REGULAR INFINITE ECONOMIES

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Abstract. Smooth infinite economies with separable utilities have individual demand functions described by Fredholm maps. Interpreting the aggregate excess demand function as a Z-Rothe vector field on the price space, allows us to use Tromba's extension of the Poincaré-Hopf theorem. We study parametric transversality of excess demand functions and we study the number of equilibria, showing that generically they are odd.

1. Introduction

Determinacy of competitive equilibria for economies with an infinite number of commodities has presented us with many challenges. Araujo [2], loosely speaking, shows that for general Banach spaces a demand function will exist if and only the commodity space is reflexive. He also shows that even if the demand function exists, it will be C^1 if and only if the commodity space is actually a Hilbert space.

Different approaches exist then to attack this problem. Because of Araujo's results, Kehoe et al [7] study determinacy of equilibrium where the commodity set is a Hilbert space. The disadvantage of this approach, as they put it, is that the price domain (and, implicitly, the consumption set) has an empty interior. This means that they are allowing, to some extent, negative prices and consumption.

A second approach consists in using a weakened version of differentiability. Shannon [10] and Shannon and Zame [11] introduce the

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notion of quadratic concavity and demonstrate that Lipschitz continuity of the excess spending map is sufficient to yield generic determinacy.

A third approach is to assume separable utilities so that equilibrium conditions are described by Fredholm maps which extend differential topology to infinite dimensions. Chichilnisky and Zhou [4] point out that the literature typically takes as the price space the natural positive cone of the dual space of the commodity set. However, with separable utility functions, only a small subset of the price space can support equilibria. There is no loss of information from discarding those elements that do not support equilibria. Chichilnisky and Zhou show that smooth infinite economies with separable utilities have locally unique equilibria and this is the path that we wish to follow.

It is our aim to show, in the spirit of Dierker [5] (see also [14]), that the number of equilibria of infinite economies is odd. We do this by showing that the aggregate excess demand function defines a Z-Rothe vector field which allows constructing a Poincaré-Hopf index on the infinite dimensional price space.

In section 2, we study aggregate excess demand functions; our main result is that they define a Z-Rothe vector field. In section 3 we study parametric transversality and we then show that most economies have an aggregate excess demand function with isolated zeros. Then, in section 4, we construct an index theorem for smooth infinite economies, showing that the number of equilibria is generically odd.

2. The Structure of Aggregate Excess Demand Functions

2.1. **The Market.** Let M be a compact Riemannian manifold. Then $C(M, \mathbb{R}^n)$ is a separable topological vector space for which the interior of the positive cone is non-empty (see [4]). The **consumption space** is $X = C^{++}(M, \mathbb{R}^n)$, the positive cone of $C(M, \mathbb{R}^n)$, and the **price space** $S = \{P \in C^{++}(M, \mathbb{R}^n) : \|P\| = 1\}$. We denote by $\langle \cdot, \cdot \rangle$ the inner product on $C(M, \mathbb{R}^n)$.

We consider a finite number I of agents. An **exchange economy** is parametrized for each agent $i = 1, \ldots, I$ by their initial endowments $\omega_i \in X$ and their individual demand functions $f_i : S \times (0, \infty) \to X$. The maps $f_i(P(t), w)$ are solutions to the optimization problem

$$\max_{\langle P(t),y\rangle = w} W_i(y)$$

where $W_i(x)$ is a separable utility function, i.e., it can be written as

$$W_i(x) = \int_M u^i(x(t),t)dt$$

We assume $u^i(x(t),t) : \mathbb{R}^n_{++} \times M \to \mathbb{R}$ is a strictly monotonic, concave, C^2 function where $\{y \in R^n_{++} : u^i(y,t) \geq u^i(x,t)\}$ is closed. This implies (see [4]) that $W_i(x)$ is strictly monotonic, concave and twice Fréchet differentiable.

2.2. **Fredholm Index Theory.** We will be using tools of differential topology in infinite dimensions. Therefore, we would like our maps to be Fredholm as introduced by S. Smale (see [12]).

A (linear) **Fredholm operator** is a continuous linear map $L : E_1 \to E_2$ from one Banach space to another with the properties:

(1) dim ker $L < \infty$
(2) range L is closed
(3) coker $L = E_2/\text{range}L$ has finite dimension

If L is a Fredholm operator, then its **index** is dim ker$L-$ dim coker$L$, so that the index of L is an integer.

A **Fredholm map** is a C' map $f : M \to V$ between differentiable manifolds locally like Banach spaces such that for each $x \in M$ the derivative $Df(x) : T_xM \to T_{f(x)}V$ is a Fredholm operator. The **index** of f is defined to be the index of $Df(x)$ for some x. If M is connected, this definition does not depend on x.

2.3. **Individual Demand Functions.** Chichilnisky and Zhou [4] show that for separable utilities the individual demand functions satisfy

(1) $\langle P, f_i(P,w)\rangle = w$ for any $P \in S$ and for any $w \in (0,\infty)$
(2) $u^i_x(f_i(P(t),w),t) = \lambda P(t)$ for some $\lambda > 0$
(3) $f_i : S \times (0,\infty) \to X$ is a diffeomorphism
(4) $f_i : S \times (0,\infty) \to X$ is a Fredholm map of index zero

2.4. **Aggregate Excess Demand Functions.** In this paper we assume that the individual demand functions are fixed, so that the only parameters defining an economy are the initial endowments. Denote $\omega = (\omega_1, \ldots, \omega_I) \in \Omega = X^I$. For a fixed economy $\omega \in \Omega$ the **aggregate excess demand function** is a map $Z_\omega : S \to C(M, \mathbb{R}^n)$ defined by

$$Z_\omega(P) = \sum_{i=1}^{I} (f_i(P, \langle P, \omega_i \rangle) - \omega_i)$$

We also define $Z : \Omega \times S \to C(M, \mathbb{R}^n)$ by the evaluation

$$Z(\omega, P) = Z_\omega(P)$$

Definition 1. *We say that $P \in S$ is an* ***equilibrium*** *of the economy $\omega \in \Omega$ if $Z_\omega(P) = 0$. We denote the* ***equilibrium set***

$$\Gamma = \{(\omega, P) \in \Omega \times S : Z(\omega, P) = 0\}$$

We wish to explore the structure of aggregate excess demand functions. Below, we show that it can be interpreted as a vector field on the price space and that it is a Fredholm map.

Proposition 1. *The excess demand function $Z_\omega : S \to C(M, \mathbb{R}^n)$ of economy $\omega \in \Omega$ is a vector field on S.*

Proof. Since $\langle P, f_i(P, y) \rangle = y$ for any $P \in S$ and for any $y \in (0, \infty)$, then

$$\begin{aligned}
\langle P, Z_\omega(P) \rangle &= \langle P, \sum_{i=1}^{I} (f_i(P, \langle P, \omega_i \rangle) - \omega_i) \rangle \\
&= \sum_{i=1}^{I} \langle P, f_i(P, \langle P, \omega_i \rangle) \rangle - \sum_{i=1}^{I} \langle P, \omega_i \rangle \\
&= \sum_{i=1}^{I} \langle P, \omega_i \rangle - \sum_{i=1}^{I} \langle P, \omega_i \rangle \\
&= 0
\end{aligned}$$

□

Denote by TS the tangent bundle of S and TS_0 its zero section. We can then interpret Z_ω as a section of TS and an equilibrium as a point where this section intersects TS_0.

2.5. **The Fredholm Index of the Excess Demand.**

Proposition 2. *The excess demand function $Z_\omega : S \to C(M, \mathbb{R}^n)$ of economy $\omega \in \Omega$ is a Fredholm map of index zero.*

Proof. Chichilnisky and Zhou [4] have shown that the derivative Df_i of each individual demand function is a linear Fredholm operator of index zero. This is because Df_i can be written as the sum of the finite rank operator

$$-\frac{\lambda\langle P(t), (u^i_{xx})^{-1} DP(t)\rangle + \langle DP(t), f_i\rangle}{\langle P(t), (u^i_{xx})^{-1} P(t)\rangle}(u^i_{xx})^{-1} P(t)$$

and the invertible operator

$$\frac{(u^i_{xx})^{-1} P(t)}{\langle P(t), (u^i_{xx})^{-1} P(t)\rangle} Dw + \lambda (u^i_{xx})^{-1} DP(t)$$

In general the sum of two Fredholm operators of index zero is not again a Fredholm operator of index zero. However, the matrix (u^i_{xx}) is negative definite, and every negative definite matrix is invertible and its inverse is also negative definite.

□

2.6. **Z-Rothe vector fields.** Knowing that the excess demand function is a vector field on the price space, and that is a Fredholm map for which we know its index, we would like to give it the structure of a Z-Rothe vector field (see [13]). In section 4 it will become clear that we need a vector field that is outward pointing, so we insist $-Z_\omega$ to be Z-Rothe.

Let E be a Banach space and $\mathcal{L}(E)$ be the set of linear continuous maps from E to itself. Denote by $G\mathcal{L}(E)$ the general linear group of E; that is, the set of invertible linear maps in $\mathcal{L}(E)$. Let $C(E)$ be the linear space of compact linear maps from E to itself.

We write $\mathcal{S}(E) \subset G\mathcal{L}(E)$ to denote the maximal starred neighborhood of the identity in $G\mathcal{L}(E)$. Formally

$$\mathcal{S}(E) = \{T \in G\mathcal{L}(E) : (\alpha T + (1-\alpha)I) \in G\mathcal{L}(E), \forall \alpha \in [0,1]\}$$

The **Rothe set** of E is defined as

$$\mathcal{R}(E) = \{A : A = T + C, T \in \mathcal{S}(E), C \in C(E)\}$$

and its invertible members by $G\mathcal{R}(E) = \mathcal{R}(E) \cap G\mathcal{L}(E)$.

A C^1 vector field X on a Banach manifold M is **Z-Rothe** if whenever $X(p) = 0$, $DX(P) \in \mathcal{R}(T_P M)$

Proposition 3. *The negative of the excess demand function, $-Z_\omega : S \to TS$ is a Z-Rothe vector field.*

Proof. From the proof of proposition 2, we know that $-Z_\omega$ can be written as the sum of a finite rank operator and an invertible operator. All we need to show then is that

$$\alpha \left[-\frac{(u^i_{xx})^{-1}P(t)}{\langle P(t), (u^i_{xx})^{-1}P(t)\rangle} Dw - \lambda (u^i_{xx})^{-1} DP(t) \right] + (1-\alpha)I$$

is invertible for all $\alpha \in [0,1]$. But this sum is just a homotopy of positive-definite operators.

□

3. Determinacy of equilibrium

In this section we wish to show parametric transversal density. We first need to give a manifold structure to the equilibrium set Γ.

3.1. **Regular Values.**

Proposition 4. *The derivative of the map $Z : \Omega \times S \to TS$ is a surjective map. In particular, it has 0 as a regular value.*

Proof. We need to compute the derivative $DZ : T(\Omega \times S) \to T(TS)$. Linearizing $Z(\omega, P)$ to first order in ϵ and letting $y_i = \langle P, \omega_i \rangle$, we get

$$
\begin{aligned}
&Z(\omega_1+\epsilon k_1,\dots,\omega_I+\epsilon k_I,P+\epsilon h)\\
&=\sum f_i(P+\epsilon h,\langle P+\epsilon h,\omega_i+\epsilon k_i\rangle)-\sum(\omega_i+\epsilon k_i)\\
&=\sum f_i(P+\epsilon h,\langle P,\omega_i\rangle+\epsilon\langle P,k_i\rangle+\epsilon\langle h,\omega_i\rangle)-\sum\omega_i-\epsilon\sum k_i\\
&=\sum[f_i(P,\langle P,\omega_i\rangle)+\epsilon(D_{y_i}f_i)_{(P,\langle P,\omega_i\rangle)}(\langle P,k_i\rangle)+\\
&+\epsilon(D_{y_i}f_i)_{(P,\langle P,\omega_i\rangle)}(\langle h,\omega_i\rangle)+\epsilon(D_Pf_i)_{(P,\langle P,\omega_i\rangle)}(h)]-\sum\omega_i-\epsilon\sum k_i\\
&=Z(\omega_1,\dots,\omega_I,P)+\\
&+\epsilon\sum[(D_Pf_i)_{(P,\langle P,\omega_i\rangle)}(h)+(D_{y_i}f_i)_{(P,\langle P,\omega_i\rangle)}(\langle P,k_i\rangle+\langle h,\omega_i\rangle)-k_i]
\end{aligned}
$$

So

$$
\begin{aligned}
&DZ_{(\omega,P)}(k_1,\dots,k_I,h)\\
&=\sum_{i=1}^{I}\left[(D_Pf_i)_{(P,\langle P,\omega_i\rangle)}(h)+(D_{y_i}f_i)_{(P,\langle P,\omega_i\rangle)}(\langle P,k_i\rangle+\langle h,\omega_i\rangle)-k_i\right]
\end{aligned}
$$

Or in matrix form, $DZ_{(\omega,P)}=$

$$
\left(\begin{array}{c|c}
\overbrace{0\dots0}^{I} & 1\\
\hline
\underbrace{(D_{y_i}f_i)_{(P,\langle P,\omega_i\rangle)}(\langle P,-\rangle)-Id}_{i=1,\dots,I} & \begin{array}{c}\sum_i(D_Pf_i)_{(P,\langle P,\omega_i\rangle)}(-)+\\ \\ +\sum_i(D_{y_i}f_i)_{(P,\langle P,\omega_i\rangle)}(\langle -,\omega_i\rangle)\end{array}
\end{array}\right)
$$

To compute the cokernel let

$$DZ_{(\omega,P)}(k_1,\dots,k_I,h)=(Q,\dot{Q})\in T(TS)$$

We need to solve for $(k_1,\dots,k_I,h)$. We first observe that $h=Q$. The second row would then be,

$$\sum\{[(D_{y_i}f_i)(\langle P,k_i\rangle)-(k_i)]+[(D_Pf_i)(Q)]+[(D_{y_i}f_i)(\langle Q,\omega_i\rangle)]\}=\dot{Q}$$

Then

$$\sum[(D_{y_i}f_i)(\langle P,k_i\rangle)-(k_i)]=H(Q,\dot{Q}) \tag{1}$$

where

$$H(Q,\dot{Q}) = \dot{Q} - \sum\{[(D_P f_i)(Q)] + [(D_{y_i} f_i)(\langle Q, \omega_i\rangle)]\}$$

But for every $i = 1, \ldots, I$, $(D_{y_i} f_i)(\langle P, k_i\rangle) - (k_i)$ is onto. And, therefore, so is DZ.

□

3.2. **The Infinite Equilibrium Manifold.** Knowing that 0 is a regular value of Z we would like to give the equilibrium set Γ the structure of a Banach manifold.

Quinn [9] defines a C^∞ **representation of maps** $\rho : A : M \to N$ consisting of a Banach manifold A together with a function $\rho : A \to C^\infty(M,N)$ such that the evaluation map

$$Ev_\rho : A \times M \to N; (a,m) \mapsto \rho_a(m)$$

is C^∞. In our situation, $Ev_\rho : A \times M \to N$ corresponds to $Z : \Omega \times S \to TS$.

Suppose we have a C^∞ map $F : W \to N$ which is transversal to Ev_ρ. If we form the pullback diagram

$$\begin{array}{ccc} P & \xrightarrow{g} & W \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle F} \\ A \times M & \xrightarrow{Ev_\rho} & N \\ {\scriptstyle \pi_A}\downarrow & & \\ A & & \end{array} \tag{2}$$

where $P = (Ev_\rho \times F)^{-1}(\Delta_N)$, then P is a C^∞ Banach manifold, and $\pi_a \circ h$ is a C^∞ map.

Proposition 5. *The equilibrium set Γ is a C^∞ Banach manifold of dimension equal to dim ker DZ. We shall call it the **equilibrium manifold**. Furthermore the natural projection map $pr_\Omega : \Omega \times S|_\Gamma \to \Omega$ is a C^∞ map.*

Proof. Notice that the inclusion $0 \to C(M,\mathbb{R}^n)$ is a C^∞ map. We also know from Proposition 4 that DZ is surjective, so it has 0 as a regular value. Then, we can form the pullback diagram

$$\begin{array}{ccc} \Gamma & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \Omega \times S & \xrightarrow{Z} & TS \\ {\scriptstyle pr_\Omega}\downarrow & & \\ \Omega & & \end{array}$$

and as in diagram (2) we get that Γ is a C^∞ Banach manifold and the natural projection map is a C^∞ map. $\square$

3.3. Regular Economies.

Definition 2. *We say that an economy is* ***regular*** *(resp.* ***critical****) if and only if ω is a regular (resp. critical) value of the projection $pr : \Gamma \to \Omega$.*

Definition 3. *Let Z_ω be the excess demand of economy ω. A price system $P \in S$ is a* ***regular equilibrium price*** *system if and only if $Z_\omega(P) = 0$ and $DZ_\omega(P)$ is surjective.*

We would like to compare the set of regular economies with those economies whose excess demand function has only regular prices. Quinn [9] will tell us that these two sets coincide; precisely, in diagram (2), $\rho_a \pitchfork F$ if and only if a is a regular value of $\pi_A \circ h$

And so we get,

Proposition 6. *The economy $\omega \in \Omega$ is regular if and only if all equilibrium prices of Z_ω are regular.*

Proof. Consider the diagram

$$\begin{array}{ccc} \Gamma & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \Omega \times S & \xrightarrow{Z} & TS \\ {\scriptstyle pr_\Omega}\downarrow & & \\ \Omega & & \end{array}$$

Quinn's result says that the excess demand Z_ω is transversal to zero if and only if 0 is a regular value of pr_Ω. □

3.4. **Determinacy of equilibrium.** We would now like to understand how big is the set of economies that give an excess demand function with all equilibrium prices being regular. For that, we need a couple of definitions.

A **left Fredholm map** is a map of Banach manifolds of class at least C^1 whose derivative at each point has closed image and finite dimensional kernel.

A map is $\sigma-$**proper** if its domain is the countable union of sets, restricted to each of which the function is proper.

Quinn [9] has also proves that a transversal density theorem holds in infinite dimensions.

Theorem 1. [9] *Let $\rho : A : M \to N$ be a C^∞ representation of left Fredholm maps, M separable, and $F : W \to N$ a C^∞ σ-proper left Fredholm map. If further*

1. *F is transversal to Ev_ρ, and*
2. *each ρ_a satisfies that for each $m \in M$ and $w \in W$ such that $\rho_a(m) = F(w)$, then $(imT_m\rho_a) \cap (imT_wF)$ is finite dimensional*

then the set of a with $\rho_a \pitchfork F$ is residual in A.

The infinite-dimensional transversal density theorem can be used to give us an alternative proof that almost all economies are regular.

Proposition 7. *Almost all economies are regular. That is, the set of economies $\omega \in \Omega$ transversal to zero are residual in Ω.*

Proof. We have seen that $Z_\omega : S \to TS$ is a Fredholm map. In particular it is a left Fredholm map. Observe that the inclusion $0 \to C(M, \mathbb{R}^n)$ is σ-proper.

We also know that $Z(\omega, P)$ has 0 as a regular value since $DZ(\omega, P)$ is surjective. Finally, for each $P \in S$ such that $Z_\omega(P) = 0$ we have

$$(\mathrm{im}T_PZ_\omega) \cap (\mathrm{im}T_0 0)$$

is finite dimensional since obvioulsy 0 is finite. Therefore, theorem 1 implies the result. □

4. The Index Theorem of Smooth Infinite Economies

Knowing that most economies are regular we need to find a right way of counting the number of equilibria. With an excess demand function that is a Fredholm map, we may use tools of infinite-dimensional differential topology that resembles the finite dimensional case. This is why the proof of the index theorem in our setting is very similar to that of Dierker [5].

Suppose that the excess demand satisfies the boundary assumption of [5], namely that if $P_n \in S$ and $P_n \to P \in \partial S$, then $\|Z_\omega(P_n)\| \to \infty$. Suppose also that Z_ω is bounded below. Then $-Z_\omega$ points outwards. Finally, assume that there are only finitely many zeros.

4.1. **Euler Characteristic.** A zero P of a vector field X is **nondegenerate** if $DX(P) : T_P M \to T_P M$ is an isomorphism.

Suppose that a Z-Rothe vector field X has only nondegenerate zeros, and let P be one of them. Then, $DX(P) \in G\mathcal{R}(E)$. Tromba [13] shows that $G\mathcal{R}(E)$ has two components; $G\mathcal{R}^+(E)$ denotes the component of the identity. Define

$$sgnDX(P) = \begin{cases} +1, & \text{if } DX(P) \in G\mathcal{R}^+(T_P M) \\ -1, & \text{if } DX(P) \in G\mathcal{R}^-(T_P M) \end{cases}$$

The **Euler characteristic** is then given by the formula

$$\chi(X) = \sum_{P \in Zeros(X)} sgnDX(P)$$

Tromba also shows that this Euler characteristic is invariant under homotopy of vector fields. All we have to do is to construct a vector field on S that has only one singularity and that is homotopic to Z_ω.

4.2. **The Index Theorem of Smooth Infinite Economies.**

Proposition 8. *Suppose that an aggregate excess demand function Z_ω is bounded from below and that it satisfies the boundary assumption. Suppose also that Z_ω has only finitely many singularities and that they are all nondegenerate. Then,*

$$\sum_{P \in Zeros Z_\omega} sgn\left[-DZ_\omega(P)\right] = 1$$

Proof. For any fixed $Q \in C^{++}(M, \mathbb{R}^n)$ define the vector field $Z^Q : \bar{S} \to TS$ given by

$$Z^Q(P) = \left[\frac{Q(t)}{\langle P(t), Q(t)\rangle}\right] - P(t)$$

By construction, $Z^Q(P)$ has only one zero and is inward-pointing. Its derivative $DZ^Q_{(P)} : T\bar{S} \to T(TS)$ is given by

$$DZ^Q_{(P)}(h) = -\frac{Q\langle h, Q\rangle}{\langle P, Q\rangle^2} - h$$

where $-\frac{Q\langle h,Q\rangle}{\langle P,Q\rangle^2}$ is compact and $-h$ is invertible; then $DZ^Q \in \mathcal{R}(T_P S)$. Now let

$$-\frac{Q\langle h, Q\rangle}{\langle P, Q\rangle^2} - h = h' \tag{3}$$

We need to solve for h. Then,

$$Q\langle h, Q\rangle + h\langle P, Q\rangle^2 = -h'\langle P, Q\rangle^2$$

Acting Q on both sides we get,

$$\langle Q, Q\rangle\langle h, Q\rangle + \langle h, Q\rangle\langle P, Q\rangle^2 = -\langle h', Q\rangle\langle P, Q\rangle^2$$

Solving for $\langle h, Q\rangle$ we get

$$\langle h, Q\rangle = \frac{-\langle h', Q\rangle\langle P, Q\rangle^2}{\langle Q, Q\rangle + \langle P, Q\rangle^2}$$

where the denominator never vanishes since $Q \in C^{++}(M, \mathbb{R}^n)$. Substituting $\langle h, Q\rangle$ in 3 we then get

$$h = h' + \frac{Q}{\langle P, Q\rangle^2}\left[\frac{\langle h', Q\rangle\langle P, Q\rangle^2}{\langle Q, Q\rangle + \langle P, Q\rangle^2}\right]$$

This shows that DZ^Q is invertible and therefore $DZ^Q \in G\mathcal{R}(T_P S)$. Furthermore, since it is not in the same component of the identity it has to be in $G\mathcal{R}^-(T_P S)$ and its only zero has index -1. The vector field Z^Q is inward pointing so reversing orientation will make outward poiting with index of +1.

□

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