k(q, \theta) - \lambda/3 \) when the status quo is \( q \); the second follows from (ii) and the first part of the proof; and the third follows from (iii).

**Proof of Theorem 6** Let \( \{\gamma^m\} \) be an approximately canonical sequence, consider a corresponding sequence \( \{\sigma^m\} \) of stationary legislative equilibria, and let \( \{\mu^m\} \) be any selection of invariant distributions corresponding to these equilibria. We claim that \( \{\xi^m\} \) converges weakly to the unit mass on \( x^* \), i.e., for all \( \eta > 0 \), we have \( \xi^m(B_\eta(x^*)) \to 1 \). To prove this, consider any \( \eta > 0 \). Part 2 of Theorem 4 establishes that for all \( m \), the equilibrium transition \( P^m(x, A) \) satisfies Doeblin’s condition, so there is a finite number of ergodic sets, \( E^m_1, \ldots, E^m_q \), and each \( E^m_j \) admits a unique invariant distribution \( \xi^m_j \). For each \( m \), let \( E^m_{jm} \) satisfy

\[
\xi^m_j(B_\eta(x^*)) = \min \{\xi^m_j(B_\eta(x^*)) \mid j = 1, \ldots, m\}.
\]

By Theorem 5.7 of Doob (1953), \( \xi^m \) is a convex combination of \( \{\xi^m_j\} \), and it follows that \( \xi^m(B_\eta(x^*)) \geq \xi^m_j(B_\eta(x^*)) \). Note that by part 3 of Theorem 4, the transition probability \( P_m \) is aperiodic, and restricting the transition probability to \( E^m_{jm} \), we are in Case (b) of Doob (1953) (see his discussion on p.203 for \( d = 1 \)), and it follows that for each \( x \in E^m_{jm} \), the sequence \( \{P^m_t(x, \cdot)\}_t \) converges in the total variation norm to \( \xi^m_{jm} \), and in particular, we have \( \lim_{t \to \infty} P^m_{jm}(x, B_\eta(x^*)) = \xi^m_{jm}(B_\eta(x^*)) \).

We claim that for all \( \zeta > 0 \), there exists \( \overline{m} \) such that for all \( m \geq \overline{m} \), all \( q \in \tilde{X}^m \), and all \( \theta \in \text{supp}\, f^m \), we have \( \pi^m_k(q, \theta) \in B_\zeta(x^*) \). To prove this, first extract a sequence \( \{x^m\} \) in \( \tilde{X} \) such that \( x^m \in X^m \) for all \( m \) and \( x^m \to x^* \). Choose \( \lambda, \kappa > 0 \) such that \( \lambda + \kappa < \zeta^2(1 - \delta^*)/\delta^* \), which implies \( \Delta = (1 - \delta^*)\zeta^2/2 - \delta^*(\lambda + \kappa)/2 > 0 \). Letting \( \delta \to 1 \) satisfy \( \delta^* < \delta \), Theorem 5 implies that for sufficiently high \( m \) and all \( y \in \tilde{X}^m \), we have \( v^m_k(y; \sigma^m) \geq v^m_k(y) - \lambda \). Furthermore, writing \( u^c_k \) for the quadratic utility function with ideal point \( \tilde{x}^m_k \), in the canonical neighbor of \( \gamma^m \), the maximized value of \( u^c_k \) is zero, and so for high enough \( m \) and all \( y \in \tilde{X}^m \), we have \( u^c_k(y) < \kappa \), the latter implying \( v^m_k(y; \sigma^m) \leq \kappa \). Note that for all \( m \), all \( \theta \in \text{supp}\, f^m \), and all \( y \in \tilde{X}^m \), we have

\[
U^m_k(x^m, \theta; \sigma^m) - U^m_k(y, \theta; \sigma^m) \\
\geq (1 - \delta^m_k) \left[ (u^c_k(x^m) - u^c_k(y)) + (u^c_k(x^m) - u^c_k(y)) + (u^c_k(y) - u^c_k(y)) \right] \\
+ (x^m - y) \cdot \theta_k - \delta^m_k \left[ v^m_k(x^*; \sigma^*) - v^m_k(y; \sigma^m) \right] \\
\geq (1 - \delta^m_k) \left[ -e^m + u^c_k(x^m) - u^c_k(y) - e^m - p(e^m) \right] \\
+ \delta^m_k \left[ u^c_k(x^m) - \lambda - \kappa \right]
\]

where we assume without loss of generality that \( k^* = k^m \) for all \( m \), and we use the notation \( p(e^m) \) from the proof of Theorem 5. Note that \( p(e^m) \to 0 \) and \( u^c_k(x^m) \to 0 \).
Therefore, letting \( \{\theta^m\} \) be any sequence such that \( \theta^m \in \text{supp} f^m \) for all \( m \) and letting \( \{y^m\} \) be any sequence such that \( y^m \in \bar{X} \setminus B_\gamma(x^*) \) for all \( m \), we have

\[
\liminf_{m \to \infty} U_k^m(x^m, \theta^m; \sigma^m) - U_k^m(y^m, \theta^m; \sigma^m) \geq (1 - \delta^* \zeta [\varepsilon^2] + \delta^* [-\lambda - \kappa] > \Delta.
\]

Above, we have established that for \( e^m < \zeta(\Delta) \), legislator \( k^* \) can obtain dynamic utility no less than \( U_k^m(x^m, \theta^*; \sigma^m) - \Delta \) by proposing \( x^m \), and it follows that for high enough \( m \), all \( q \in \bar{X} \setminus B_\gamma(x^*) \), all \( \theta \in \text{supp} f^m \), and all \( y \in \bar{X} \setminus B_\zeta(x^*) \), we have

\[
U_k^m(\pi_k^m(q, \theta), \theta^*; \sigma^m) \geq U_k^m(x^m, \theta^*; \sigma^m) - \Delta > U_k^m(y, \theta; \sigma^m).
\]

We conclude that \( \pi_k^m(q, \theta) \in B_\zeta(x^*) \), as claimed.

Next, we claim that for all \( m \geq \overline{m} \), all \( x \in E_{j_m}^m \), and all \( t \), we have

\[
|P^t_m(x, B_\eta(x^*)) - \xi^m_m(B_\eta(x^*))| \leq (1 - p_k^m)^t,
\]

where \( p_k^m \) is the recognition probability of legislator \( k^* \) in model \( \gamma^m \). We prove this by at first fixing model \( \gamma^m \) and constructing a sequence \( \{p^t\} \) of probability distributions with binary support, where \( p^t = (p^t_1, p^t_2) \) and \( p^t_1 \) corresponds to the probability in equilibrium \( \sigma^m \) that the policy lies in \( B_\eta(x^*) \) in period \( t \) given initial state \( x \in E_{j_m}^m \), i.e., \( p^t_1 = P^t_m(x, B_\eta(x^*)) \) and \( p^t_2 = 1 - p^t_1 \). Note that \( p^t_1 \to \xi^m_m(B_\eta(x^*)) \). Viewing \( p^t \) as a \( 1 \times 2 \) row vector, we can express \( \{p^t\} \) recursively as \( p^{t+1} = p^t A^t \) by defining

\[
A^t = \begin{bmatrix}
P_m(x, B_\eta(x^*)) & 1 - P_m(x, B_\eta(x^*)) \\
0 & 0
\end{bmatrix},
\]

and for \( t \geq 2 \),

\[
A^t = \begin{bmatrix}
\frac{\int_{B_\eta(x^*)} P_m(z, B_\eta(x^*)) P_m^{-1}(x, dz)}{P_m^t(x, B_\eta(x^*))} & \frac{\int_{B_\eta(x^*)} (1 - P_m(z, B_\eta(x^*)) P_m^{-1}(x, dz)}{1 - P_m^t(x, B_\eta(x^*))} \\
\frac{\int_{B_\eta(x^*)} P_m(z, B_\eta(x^*)) P_m^{-1}(x, dz)}{P_m^t(x, B_\eta(x^*))} & \frac{\int_{B_\eta(x^*)} (1 - P_m(z, B_\eta(x^*)) P_m^{-1}(x, dz)}{1 - P_m^t(x, B_\eta(x^*))}
\end{bmatrix},
\]

where in case the denominator in some row is zero, we simply replace the entries of that row with zeros. Defining \( B^t = \prod_{k=1}^t A^k \) for each \( t \), we have \( p^t = p^1 A^t \).

Furthermore, we have \( B^{t+1} = A^t B^t \), and letting \( a^t_{k, \ell} \) and \( b^t_{k, \ell} \) denote the entries in row \( k \), column \( \ell \) of \( A^t \) and \( B^t \), respectively, this yields

\[
a^{t+1}_{1,1} = a^t_{1,1} b^t_{1,1} + a^t_{1,2} b^t_{2,1} = a^t_{1,1} b^t_{1,1} + (1 - a^t_{1,1}) b^t_{2,1} = a^t_{1,1} (b^t_{1,1} - b^t_{2,1}) + b^t_{2,1},
\]

\[
b^{t+1}_{2,1} = a^t_{2,1} b^t_{1,1} + a^t_{2,2} b^t_{2,1} = a^t_{2,1} b^t_{1,1} + (1 - a^t_{2,1}) b^t_{2,1} = a^t_{2,1} (b^t_{1,1} - b^t_{2,1}) + b^t_{2,1}.
\]

In particular, the sequences \( \{b^t_1\} \) and \( \{\overline{b} \} \) defined by \( \overline{b} = \max\{b^t_{1,1}, b^t_{2,1}\} \) and \( \overline{b} = \min\{b^t_{1,1}, b^t_{2,1}\} \) are, respectively, weakly decreasing and weakly increasing. We have argued that for \( m \geq \overline{m} \), we have \( \pi_k^m(q, \theta) \in B_\eta(x^*) \), and thus \( P_m(z, B_\eta(x^*)) \geq p_k^m \).

Therefore,

\[
|b^{t+1}_{1,1} - b^{t+1}_{2,1}| = |(a^t_{1,1} - a^t_{2,1})(b^t_{1,1} - b^t_{2,1})| \leq (1 - p_k^m)|b^t_{1,1} - b^t_{2,1}|,
\]

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and a simple induction argument yields \( |b_{1,t}^{t+1} - b_{2,t}^{t+1}| \leq (1 - p_k)^{t+1} \). This, with monotonicity of \( \{b_t\} \) and \( \{b_k\} \), implies that \( \{b_{1,t}\} \) and \( \{b_{2,t}\} \) are convergent with common limit, say, \( b^* \). But \( p_t = p_1b_{1,t} + p_2b_{2,t} \) and \( p_t \to \xi_j \), which implies \( b^* = \xi_j(B_\eta(x^*)) \). Furthermore, because \( b_t \leq b^* \), we have \( |b_{1,t} - b^*| \leq (1 - p_k)^t \) and \( |b_{2,t} - b^*| \leq (1 - p_k)^t \) for all \( t \). Finally, we have
\[
|p_t^{t+1} - b^*| = |p_1^t(b_{1,t}^{t+1} - b^*) + p_2^t(b_{2,t}^{t+1} - b^*)| \leq (1 - p_k)^t,
\]
as claimed.

Now suppose in order to deduce a contradiction that \( \liminf_{m \to \infty} \xi^m_{jm}(B_\eta(x^*)) < 1 \).

Going to a subsequence (still indexed by \( m \)), we may assume that \( \lim_{m \to \infty} p_{k*}^m = p_k^* > 0 \) and \( \lim_{m \to \infty} \xi^m_{jm}(B_\eta(x^*)) = \beta < 1 \). We claim that there are sequences \( \{y^\ell\} \) and \( \{m_\ell\} \) such that \( y^\ell \in E_{jm_\ell}^m \) for all \( \ell \), that \( m_\ell \to \infty \), and that \( y^\ell \to x^* \). Indeed, let \( \{\zeta_\ell\} \) be a sequence such that \( \zeta_\ell > 0 \) for all \( \ell \) and \( \zeta_\ell \to 0 \). For each \( \ell \), we can choose \( m_\ell \) by the above claim so that for all \( q \in \tilde{X}^m \) and all \( \theta \in \text{supp}^m \), we have \( \pi_{k*}^m(q, \theta) \in B_{\epsilon_\ell}(x^*) \); furthermore, we can choose \( m_\ell \) so that \( m_\ell > m_{\ell-1} \) and \( p_{k*}^m > 0 \). Given any \( x \in E_{jm_\ell}^m \), we must have \( P_{m_\ell}(x, E_{jm_\ell}^m) = 1 \), and therefore, since \( p_{k*}^m > 0 \), we have \( \pi_{k*}^m(q, \theta) \in E_{jm_\ell}^m \) for almost all \( (q, \theta) \in (\text{supp}^m \times \text{supp}^m) \). Choosing any \( (q^\ell, \theta^\ell) \) in that set of full measure, we then set \( y^\ell = \pi_{k*}^m(q^\ell, \theta^\ell) \in B_{\zeta_\ell}(x^*) \cap E_{jm_\ell}^m \) to fulfill the claim.

Theorem 5 then allows us to choose a sequence \( \{\lambda_\ell\} \) such that \( \lambda_\ell > 0 \) and
\[
(1 - \delta_{k*}^m) \sum_{t=0}^\infty \lambda_\ell = \int \left[ \sum_{t=0}^\infty \lambda_\ell \right] (1 - P_{m_\ell}(y^\ell, B_\eta(x^*)))(-\eta^2) + \epsilon_{m_\ell} + \rho(\epsilon_{m_\ell})
\]
\[
\geq (1 - \delta_{k*}^m) \sum_{t=0}^\infty \lambda_\ell \left[ \int u_{k*}^m(y^\ell, \sigma_{m_\ell}) + \epsilon_{m_\ell} + \rho(\epsilon_{m_\ell}) \right]
\]
\[
\geq \lambda_\ell \left[ \int u_{k*}^m(y^\ell) - \lambda_\ell \right] \geq -||y^\ell - x^*||^2 - \epsilon_{m_\ell} - \lambda_\ell
\]
for all \( \ell \) and such that \( \lambda_\ell \to 0 \). Taking limits, the righthand side of the above inequality goes to zero, which yields
\[
\lim_{\ell \to \infty} (1 - \delta_{k*}^m) \sum_{t=0}^\infty \lambda_\ell = 1.
\]
In particular, for all \( t \), we have \( \lim_{\ell \to \infty} P_{m_\ell}(y^\ell, B_\eta(x^*)) = 1 \). Now choose \( t' \) sufficiently large that \( (1 - p_{k*}^m)^{t'} < 1 - \beta \), choose \( \phi > 0 \) such that \( \phi < [1 - \beta - (1 - p_{k*}^m)^{t'}]/2 \), and choose \( \ell \) such that \( m_\ell > \overline{m} \), that \( \phi < [1 - \beta - (1 - p_{k*}^m)^{t'}/2 \), and that \( P_{m_\ell}(y^\ell, B_\eta(x^*)) > 1 - \phi \). Then
\[
(1 - p_{k*}^m)^{t'} \geq |P_{m_\ell}(y^\ell, B_\eta(x^*)) - \xi_jm_\ell(B_\eta(x^*))| > 1 - \phi - (1 - \beta - \phi) > (1 - p_{k*}^m)^{t'}
\]
a contradiction. This establishes the theorem.
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