The Dynamic Assignment of Heterogenous Objects: A Mechanism Design Approach

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Abstract

We study the allocation of several heterogenous, commonly ranked objects to impatient agents with privately known characteristics who arrive sequentially according to a Poisson or renewal process. We analyze and compare the policies that maximize either welfare or revenue. We focus on two cases: 1. There is a deadline after which no more objects can be allocated; 2. The horizon is potentially infinite and there is time discounting. We first characterize all implementable allocation schemes, and we compute the expected revenue for any implementable, deterministic and Markovian allocation policy. These properties are shared by the welfare and revenue maximizing policies. Moreover, we show that these policies do not depend on the characteristics of the available objects at each point in time. The revenue-maximizing allocation scheme is obtained by a variational argument which sheds somewhat more light on its properties than the usual dynamic programming approach. We also obtain several properties of the welfare maximizing policy using stochastic dominance measures of increased variability and majorization arguments. These results yield upper/lower bounds on efficiency/revenue for large classes of distributions of agents' characteristics or of distributions of inter-arrival times for which explicit solutions cannot be obtained in closed form.

1 Introduction

We study the following dynamic mechanism design problem in continuous time: a designer has to allocate (or assign) a fixed, finite set of heterogenous objects with known characteristics to a stream of randomly arriving agents with privately known characteristics. The objects are substitutes, and each agent is willing to get at most one object. Moreover, all agents rank the available objects

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in the same way, and values for objects have a multiplicative structure involving the agents' and objects' types . In one formulation we assume that there is a deadline by which all objects must be sold, in another we assume a discounted infinite horizon. Under the assumption that monetary transfers are feasible, we analyze and compare two distinct goals for the designer: maximization of expected welfare, and maximization of expected revenue.

For both considered goals, the main trade-off is as follows: assigning an object today means that the valuable option of assigning it in the future - possibly to an agent who values it more-, is foregone; on the other hand, since the arrival process of agents is stochastic, the "future" may never materialize (if there is a deadline) or it may be farther away and thus discounted.

The basic dynamic assignment problem has numerous applications such as the retail of seasonal and style goods, the allocation of fixed capacities in the travel and leisure industries (e.g., airlines, trains, hotels, rental cars), the allocation of priorities in a queue (e.g., for medical procedures), the assignment of personnel to incoming tasks. More recently, dynamic pricing methods plays an increased role in the allocation of electricity and bandwidth.

Whereas a large literature on *yield or revenue management* has directly focused on revenue-maximizing pricing (mostly for the special case of linear, private values for identical objects) our approach starts by a characterization of **all** dynamically implementable deterministic allocation policies. All deterministic, implementable policies are described by partitions of the set of possible agent types: an arriving agent gets the best available object if his type lies in the highest interval of the partition, the second best available object if his type lies in the second highest interval, and so on... These intervals generally depend on the point in time of the arrival and on the composition of the set of available objects at that point in time.

For any implementable allocation policy we derive the associated menus of prices (one menu for each point in time, and for each subset of remaining objects) that implement it, and show that these menus have an appealing recursive structure: each agent who is assigned an object has to pay the value he displaces in terms of the chosen allocation.

A first application is to the implementation of the dynamically welfare-maximizing allocation policy under Poisson arrivals, which has been characterized for the complete information case - via a system of differential equations - by Albright [1]. Since that policy is deterministic, Markovian and has the form of a partition, it can be implemented also in our private information framework by the dynamic price schedules identified above, which coincide then with the dynamic version of the Vickrey-Clarke-Groves mechanism. A rather surprising feature is that the cutoff curves defining the intervals in the time-dependent partitions that characterize the dynamic welfare maximizing policy depend only on the cardinality of the set of available objects, but not on the exact composition of that set. This is due here to the multiplicative structure of the agents' valuations for objects.

The dynamically efficient allocation policy can be explicitly computed if the distribution of agents' types is exponential, while this is not often the case for general distributions. But, we can use comparative static results in order to bound the cutoff curves in the welfare maximizing policy (and the associated expected welfare) for the important and large, non-parametric classes of distributions that second-order stochastically dominate (are dominated by) the exponential distribution - these are the so called *new better (worse) than old in expectation* distributions. We show that a decrease in the second order stochastic sense in the distribution of agents' types (which implies an increase in variability) leads to an increase in expected welfare in the dynamic assignment problem. The proof of this result uses several simple insights from *majorization theory.* [1]

While the above comparative static result holds for both the deadline model and for the discounted, infinite horizon model, in the latter case - where the welfare maximizing policy can be characterized and turns out to be time-independent for general *renewal* arrival processes - we also examine the effect on expected welfare of a stochastic increase in the distribution of inter-arrival times in the sense of the *Laplace-transform order.* This stochastic order is much weaker than second order stochastic dominance. In particular, more variability in inter-arrival times leads to higher expected welfare. Here bounds on expected welfare relative to the case with an exponential distribution of agents' types and with a Poisson arrival process can be expressed in terms of the well known *Lambert-W function*.

We next switch attention to revenue-maximization. Using several basic results about the Poisson stochastic process, we first compute the revenue generated by any individual-rational, deterministic, Markovian and implementable allocation policy. Then, we can directly use variational arguments in order to characterize the revenue-maximizing policy. The associated optimal prices are of "secondary importance" since they are completely determined by the implementation conditions. Whereas the optimal prices necessarily depend on the composition of the set of available objects, our main result is that, at each point in time, the revenue maximizing allocation policy depends only on the size of the set of available objects, but not on the exact composition of that set. The argument for this somewhat surprising result is somewhat subtler than that encountered for welfare-maximization.

To understand the meaning of this result, consider the same model, but with identical objects. Then, for each size of available inventory, and for each point in time, the revenue maximizing allocation policy is characterized by a single cut-off type: only an arriving agent with type above that cut-off obtains one of the objects. In contrast, when objects are heterogenous, the revenue maximizing policy is, at each point in time, and for each subset of available objects, characterized by several cut-off types which determine if the arriving agent gets the best available object, the second best, etc... Our result implies that, for any subset of k available heterogenous objects, and for any point in time, the highest cut-off coincides with the optimal cut-off in a situation with one available object, the second-highest cut-off coincides with the optimal cutoff

[1] See Hardy, Littlewood and Polya, [17].

in a situation with two identical objects, and so on till the lowest cut-off which coincides with the optimal cut-off in a situation with an inventory of k identical objects.

Our last result is devoted to a comparison of the welfare maximizing and revenue maximizing allocation policies. We show that, for distributions of agents' types that have an increasing failure (or hazard) rate, the revenue maximizing policy is, overall, strictly more conservative: at any point in time, the cutoff curves defining the revenue maximizing dynamic policy are strictly above the respective cutoff curves defining the welfare maximizing policy[2]. This result can be combined with the comparative statics results obtained for welfare maximization in order to obtain bounds on revenue for many cases where an explicit computation is not feasible.

Finally, we want to note that our focus on implementable allocation policies rather than on prices - inspired by the "mechanism design philosophy" and the payoff/revenue equivalence principle - is crucial for our results. Even for the much studied case of dynamic revenue-maximization for identical goods, our approach yields new insights and formulas relative to the approach that uses dynamic programming and Bellman's equations to characterize optimal prices. Moreover, our approach facilitates the calculation and assessment of the revenue associated to simple policies that may be used in practice, even if they are not optimal in some sense, e.g., the use of a finite set of prices/price adjustments.

The rest of the paper is organized is follows: In the remainder of this Section we review the related literature.

In Section 2 we present the continuous-time model of sequential assignment of several heterogenous objects to randomly arriving, privately informed agents.

Section 3 focuses on a characterization of implementable, deterministic and Markovian policies, and of the associated menus of dynamic prices that implement such policies.

In Section 4 we present a Theorem, due to Albright [1] that determines the dynamic welfare maximization policy in a framework with complete information and Poisson arrivals. In Subsection 4-1 we apply Albright's theorem to a setting where the allocation of all available objects must occur before a known deadline, and we consider the effect of changes in the distribution of agents' types. Subsection 4-2 deals with an infinite horizon model with exponential discounting. There we consider both the effects of changes in the distribution of agents' types and in the distribution of inter-arrival times.

In Section 5 we turn to revenue maximization, and focus first on the deadline case and Poisson arrivals. We obtain a general expression for expected revenue, and we use a variational argument in order to derive functional equations that characterize the revenue maximizing allocation policy and the expected revenue generated by this policy. For the variational argument we need to assume that the agents' virtual valuation function is increasing. In Subsection 5-1 we briefly

[2]We also show that this result may fail if the hazard rate condition is not fulfilled, even if virtual valuations are increasing.

characterize the (stationary) revenue maximizing policy for the infinite horizon case with exponential discounting and Poisson arrivals.

Section 6 uses the above results for a comparison of the welfare maximizing and revenue maximizing allocation policies.

Most of the proofs are relegated to an Appendix.

1.1 Related Literature

There is a large theoretical and applied literature on dynamic pricing of inventories (sometimes called *revenue or yield management*) in the fields of Management and Operations Research[3]. We refer the reader to the surveys by Bitran and Caldentey [6] and Elmaghraby and Keskinocak [12], and to the book by Talluri and Van Ryzin [25]. McAfee and te Velde [20] survey the applications to the airline industry. That industry pioneered many of the modern practices in revenue management.

As Bitran and Caldentey [6] note, due to the technical complexity, the literature on dynamic revenue maximization with stochastic demand has focused on models with identical objects[4]. In a continuous-time framework with stochastic arrivals of agents, Kincaid and Darling [18], and Gallego and Van Ryzin [13] use dynamic programming in order to characterize - implicitly via Bellman's equations - the revenue maximizing pricing policy for a set of identical objects that need to be sold before a deadline. A main result is that the expected revenue in the optimal policy - which is characterized for each size of inventory by a single posted price - is increasing and concave both in the number of objects and in the length of time left till the deadline. Moreover, each relevant cutoff price drops with time as long as there is no sale, but jumps up after each sale[5]. These authors were able to calculate in closed form the solution for what amounts (in our terms) to an exponential distribution of agents' values. Generally, a closed form solution is not available, and even the general expression of expected revenue as a function of the optimal cutoff prices is not available in the literature.

Arnold and Lippman [2], Das Varma and Vettas [9] and Gallien [15] consider the same basic problem as above, but in a framework with an infinite horizon and discounting. In this case, the revenue maximizing posted prices - again one price for each size of inventory -, turn out to be stationary, i.e., do not depend on time[6].

In contrast to the above focus on revenue maximization, the mechanism

[3] The relevant Economics literature is much smaller. We do not discuss here the literature on the so-called "Coase Conjecture" where the inventory can be replenished and agents are strategic about their arrival.

[4] See Gallego and Van Ryzin [14] for an exception. Some models assume identical objects but assume that customers belong to several known classes which allows the use of price discrimination.

[5] Bitran and Mondschein [5] obtain similar results in a discrete time framework.

[6] Das Varma and Vettas consider a model with discrete time and deterministic arrivals, whereas Gallien and Arnold and Lippman have continuous time models with stochastic arrivals. The latter authors assume (rather than derive) the stationarity of posted prices, and also compare these to reservation prices in a model where arriving agents announce bids.

design literature on dynamic welfare maximization is somewhat less well developed. An early paper which uses optimal stopping theory to characterize the efficient assignment of a single object to randomly arriving agents in continuous time is Elfving [11].

Derman, Lieberman and Ross [8] introduced a model where a set of distinct, but commonly ranked, objects needs to be assigned to a set of sequentially arriving agents. There is a finite number of periods (time is discrete), and one agent arrives at each period. Both objects and agents can have different types and these determine the agents' valuations for the objects via a supermodular function. The objects' types are fixed, and the agents' types becomes common knowledge only upon arrival. In other words, at each decision node there is complete information about the present, and uncertainty about the future. These authors characterize the welfare maximizing assignment policy, and show that, surprisingly, under a multiplicative specification of values, the optimal policy at each stage does not depend on the characteristics of the available objects (this policy does depend though on number of periods and objects which are left). Albright [1] extends the Derman-Lieberman-Ross model to a continuous-time framework with random arrivals of agents[7]. Albright's model and results form the basis for the present paper.

In Gallego and van Ryzin's framework with identical objects, McAfee and te Velde [19] compute the dynamic welfare maximizing policy for a Pareto distributions of agents' values, and show that it coincides there with the revenue optimizing policy.

Finally, note that if the objects to be allocated in the D-L-R or Albright model are places (or ranks) in a queue, one obtains an instance of a queueing problem with priorities. Dolan [10] pioneered the use of dynamic versions of the Vickrey-Clarke-Groves mechanisms in order to achieve welfare maximization in queues with random arrivals and with incomplete information about the agents' characteristics. Dynamic extensions of VCG schemes (for much more general situations than those considered by Dolan) have recently attracted a lot of interest (see for example Athey and Segal [3], Bergemann and Valimäki [4], and Parkes and Singh [21]).

2 The Model

There are n items (or objects). Each item i is characterized by a "type" p_i. Each agent j is characterized by a "type" x_j. Agents arrive according to a (possibly non-homogenous) Poisson process with intensity $\lambda(t)$, and each can only be served upon arrival (i.e., agents are impatient). After an item is assigned, it cannot be reallocated in the future. For some results we relax the Poisson assumption, and we allow for a more general renewal stochastic process that describe arrivals.

[7] An early paper dealing with the efficient assignment of a single object to randomly arriving agents in continuous time is Elfving [11].

An agent with type x_j who obtains an object with characteristic p_i enjoys an utility of $p_i x_j$. If an item with type p_i is assigned to an agent with type x_j at time t, then the utility for the designer is given by $r(t)p_i x_j$ where r is a piecewise continuous, non-negative, non-increasing discount function which satisfies $r(0) = 1$.

While the items' types $0 \leq p_n \leq p_{n-1} \leq ... \leq p_1$ are assumed to be known constants, the agents' types are assumed to be represented by independent and identically distributed random variables X_i on $[0, +\infty)$ with common c.d.f. F. The realization of X_i is private information of agent i. We assume that each X_i has a finite mean, denoted by μ, and a finite variance.

3 Implementable Policies

Without loss of generality, we restrict attention to direct mechanisms where every agent, upon arrival, reports his characteristic x_i and where the mechanism specifies an allocation (which item, if any, the agent gets) and a payment. As we shall see, the schemes we develop also have an obvious and immediate interpretation as indirect mechanisms, where the designer sets a (possibly time-dependent) menu of prices, one for each item, and the arriving agents are free to choose out that menu.

An allocation policy is called *deterministic* and *Markovian* if, at any time t, and for any possible type of agent arriving at t, it uses a non-random allocation rule that only depends on the arrival time t, on the declared type of the arriving agent, and on the set of items available at t, denoted by Π_t. Thus, the policy depends on past decisions only via the state variable Π_t. Gihman and Skorohod [16] give sufficient conditions ensuring that a controlled stochastic process has an optimal policy that is deterministic and Markovian. These conditions are satisfied for the type of problems (e.g., welfare or revenue maximization) discussed here.

Denote by $p_t : [0, +\infty) \times \Pi_t \to \Pi_t \cup \emptyset$ a deterministic Markovian allocation policy for time t and by $q_t : [0, +\infty) \times \Pi_t \to \mathbb{R}$ the associated payment rule. Denote also by k_t the cardinality of set Π_t. Finally, we restrict attention to interim-individually rational policies, where no agent ever pays more than the utility obtained from the physical allocation.

The next Proposition shows that a deterministic Markovian allocation policy is implementable if and only if it based on a partition of the agents' type space.[8]

Proposition 1 *Assume that Π_t is the set of objects available at time t, and assume that $p_j \neq p_k$ for any $p_j, p_k \in \Pi_t$, $j \neq k$. A deterministic, Markovian policy p_t is implementable if and only if there exist k_t+1 functions $\infty = y_{0,\Pi_t}(t) > y_{1,\Pi_t}(t) \geq y_{2,\Pi_t}(t) \geq \cdots \geq y_{k_t,\Pi_t}(t) \geq 0$, such that $x_i \in (y_{j,\Pi_t}(t), y_{j-1,\Pi_t}(t))$ $\Rightarrow p_t(x_i, \Pi_t) = p_{(j)}$ where $p_{(j)}$ denotes the j'th highest element of the set Π_t, and*

[8] In fact, the result holds for any deterministic policy. But, since the rest of the analysis focuses on the Markov case, and in order to save on notational complexity, we consider only this case also here.

such that $x_i < y_{k_t,\Pi_t}(t) \Rightarrow p_t(x_i,\Pi_t) = \emptyset$.[9] *Moreover, the associated payment scheme must satisfy* $q_t(x_i,\Pi_t) = q_t(\widetilde{x_i},\Pi_t)$ *if* $p_t(x_i,\Pi_t) = p_t(\widetilde{x_i},\Pi_t)$.

Proof. $\Longrightarrow$ Note first that, in any incentive compatible mechanism, if two reports lead to the same physical allocation, then the payment should be the same as well. Second, a direct mechanism is equivalent here to a mechanism where the arriving agent at time t chooses an object and a payment from a menu $(p_j,q_j)_{j=1}^{k_t}$. In addition, note that if some type x_i prefers the pair (p_k,q_k) over any other pair (p_l,q_l) with $p_k > p_l$, then any type $\widetilde{x_i} > x_i$ also prefers (p_k,q_k) over (p_l,q_l). Similarly, if some type x_i prefers (p_k,q_k) over (p_l,q_l) with $p_k < p_l$ then any type $\widetilde{x_i} < x_i$ also prefers (p_k,q_k) over (p_l,q_l). These observations allow us to conclude that an implementable policy must partition the set of the agents' reported types as in the Proposition's statement. Otherwise, there exist t,Π_t and $\overline{x_i} > x_i > \underline{x_i}$ such that $p_t(\overline{x_i},\Pi_t) = p_t(\underline{x_i},\Pi_t) \neq p_t(x_i,\Pi_t)$. Assume first that $p_t(\overline{x_i},\Pi_t) > p_t(x_i,\Pi_t)$. Since type x_i prefers $(p_t(x_i,\Pi_t), q_t(x_i,\Pi_t))$ over $(p_t(\overline{x_i},\Pi_t), q_t(\overline{x_i},\Pi_t))$, the same should hold also for type $\underline{x_i}$, a contradiction to incentive compatibility. Assume next that $p_t(\underline{x_i},\Pi_t) < p_t(x_i,\Pi_t)$. Since type x_i prefers $(p_t(x_i,\Pi_t), q_t(x_i,\Pi_t))$ over $\left(p_t(\underline{x_i},\Pi_t), q_t(\underline{x_i},\Pi_t)\right)$, the same should hold also for type $\overline{x_i}$, which yields again a contradiction.

That is, an agent who arrives at time t gets object $p_{(k)}$ if he reports a type contained in the interval $(y_{k+1,\Pi_t}(t), y_{k,\Pi_t}(t))$. Moreover, if $\widetilde{x_i} > x_i$, then $p_t(\widetilde{x_i},\Pi_t) \geq p_t(x_i,\Pi_t)$. A similar argument shows that $p_t(y_{i,\Pi_t}(t),\Pi_t) \in \{p_{(i)}, p_{(i-1)}\}$ for $i \in \{1,2,...,k_t\}$.

$\Longleftarrow$ The proof is constructive. That is, given a policy which is based on a partition, we design a payment scheme q_t that, for any $j \in \{1,...,k_t\}$, will lead type $x_i \in (y_{j,\Pi_t}(t), y_{j-1,\Pi_t}(t)]$ to choose the object with type $p_{(j)}$. Without loss of generality, we assume that an agent whose type is on the boundary between two intervals in the partition chooses the item with higher type. Consider then the following payment scheme

$$q_{j,\Pi_t}(t) = \sum_{i=j}^{k_t}(p_{(i)} - p_{(i+1)})y_{i,\Pi_t}(t). \tag{1}$$

Note that type $x_i = y_{j,\Pi_t}(t)$ is indifferent between $(p_{(j)}, q_j)$ and $(p_{(j+1)}, q_{j+1})$. Moreover, any type above $y_{j,\Pi_t}(t)$ prefers $(p_{(j)}, q_j)$ over $(p_{(j+1)}, q_{j+1})$, while any type below prefers $(p_{(j+1)}, q_{j+1})$ over $(p_{(j)}, q_j)$. Therefore, any type $x_i \in (y_{j,\Pi_t}(t), y_{j-1,\Pi_t}(t)]$ prefers $(p_{(j)}, q_j)$ over any other pairs in the menu.[10] ■

We assumed above that the set of objects available at t contains only objects with distinct types. If there are some identical objects, there exist other implementable policies that do not take the form of partitions. But, for each such

[9] Types at the boundary between two intervals can be assigned to either one of the neighboring elements of the partition. That is, if $x_i \in \{y_{k_t,\Pi_t}(t), y_{k_t-1,\Pi_t}(t), ..., y_{2,\Pi_t}(t), y_{1,\Pi_t}(t)\}$, then $p_t(y_{i,\Pi_t}(t),\Pi_t) \in \{p_i, p_{i+1}\}$, $i = 1,2,..,k_t$.

[10] The payment given in (1) is not the only one implementing the partition $\infty > y_{1,\Pi_t}(t) \geq y_{2,\Pi_t}(t) \geq \cdots \geq y_{k_t,\Pi_t}(t) \geq 0$. Adding to the payment any function that does not depend on the reported type of the agent will not change the implemnted partition.

policy, there exists another implementable policy that is based on a partition, and that generates the same expected utility for all agents and for the designer.

4 The Dynamically Efficient Policy

Albright [1] characterized the allocation policy that maximizes the total expected welfare from the designer's point of view in a complete-information model. That is, upon arrival, the type of the arriving agent becomes public information. His main result is:

Theorem 1 *(Albright, [1]) There exist n unique functions $y_n(t) \leq y_{n-1}(t)... \leq y_1(t)$, $\forall t$, which do not depend on the p's such that:*

1. *If an agent with type x arrives at a time t, it is optimal to assign to that agent the j'th highest element of Π_t if $x \in (y_j(t),\ y_{j-1}(t)]$,where $y_0 \equiv \infty$, and not to assign any object if $x < y_{k_t}(t)$.*

2. *For each k, the function $y_k(t)$ satisfies :*

   (a) $lim_{t\to\infty} r(t)y_k(t) = 0$

   (b) $\frac{d[r(t)y_k(t)]}{dt} = -\lambda(t)r(t)\int_{y_k}^{y_{k-1}}(1-F(x))dx \leq 0$.

3. *The expected welfare starting from time t is given by $\left[\sum_{i=1}^{k_t} r(t)p_{(i)}y_i(t)\right]$, where $p_{(i)}$ is the i'th highest element of Π_t.*

Remark 1 *The most surprising element in the above result is that the dynamic welfare maximizing cutoff curves $y_j(t)$ do not depend on the items' characteristics. In other words, although the Markov decision problem is one with $2^n - 1$ states, corresponding to all possible non-empty subsets of items Π_t, the welfare maximizing policy is such that, at each point in time t , and for each type of the arriving agent, the allocation decision is only contingent on the cardinality of Π_t, k_t. Moreover, since selling one object is equivalent to exchanging one of the currently available items with an item having a type equal to zero, the above observation implies that after any sale at time t, the $k_t - 1$ curves that determine the optimal allocation from time t on, coincide with the $k_t - 1$ highest curves that were relevant for the decision at time t. Thus, in effect, there are only n relevant states for the decision maker instead of $2^n - 1$. To understand the intuition behind this result, assume for simplicity that at time t there are two objects $p_1 > p_2$ and that the relevant cutoffs are $y_1^e > y_2^e$. Consider the effect of a small shift in the highest cut-off from y_1^e to $y_1^e + \epsilon$. Note first that this shift has any effect only if an agent indeed arrives at t. Second, the shift has no effect on designer's welfare if the arriving agent has a value above $y_1^e + \epsilon$ or below y_1^e. If, however, at time t an agent with value y_1^e arrives, then this shift switches the object he gets from p_2 to p_1 and therefore switches the object available for future* *allocation from p_1 to p_2. Therefore, the effect of the shift on the social welfare is*

$$\begin{aligned} & f\left(y_1^e\right)\left(p_1 y_1^e + W\left(p_2, t\right) - p_2 y_1^e - W\left(p_1, t\right)\right) \\ = \quad & \left(p_1 - p_2\right) f\left(y_1^e\right)\left(y_1^e - W\left(1, t\right)\right) \end{aligned}$$

where $W(p,t)$ denotes here the expected welfare at time t if only one object with type p remains, given that the optimal policy is followed from time t on and the equality follows since $W(p,t)$ is linear in p. Note that the above expression is linear and separable in the difference $(p_1 - p_2)$, and therefore the optimal cutoff - where the total effect of a shift should be equal to zero - will not depend on this difference.

Since the Markovian, deterministic policy described in Theorem 1 has the form of a partition, it can be implemented by the payments (or by prices in an indirect mechanism) described in Proposition 1. Note that Theorem 1 -3 implies that the payment

$$q_{j,\Pi_t}\left(t\right) = \sum_{i=j}^{k_t} (p_i - p_{i+1}) y_{i,\Pi_t}\left(t\right)$$

can be interpreted as the expected externality imposed on other agents by an agent that arrives at time t who gets the object with the j-th highest type among those remaining at time t. In other words, for the dynamically efficient policy, our implementing mechanism coincides with a dynamic Clarke-Groves-Vickrey mechanism, as studied by Athey and Segal ([3]), Bergemann and Valimäki ([4]), and Parkes and Singh ([21]).

4.1 The Dynamic Efficient Allocation with a Deadline

In this Section we apply Theorem 1 to a framework with deadline T after which all objects perish. Our main result shows that, under the dynamic welfare-maximizing policy, an increase in the variability of the distribution of the agents' values (while keeping a constant mean) increases expected welfare. We want to emphasize that this result holds even if all available objects are identical !

Besides its intrinsic interest, this result allows us to bound the welfare maximizing cutoff curves (and thus the expected welfare) for large and important families of distributions for which an explicit solution of the system of differential equations that characterizes the efficient policy is not available (see Theorem 1-2b).

We assume that the discount rate satisfies:

$$r(t) = 1 \text{ for } 0 \leq t \leq T \text{ and } r(t) = 0 \text{ for } t > T$$

It is then obvious that the dynamically efficient policy needs to satisfy

$$y_1(T) = y_2(T) = ... = y_n(T) = 0$$

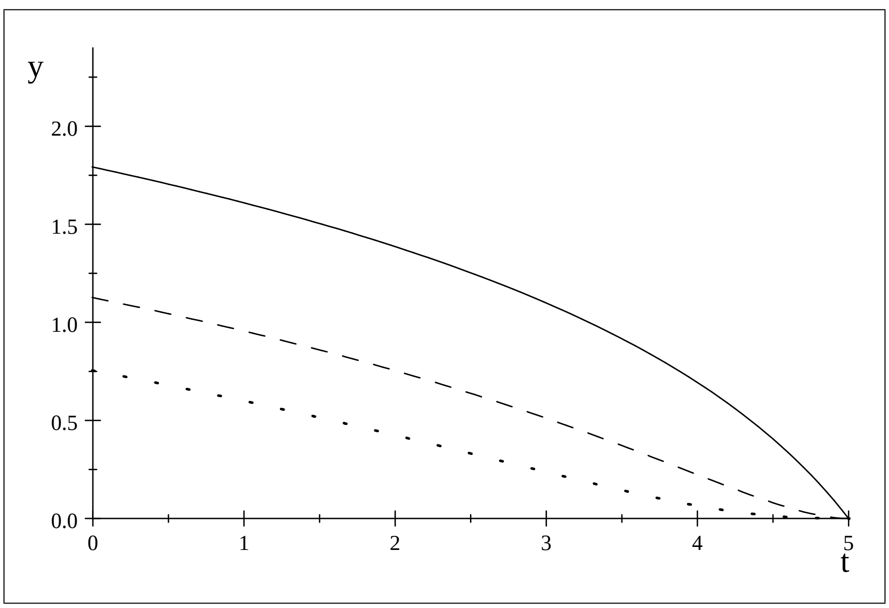
Example 1 *Assume that there are three objects, that the arrival process is homogenous with rate $\lambda(t) = \lambda$ normalized to be 1, and that the distribution of agents' types is exponential, i.e., $F(x) = 1 - e^{-x}$. From Theorem 1, we obtain the following system of differential equations that characterize cut-off curves in the dynamic welfare maximizing policy:*

$$\begin{aligned} y_1' &= -\int_{y_1}^{\infty} e^{-x}dx = -e^{-y_1} \\ y_2' &= -\int_{y_2}^{y_1} e^{-x}dx = e^{-y_1} - e^{-y_2} \\ y_3' &= -\int_{y_3}^{y_2} e^{-x}dx = e^{-y_2} - e^{-y_3} \end{aligned}$$

with initial conditions $y_1(T) = y_2(T) = y_3(T) = 0$. The solution to this system is given by:

$$\begin{aligned} y_1(t) &= \ln(1 + T - t) \\ y_2(t) &= \ln\left(1 + \frac{(T-t)^2}{2(1+T-t)}\right) \\ y_3(t) &= \ln\left(1 + \frac{(T-t)^3}{3[(T-t)^2 + 2(1+T-t)]}\right) \end{aligned}$$

The following figure depicts the solution for $T = 5$:



y_1 *- solid line;* y_2 *- dashed line;* y_3 *- the dotted line*

For the main result in this Section, we first need a well- know concept, due to Hardy, Littlewood and Polya [17] and a Lemma:

Definition 1 *For any $n-$tuple $\gamma = (\gamma_1, \gamma_2, .., \gamma_n)$ let $\gamma_{(j)}$ denote the jth largest coordinate (so that $\gamma_{(n)} \leq \gamma_{(n-1)} \leq ... \leq \gamma_{(1)}$. Let $\alpha = (\alpha_1, \alpha_2, .., \alpha_n)$ and $\beta = (\beta_1, \beta_2, .., \beta_n)$ be two $n-$tuples. We say that α is* majorized *by β and we write $\alpha \prec \beta$ if the following system of $n-1$ inequalities and one equality is satisfied:*

$$\begin{aligned} \alpha_{(1)} &\leq \beta_{(1)} \\ \alpha_{(1)} + \alpha_{(2)} &\leq \beta_{(1)} + \beta_{(2)} \\ ... &\leq ... \\ \alpha_{(1)} + \alpha_{(2)} + ..\alpha_{(n-1)} &\leq \beta_{(1)} + \beta_{(2)} + \beta_{(n-1)} \\ \alpha_{(1)} + \alpha_{(2)} + .. + \alpha_{(n)} &= \beta_{(1)} + \beta_{(2)} + .. + \beta_{(n)} \end{aligned}$$

We say that α is weakly sub-majorized *by β and we write $\alpha \prec_w \beta$ if all relations above hold with weak inequality.*

Lemma 1 *Let $\alpha = (\alpha_1, \alpha_2, ..\alpha_n)$ and $\beta = (\beta_1, \beta_2, .., \beta_n)$ be two $n-$tuples such that $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i$. Then $\alpha \prec \beta$ if and only if $\sum_{i=1}^n p_i \alpha_{(i)} \leq \sum_{i=1}^n p_i \beta_{(i)}$ for any constants $p_n \leq p_{n-1} \leq ... \leq p_1$.*

Proof. See Appendix. ■

Theorem 2 *Consider two distributions of agents' types F and G such that $\mu_F = \mu_G = \mu$ and such that F second-order stochastically dominates G (in particular F has a lower variance than G). Then it holds that:*

1. $$\forall k, t, \ \sum_{i=1}^k y_i^F(t) \leq \sum_{i=1}^k y_i^G(t)$$

2. *For any time t and for any set of available objects at t, $\Pi_t \neq \emptyset$, the expected welfare in the efficient dynamic allocation under F is lower than that under G.*

Proof. See Appendix. ■

A main application of the above Theorem follows: For a constant arrival rate, the system of differential equations that characterizes the efficient dynamic allocation can be solved explicitly for any number of objects if the distribution of the agents' types is exponential (see Example above), while this is rarely the case for other distributions. Together with the above result, that solution can be used to bound the optimal policy and the associated welfare for large, non-parametric classes of distributions that are often used in applications.

Definition 2 *A non-negative random variable X is said to be* new better than used in expectation - *NBUE (*new worse than used in expectation - *NWUE) if*

$$E[X - a \mid \ X > a] \leq (\geq) \ E[X] \ , \forall a \geq 0$$

The classes of NBUE (NWUE) distributions are large and contain most of the distributions that appear in applications. For example, any distribution with an *increasing failure (or hazard) rate* is NBUE, while any distribution with a *decreasing failure rate* is NWUE.

Theorem 3 *Let F, the distribution of agents' types be NBUE (NWUE) with mean μ. Then, for any t and $\Pi_t \neq \varnothing$, the expected welfare in the efficient dynamic allocation under F is lower (higher) than that under the exponential distribution $G(x) = 1 - e^{-\frac{x}{\mu}}$.*

Proof. The result follows directly from Theorem 2 by noting that F second order stochastically dominates $G(x) = 1 - e^{-\frac{x}{\mu}}$ (is second-order stochastically dominated by $G(x) = 1 - e^{-\frac{x}{\mu}}$) is equivalent to F being NBUE (NWUE) - see Theorem 8.6.1 in Ross [24]. In other words,

$$\forall y \geq 0, \int_y^\infty (1 - F(x))dx \leq (\geq)\mu e^{-\frac{x}{\mu}} \text{ if } F \text{ is NBUE (NWUE)}$$

■

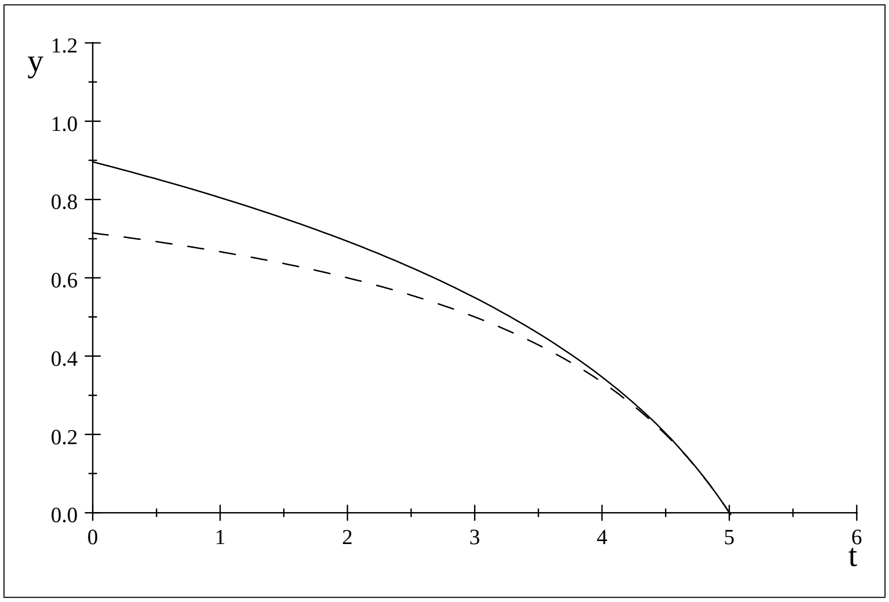
Example 2 *Let $F(x) = x$ on $[0, 1]$ so that F is IFR and thus NBUE, and let $\lambda(t) = \lambda = 1$. Assume that there is one object with type $p_1 = 1$. The optimal cut-off curve satisfies*

$$y_F' = -\int_{y_F}^1 (1 - x)dx = -\frac{1}{2} + y_F - \frac{y_F^2}{2}$$

with initial condition $y_F(T) = 0$. The solution to this differential equation is

$$y_F(t) = 1 - \frac{2}{T - t + 2}$$

The figure below illustrates the Theorem for $T = 5 : y_F(t) = 1 - \frac{2}{7-t}$ is the dashed line, $y_G(t) = \frac{1}{2}\ln[6 - t]$ is the solid line corresponding to the exponential distribution $G(x) = 1 - e^{-2x}$ which has mean $\frac{1}{2}$.



4.2 The Dynamic Efficient Allocation with Infinite Horizon and Discounting

In this Section we assume that $r(t) = e^{-\alpha t}$. Given this "memoryless" specification, the arrival process can be more general, and it is assumed here to be a *renewal* process with general inter-arrival distribution B (instead of a Poisson process where the inter-arrival distribution is exponential).[11] We start with a simple, complete information example that illustrates the main insight - the stationarity of the welfare maximizing dynamic policy.

Example 3 *Assume that the arrival process is Poisson with rate λ, i.e., $B(t) = 1 - e^{-\lambda t}$. Let $\widetilde{B}$ denote the Laplace- transform of the inter-arrival distribution B, and note first that*[12]

$$\widetilde{B}(\alpha) = \int_0^{\infty} e^{-\alpha t} \lambda e^{-\lambda t} dt = \frac{\lambda}{\alpha + \lambda}$$

and that

$$\frac{\widetilde{B}(\alpha)}{1 - \widetilde{B}(\alpha)} = \frac{\lambda}{\alpha}$$

[11] The derived controlled stochastic process is *semi-Markov* since the Markov property is preserved only at decision points, but not between them. See Puterman (2005) for solution approaches to such problems by an *uniformization* procedure, and for conditions guaranteeing that optimal policies are deterministic and Markovian.

[12] $\widetilde{B}(\alpha)$ acts here as the effective discount rate. It represents the discounted value of one unit at the expected time of the next arrival.

Consider now the the differential equation (see Theorem 1) defining the efficient allocation curve for the case of one object $y_1(t)$:

$$\frac{d[r(t)y_1(t)]}{dt} = -\lambda r(t) \int_{y_1}^{\infty} (1 - F(x))dx$$

Plugging $r(t) = e^{-\alpha t}$ we get

$$(y_1' - \alpha y_1) = -\lambda \int_{y_1}^{\infty} (1 - F(x))dx.$$

Postulating now $y_1' = 0$ yields

$$y_1 = \frac{\lambda}{\alpha} \int_{y_1}^{\infty} (1 - F(x))dx = \frac{\widetilde{B}(\alpha)}{1 - \widetilde{B}(\alpha)} \int_{y_1}^{\infty} (1 - F(x))dx$$

On the interval of definition of F $[0, \tau]$ the identity function on the left hand side, y_1, increases from 0 to τ while the function $\frac{\lambda}{\alpha} \int_{y_1}^{\infty} (1 - F(x))dx$ decreases in y_1 from $\frac{\lambda}{\alpha}\mu$ (where μ is the mean of F) to 0. Thus, there is a unique intersection point, and the equation above has a unique solution y_1^*. Since $\lim_{t \to \infty} e^{-\alpha t} y_1^* = 0$, we obtain that the efficient dynamic cut-off curve is indeed described by the constant y_1^*. The derivations for more items follow analogously.

The complete-information efficient dynamic assignment for the general case is characterized in the following Theorem:

Theorem 4 *(Albright, [1]) Assume that $r(t) = e^{-\alpha t}$. The efficient allocation curves are constants (i.e., independent of time) $y_n \leq y_{n-1} ... \leq y_1$. These constants do not depend on the p's, and are given by the implicit recursion:*

$$(y_k + y_{k-1} + ..y_1) = \frac{\widetilde{B}(\alpha)}{1 - \widetilde{B}(\alpha)} \int_{y_k}^{\infty} (1 - F(x))dx \ , \ 1 \leq k \leq n$$

where $\widetilde{B}$ is the Laplace- transform of the inter-arrival distribution B.

The efficient dynamic allocation policy is obviously Markovian and deterministic, and can be therefore implemented by the payments of Proposition 1. The analog of Theorem 2 for this case is:

Theorem 5 *Consider two distributions of agents' types F and G such that $\mu_F = \mu_G = \mu$ and such that F second-order stochastically dominates F (in particular F has a lower variance than G). Then, for any fixed inter-arrival distribution B it holds that:*

1. $\forall k,\ \sum_{i=1}^{k} y_i^F \leq \sum_{i=1}^{k} y_i^G$
2. *For any t and any $\Pi_t \neq \emptyset$ the expected welfare in the efficient dynamic allocation under F is lower than that under G.*

Proof. See Appendix. ■

In addition to the above Theorem about the benefits of increased variability in the agents' types, we now obtain a comparative-statics result about the benefits of variability in arrival times. Interestingly, this next result holds for a stochastic order that is much weaker than second-order stochastic dominance. We first need the following definition (see Shaked and Shanthikumar, 2006):

Definition 3 *Let X, Y be two non-negative random variables. Then X is said to be smaller than Y in the Laplace transform order, denoted by $X \leq_{Lt} Y$, if*

$$E[e^{-sX}] \geq E[e^{-sY}] \quad \textit{for all } s > 0$$

Note that the function $w(x) = -e^{-sx}$ is increasing an concave for any $s > 0$. Thus, we obtain that $X \leq_{SSD} Y \;\; \Rightarrow X \leq_{Lt} Y$ since the former involves a comparison of expectations with respect to **all** increasing concave functions.

Theorem 6 *Consider two inter-arrival distributions B and E such that $B \geq_{Lt} E$. Then, for any fixed distribution of agents' characteristics F, it holds that:*

1. $\forall k,\ \sum_{i=1}^{k} y_i^B \leq \sum_{i=1}^{k} y_i^E$
2. *For any t and for any $\Pi_t \neq \emptyset$, the expected welfare in the efficient dynamic allocation under B is lower than that under E.*

Proof. See Appendix. ■

Again, we can apply the above comparative static results in order to bound the optimal cut-off curves and the associated expected welfare for large classes of distributions of the agents' types and of the inter-arrival times.

Corollary 1 *For any t and for any $\Pi_t \neq \emptyset$ we have:*

1. *For any fixed distribution of inter-arrival times, the expected welfare under an NBUE(NWUE) distribution of agents' types with mean μ is lower (higher) than the expected welfare under the exponential distribution $G(t) = 1 - e^{-\frac{t}{\mu}}$.*
2. *For any fixed distribution of agents' types, the expected welfare under an NBUE(NWUE) distribution of inter-arrival times with mean μ is lower (higher) than the expected welfare under a Poisson arrival process with rate $\frac{1}{\mu}$.*

Proof. The first claims follows from Theorems 5 and from the fact that NBUE (NWUE) distributions second order stochastically dominate (are dominated by) an exponential distribution with the same mean. The second claim follows from Theorem 6, from the above observation, and from the fact that second order stochastic dominance implies domination in the Laplace- transform order. ■

Example 4 *Assume that there is one object with* $p_1 = 1$*, and consider a situation with an NBUE(NWUE) distribution of abilities with mean* μ*, and another NBUE(NWUE) distribution of inter-arrival times with mean* ω*. Let the discount rate be* α*. Then, the expected welfare under the efficient sequential allocation policy is lower (higher) than*

$$\mu \, \mathrm{LambertW}(\frac{1}{\omega\alpha})$$

where the increasing function $\mathrm{LambertW}(x)$ *is implicitly defined by:*

$$\mathrm{LambertW}(x) e^{\mathrm{LambertW}(x)} = x$$

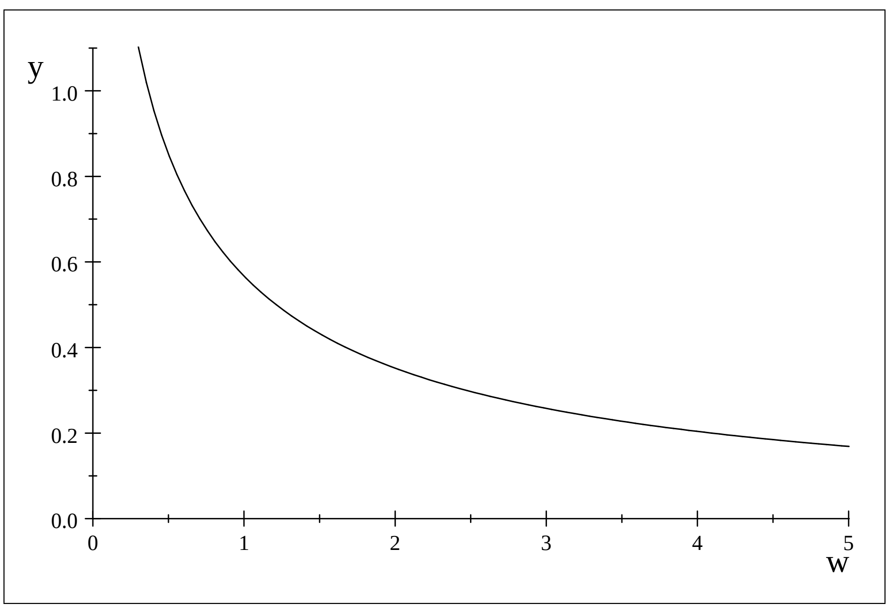
To show this, consider the exponential distributions $G_\mu = 1 - e^{-\frac{t}{\mu}}$ *for the agents' types and* $G_\omega = 1 - e^{-\frac{t}{\omega}}$ *for the inter-arrival times. For these distributions, the optimal cutoff point* y_1 *solves*

$$y_1 = \frac{\widetilde{G}_\omega(\alpha)}{1 - \widetilde{G}_\omega(\alpha)} \int_{y_1}^{\infty} e^{-\frac{t}{\mu}} dt = \frac{\mu}{\omega\alpha} e^{-\frac{y_1}{\mu}}$$

The solution to this equation is given by

$$y_1 = \mu \, \mathrm{LambertW}(\frac{1}{\omega\alpha})$$

and the result follows by the above corollary. Note that the solution is linear in μ*, the mean of the agents' distribution of types. The next figures plots the solution as a function of* ω *for* $\mu = \alpha = 1$ *(note that as intuition would suggest,* ω*, the mean inter-arrival time and* α*, the discount factor, play here analogous roles):*



5 Dynamic Revenue-Maximization with a Deadline

In this section we analyze the dynamic revenue maximization problem. A main feature that differentiates our analysis from previous ones is the fact that we use the mechanism design approach developed in Section 2 and the insight behind the celebrated payoff/revenue equivalence theorem. Thus, we focus on the dynamic allocation policy that underlies revenue maximization, while pricing plays only a "secondary" role since it is automatically induced by the implementation requirements once the allocation is fixed.

We first calculate the expected revenue for **any** given Markovian, deterministic allocation policy, and then we use a variational argument to derive the cut-off curves describing the revenue-maximizing dynamic policy. As we shall see, this approach sheds more light and is more explicit about the properties of the optimal policy and the resulting revenue than the standard dynamic programming approach that is centered around the so called Bellman's equations.

In addition to the previous assumptions, we assume here that the distribution of the agents' valuations F is twice differentiable, and we denote by f the corresponding density function. Moreover, we assume that $f(x) < \infty$ for any $x \in [0, \infty)$.

We first analyze the revenue maximizing problem in the setting with a deadline. That is, we assume that the seller's discount function satisfies:

$$r(t) = 1 \text{ for } 0 \leq t \leq T \text{ and } r(t) = 0 \text{ for } t > T.$$

In the next Section, we will briefly discuss the setting with infinite horizon and exponential discounting.

Recall from Proposition 1 that, in order to implement a Markovian, deterministic allocation which is given by $\infty = y_{0,\Pi_t}(t) > y_{1,\Pi_t}(t) \geq y_{2,\Pi_t}(t) \geq \cdots \geq y_{k_t,\Pi_t}(t) > 0$, $\forall t$, an agent that arrives at t should be charged

$$q_{j,\Pi_t}(t) = \sum_{i=j}^{k_t} (p_{(i)} - p_{(i+1)}) y_{i,\Pi_t}(t) + C(t) \tag{2}$$

if he gets the item with j-th highest characteristic among the remaining objects. Here $C(t)$ is some allocation-independent function. In any interim individually rational mechanism we must have $C(t) \leq 0$, and in order to maximize the revenue we must clearly have $C(t) = 0$.

The next Theorem calculates the expected revenue for any implementable Markovian, deterministic policy. Note that such a policy must specify an allocation decision for each possible state, i.e., for each possible subset of object $\Pi_t \neq \emptyset$ available at time t. Moreover, for each state, the policy consists of k_t $=\mid \Pi_t \mid$ cut-off curves that describe the partition of the set of agents' types - generally these curves depend on the precise composition of the set Π_t . The number of needed curves if there are n objects, $P(n)$, satisfies the recursion $P(n) = n + nP(n-1)$ with $P(1) = 1$. This yields 4 cut-off curves for two objects, 15 curves for three objects, 64 curves for 4 objects, and so on... It is obvious from the recursive formula that there are at least $n!$ curves. In order to save on notation and to keep the somewhat involved proofs more transparent, we assume below that there are only two objects with characteristics $p_1 \geq p_2$. But we will describe the completely analogous solution to revenue maximization problem for the general case with any number of distinct objects. A main result is that the dynamic revenue maximizing policy for n (possibly distinct) objects is completely described by n cutoff curves !

With slight abuse of notation, we write "2" instead of $\Pi_t = \{p_1, p_2\}$ as the second subscript of the pricing and allocation functions $q(t)$ and $y(t)$ whenever $k_t = 2$. This should not lead here to any confusions.

Theorem 7 *Assume that*

1. *the arrival process is homogenous with rate* $\lambda(t) = \lambda$
2. *If* $k_t = 2$, *the designer uses the dynamic allocation cutoff-curves* $y_{2,2}(t) \leq y_{1,2}(t)$, *i.e., the agent that arrives at time* t *gets: the object with type* p_1 *if his type is* $x_i \geq y_{1,2}(t)$; *the object with type* p_2, *if his type is* $x_i \in [y_{2,2}(t), y_{1,2}(t))$; *no object if* $x_i < y_{2,2}(t)$.
3. *If* $k_t = 1$, *the designer uses the dynamic cutoff–curves* $y_{1,p_j}(t)$, *i.e., the agent that arrives at time* t *gets the remaining object with characteristic* p_j *if* $x_i \geq y_{1,p_j}(t)$, *and no object otherwise.*

Then, the expected revenue from this policy is given by

$$\int_0^T (p_2 y_{2,2}(t) + R(p_1, t)) \lambda (1 - F(y_{2,2}(t))) e^{-\int_0^t \lambda(1-F(y_{2,2}(s)))ds} dt \;\; + \int_0^T ((p_1 - p_2) y_{1,2}(t) + R(p_2, t) - R(p_1, t)) \cdot \lambda (1 - F(y_{1,2}(t))) e^{-\int_0^t \lambda(1-F(y_{2,2}(s)))ds} dt$$

where

$$R(p_j, t) = p_j \int_t^T y_{1,p_j}(s) \lambda(1 - F(y_{1,p_j}(s)) e^{-\int_t^s \lambda[1-F(y_{1,p_j}(z))]dz} ds \qquad (3)$$

is the expected revenue at time t *if only one object with characteristic* p_j *remains, given that the dynamic allocation functions* $y_{1,p_j}(s)$ *is used from* t *on.*

Proof. See Appendix. ■

In the general case with any set of objects, the expected revenue at time t where $\Pi_t \neq \emptyset$ is given by

$$R(\Pi_t, t) = \sum_{i=1}^{k_t} \int_t^T \left(q_{i,\Pi_t}(s) + R\left(\Pi_t \backslash \{p_{(i)}\}, s\right)\right) h_{i,\Pi_t}(s) \, ds$$

where

$$h_{i,\Pi_t}(s) = \lambda \left[F\left(y_{i-1,\Pi_t}(s)\right) - F\left(y_{i,\Pi_t}(s)\right)\right] e^{-\int_t^s \lambda[1-F(y_{k_t,\Pi_t}(z))]dz}$$

is the density of the waiting time till the first arrival of an agent with a type in the interval $[y_{i,\Pi_t}(s), y_{i-1,\Pi_t}(s))$ given that no arrival that leads to a sale (e.g., type above $y_{k_t}(s)$) has occurred; and $q_{i,\Pi_t}(s)$ is given by (2).

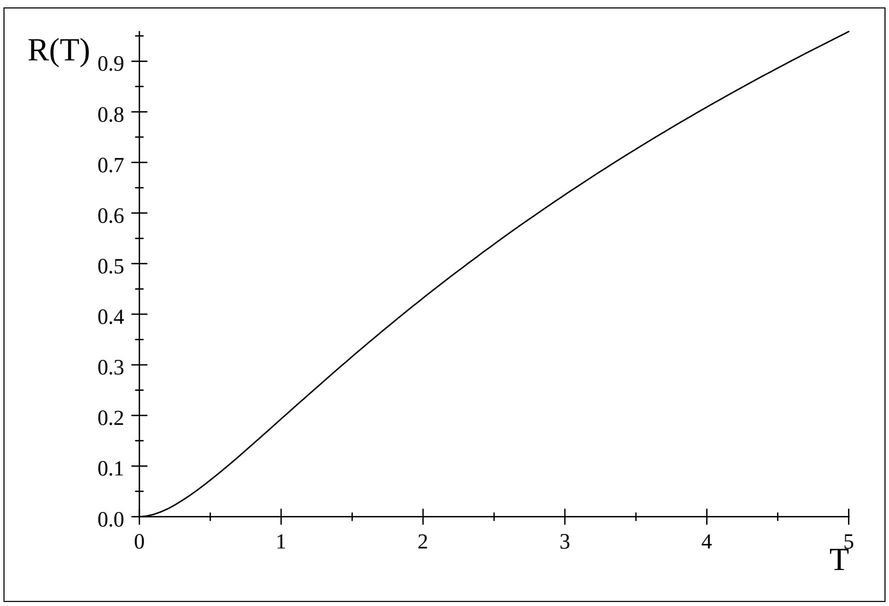
Example 5 *We now use the above Theorem to compute the expected revenue at time $t = 0$ resulting from dynamically* ***efficient*** *allocation policy in the case of one object with type $p_1 = 1$, homogenous Poisson arrivals with parameter $\lambda(t) = 1$ and an exponential distribution of agents' values. The efficient cutoff curve is given by*

$$y(t) = \ln[1 + T - t].$$

Expected revenue is then given by:

$$\begin{aligned} \int_0^T y(t)h_1(t)dt &= \int_0^T \ln[1+T-t][e^{-\ln[1+T-t]}][e^{-\int_0^t e^{-\ln[1+T-s]}ds}]dt \\ &= \ln(1+T) - \frac{T}{T+1} \end{aligned}$$

The figure displays the expected revenue as a function of the deadline T.



The next main result derives the revenue maximizing allocation policy. In particular, it shows that this policy is independent of the characteristics of the available objects.

Theorem 8 *Assume that the distribution of the agents' valuations satisfies the assumption of increasing virtual utility, i.e., the function* $x - \frac{1-F(x)}{f(x)}$ *is increasing. The dynamic revenue maximizing allocation policy is independent of the characteristics of available objects* p_1 *and* p_2*. In particular, we have:*

1. $y_{1,p_1}(t) = y_{1,p_2}(t) = y_{1,2}(t) := y_1(t)$ *where* $y_1(t)$ *is a solution of*

$$y_1(t) = \frac{1 - F(y_1(t))}{f(y_1(t))} + \lambda \int_t^T \frac{[1 - F(y_1(s))]^2}{f(y_1(s))} ds$$

2. $y_{2,2}(t) := y_2(t)$ *is a solution of*

$$y_2(t) = \frac{1 - F(y_2(t))}{f(y_2(t))} + \lambda \int_t^T \frac{[1 - F(y_2(s))]^2}{f(y_2(s))} ds - R(1,t)$$

where

$$R(1,t) = \int_t^T y_1(s)\, \lambda(1 - F(y_1(s)) e^{-\int_t^s \lambda[1-F(y_1(z))]dz} ds$$

The proof proceeds by a sequence of two Claims. First, we derive the revenue-maximizing cutoff curves when only one object remains. Afterwards, we describe the revenue maximizing allocation policy if two objects are left.

Claim 1 *If only one object remains, the dynamic revenue maximizing allocation curve* $y_1(t)$ *solves*

$$y_1(t) = \frac{1 - F(y_1(t))}{f(y_1(t))} + \lambda \int_t^T \frac{[1 - F(y_1(s))]^2}{f(y_1(s))} ds. \tag{4}$$

Moreover, the expected revenue at time t *where* $\Pi_t = p_j$ *is given by* $R(p_j, t) = p_j R(1,t)$ *where*

$$R(1,t) = \lambda \int_t^T \frac{[1 - F(y_1(s))]^2}{f(y_1(s))} ds. \tag{5}$$

Proof. See Appendix. ■

We proceed now to characterize the revenue-maximizing allocation policy if there are two objects left.

Claim 2 *If two objects remain, the dynamic revenue maximizing policy is characterized by two cutoff curves,* $y_1(t)$ *and* $y_2(t)$*, where* $y_1(t)$ *satisfies equation (4) and where* $y_2(t)$ *satisfies:*

$$y_2(t) = \frac{1 - F(y_2(t))}{f(y_2(t))} + \lambda \int_t^T \frac{[1 - F(y_2(s))]^2}{f(y_2(s))} ds - R(1,t) \tag{6}$$

Moreover, the expected revenue at time t *for the case* $\Pi_t = \{1,1\}$ *is given by*

$$R(\{1,1\},t) = \lambda \int_t^T \frac{[1 - F(y_2(s))]^2}{f(y_2(s))} ds. \tag{7}$$

Proof. See Appendix. ■

Remark 2 *The cutoff curves describing the revenue maximizing allocation policy do not depend on the p's . It is worth to compare intuition for this result to the one we gave for the analogous result for the case of welfare maximization. As we shall see, the present case is somewhat more intricate. Assume that there are two available objects* $p_1 > p_2$ *, and that at time* t *the cutoffs are* $y_1 > y_2$*.*

Again, consider the effect of small shift in the highest cut-off from y_1 *to* $y_1 + \epsilon$*. Like in welfare maximization case, this shift has any effect only if some agent arrives at* t*. Also as in welfare maximization, the shift has no effect on the expected revenue if the arriving agent has value below* y_1*. If at time* t *an agent with value* y_1 *arrives, then the shift switches the object he gets from* p_2 *to* p_1 *- which implies that he has to pay* q_1 *instead of* q_2 *- and also switches the object that remains available for the future allocation from* p_1 *to* p_2*. The effect is*

$$\begin{aligned} & f(y_1)\,(q_2 + p_1 R(1,t) - q_1 - p_2 R(1,t)) \\ = \quad & (p_1 - p_2)\, f(y_1)\,(R(1,t) - y_1) \end{aligned}$$

where $R(t)$ *denotes here the expected revenue at time* t *if only one object with* $p = 1$ *remains, assuming that the optimal policy will be followed from time* t *on. The equality in the above equation follows here from the fact that in any incentive compatible mechanism we must have* $q_2 - q_1 = (p_2 - p_1)\, y_1$*. So far, the effects are similar to those encountered in the welfare maximization case. But, a further, new effect on revenue appears here if the arriving agent's value is above* $y_1 + \epsilon$*. The reason is that in any incentive compatible mechanism, the price agents with the values above* y_1 *have to pay is given by* $q_1 = (p_1 - p_2)\, y_1 + p_2 y_2$*. Therefore, increasing the cut-off* y_1 *implies a higher revenue if an agent with value above* $y_1 + \epsilon$ *arrives. This effect is*

$$\begin{aligned} & (1 - F(y_1 + \epsilon))\,((p_1 - p_2)(y_1 + \epsilon) + p_2 (y_2 + R(1,t))) \\ & - (1 - F(y_1 + \epsilon))\,((p_1 - p_2)\, y_1 + p_2 (y_2 + R(1,t))) \\ = \quad & (p_1 - p_2)(1 - F(y_1 + \epsilon))\,\epsilon \end{aligned}$$

To sum up, the total effect of the shift on expected revenue is

$$(p_1 - p_2)\left((1 - F(y_1 + \epsilon))\,\epsilon - f(y_1)\,(y_1 - R(1,t))\right).$$

Again, the expression is linear in the difference $(p_1 - p_2)$ *and the optimal* y_1 *- where the total effect of the shift should be equal to zero - does not depend on the characteristics of the available objects.*

Remark 3 *The equations for the revenue maximizing cutoff curves have an intuitive interpretation. Assume first that only one object with* $p = 1$ *is still available. The allocation policy is described then by the equation*

$$y_1(t) - \frac{1 - F(y_1(t))}{f(y_1(t))} = \lambda \int_t^T \frac{[1 - F(y_1(s))]^2}{f(y_1(s))} ds = R(1,t).$$

On the left hand side, we have the virtual valuation of an agent with type $y_1(t)$. *As Claim 1 showed, the right hand side represents the expected revenue from time* t *on if the object is not sold at* t *given that an optimal allocation policy is followed from time* t *on. Since the seller is able to extract as revenue only the virtual valuation of an arriving buyer, the equation shows that the optimal cut-off curve satisfies an indifference condition between immediate selling and a continuation that uses the optimal policy.*

Let us now proceed to the two objects case, and assume that $p_1 = p_2 = 1$. *If both objects are still available at time* t, *then the equation*

$$y_2(t) - \frac{1 - F(y_2(t))}{f(y_2(t))} + R(1,t) = \lambda \int_t^T \frac{[1 - F(y_2(s))]^2}{f(y_2(s))} ds = R(\{1,1\},t)$$

implies that optimal cut-off at time t *is such that the seller is indifferent between selling one object - which generates a revenue given by the left hand side - and between keeping the object at time* t *and proceeding using the optimal policy - which generates a revenue given by the the right hand side.*

In the general case, if there are $k_t = |\Pi_t|$ *available objects, then, no matter what their types are, the* i*'th cut-off curve,* $1 \leq i \leq k_t$, *in the dynamic revenue-maximizing policy is given by*

$$y_i(t) - \frac{1 - F(y_i(t))}{f(y_i(t))} + \lambda \int_t^T \frac{[1 - F(y_{i-1}(s))]^2}{f(y_{i-1}(s))} ds = \lambda \int_t^T \frac{[1 - F(y_i(s))]^2}{f(y_i(s))} ds \quad (8)$$

or, equivalently, by

$$y_i(t) - \frac{1 - F(y_i(t))}{f(y_i(t))} + R(\mathbf{1}_{i-1}, t) = R(\mathbf{1}_i, t) \quad (9)$$

where $\mathbf{1}_i$ *is the set of* $1's$ *of cardinality* i *and*

$$R(\mathbf{1}_j, t) = \lambda \int_t^T \frac{[1 - F(y_j(s))]^2}{f(y_j(s))} ds \quad (10)$$

is the expected revenue at time t *from the optimal cut-off policy if* j *identical objects with* $p = 1$ *are still available.*

While equation (9) has been obtained for the case of ***identical*** *objects in the revenue-management literature (see for example Gallego and van Ryzin [13], and Bitran and Mondschein [5] for a discrete time model), the explicit expression in (10) is, to the best of our knowledge, new - a by-product of our analysis that focused on the allocation policy rather than on prices.*

For the general case with several distinct objects, note also that, if an object is sold at time t, *then the lowest among the current optimal cut-off curves becomes irrelevant* ***regardless*** *of the characteristic of the sold object, while all the other* $k_t - 1$ *cutoff curves do not change and remain relevant for the future allocation decisions. That is, the optimal cutoff curves depend only on the cardinality of* Π_t, k_t . *For any two sets of available objects* Π_t^1 *and* Π_t^2 *with* $k_t^1 = |\Pi_t^1|$ *and* $k_t^2 = |\Pi_t^2|$, *and for any* $1 \leq i \leq \min\{k_t^1, k_t^2\}$ *it holds that*

$$y_{i,\Pi_t^1}(t) = y_{i,\Pi_t^2}(t).$$

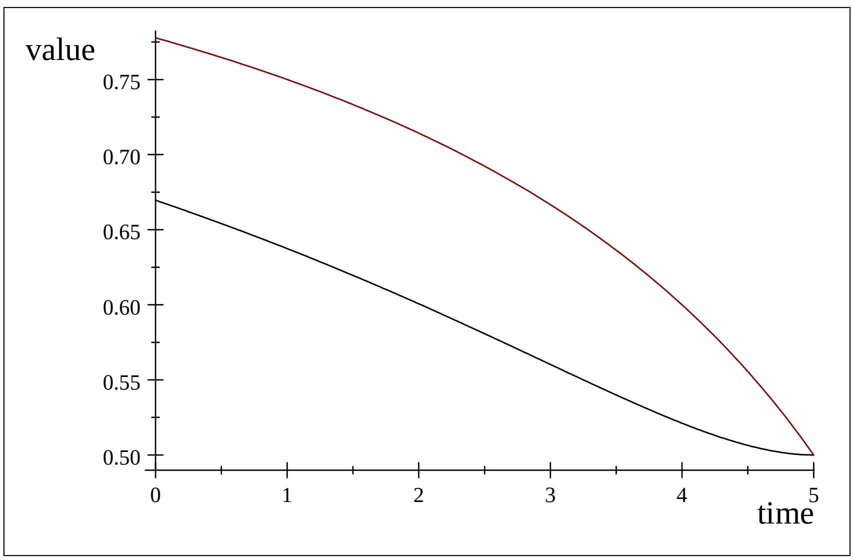
If Π_t is a set of identical objects, then only the lowest cut-off curve $y_{k_t}(t)$ where $k_t = |\Pi_t|$ is relevant for the allocation decision.

We conclude this section with an example:

Example 6 *Let $F(x) = x$ on $[0,1]$, and let $\lambda(t) = \lambda = 1$, and assume that there are two objects. Then, the revenue maximizing policy is characterized by:*

$$\begin{aligned} y_1(t) &= 1 - \frac{2}{4+T-t} \\ y_2(t) &= 1 - \frac{1-\sqrt{5}+\left(1+\sqrt{5}\right)c\,(T-t+4)^{\sqrt{5}}}{T-t+4+c\,(T-t+4)^{1+\sqrt{5}}} \end{aligned}$$

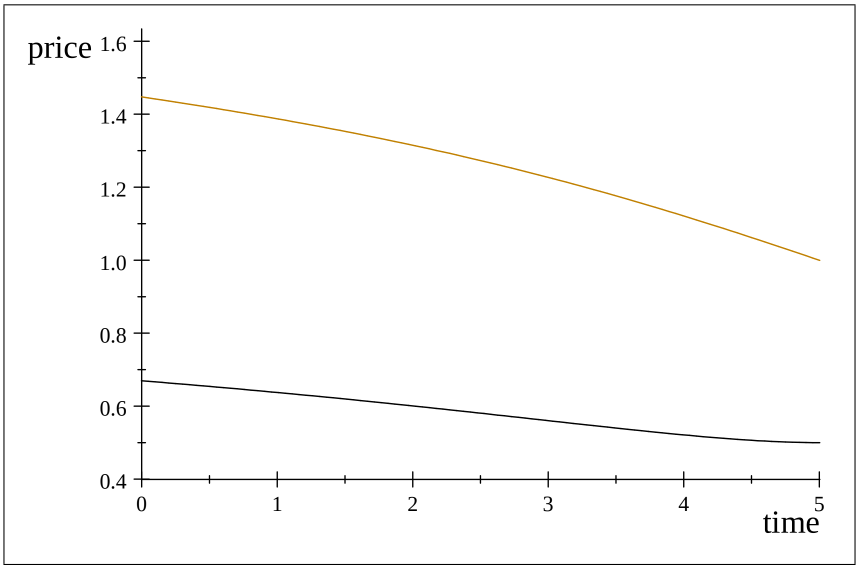
where $c = \frac{\sqrt{5}+1}{4^{\sqrt{5}}\left(\sqrt{5}-1\right)}$. The next picture plots these cut-off curves for $T = 5$:



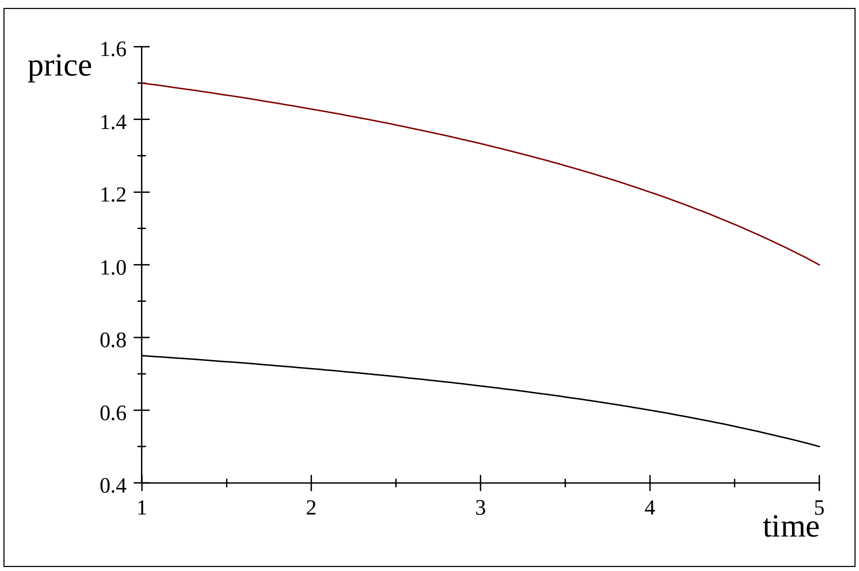
The optimal mechanism can be described as follows: Assume first that both objects are still available at time t, and consider an agent arriving at t : if his type is $x_i \in [y_2(t), y_1(t))$, he gets the object p_2 and pays $p_2 y_2(t)$; if his type is $x_i \in [y_1(t), 1]$, he gets object p_1 and pays $p_2 y_2(t) + (p_1 - p_2)\, y_1(t)$; if his type is $x < y_2(t)$, he gets nothing and pays nothing.

Assume now that an object is sold at time τ. Then the other object will be sold at some time $t > \tau$ which is the time of the first arrival of an agent with type $x_i \in [y_1(t), 1]$ (assuming that this arrival is before the deadline). The charged price depends then on the type of the available object, and is given by $p_i y_1(t)$, $i = 1, 2$.

Assume now for illustration that $T = 5$, $p_1 = 2$ and $p_2 = 1$. The next picture plots the ***price*** *dynamics if both objects are still available:*



The next picture plots the ***price*** *dynamic if only one object is available. The upper curve describes the offered price if only* p_1 *is available, while the lower curve corresponds to the offered price if only* p_2 *is available.*



Note that the prices of both objects jump up after a sale, even if the upper cutoff curve in the revenue maximizing allocation policy remains the same.

5.1 Dynamic Revenue Maximization with Infinite Horizon and Discounting

In this Section we very briefly present the characterization of the dynamic revenue maximizing allocation scheme with infinite horizon and exponential discounting. We assume that the arrival process is homogenous Poisson with $\lambda(t) = \lambda$. For the case of identical objects, Gallieni [15] has shown that the optimal cutoff curves are stationary (i.e., time independent), similarly to Albright's result [1] for the dynamic welfare maximization case (see Theorem 4).

We show below that the revenue maximizing allocation policy for the case of distinct objects does not depend on the characteristics of the available objects. Since the analysis of the discounted, infinite-horizon is similar and easier than the one we performed for the deadline case, we omit the proof of the next Proposition.

Proposition 2 *The dynamic revenue- maximizing policy consists of* n ***constants*** $y_n \leq y_{n-1} ... \leq y_1$ *which do not depend on the* p*'s such that:*

1. *If an agent with type* x_i *arrives at a time* t*, it is optimal to assign to that agent the* j*'th highest element of* Π_t *if* $x \in [y_j,\ y_{j-1})$ *where* $y_0 \equiv \infty$*, and not to assign any object if* $x_i < y_{k_t}$ *, where* $k_t = |\Pi_t|$.

2. *The constants* y_j *satisfy:*

$$y_j - \frac{1 - F(y_j)}{f(y_j)} + R(\mathbf{1}_{j-1}) = \frac{\lambda [1 - F(y_j)] (y_j + R(j-1))}{\alpha + \lambda [1 - F(y_j)]} = R(\mathbf{1}_j)$$

where

$$R(\mathbf{1}_j) = \frac{\lambda [1 - F(y_j)] (y_j + R(j-1))}{\alpha + \lambda [1 - F(y_j)]}$$

is the expected revenue at time t *if* j *identical objects with* $p_1 = p_2 = ... = p_l = 1$ *are available at time* t*, given that the designer uses the optimal allocation policy from time* t *on.*

The next example illustrates the Proposition:

Example 7 *Assume that there are three objects. Let* $F(x) = x$ *on* $[0, 1]$ *and let* $\alpha = \lambda = 1$*. Then, the constant cutoffs defining the revenue maximizing policy are:*

$$\begin{aligned}
y_1 &= 2 - \sqrt{2} \approx 0.585\,79 \\
y_2 &= 2 - \sqrt{5 - 2\sqrt{2}} \approx 0.526\,37 \\
y_3 &= 2 - \sqrt{2}\frac{\sqrt{2\sqrt{2}\sqrt{5-2\sqrt{2}} - 5\sqrt{5-2\sqrt{2}} - 13\sqrt{2} + 24}}{\sqrt{5 - 2\sqrt{2}}} \approx 0.508\,58.
\end{aligned}$$

6 Comparison of the Efficient and the Revenue Maximizing Policies

In this section we compare the efficient and revenue-maximizing dynamic allocation policies for the case where there is a deadline (the discounted infinite horizon case is analogous). Our main result in this Section shows that the curves describing the revenue-maximizing allocation are always above the respective curves describing the efficient allocation if the agents' types follow a distribution with an increasing failure (or hazard) rate - IFR. In other words, at any point in time, the revenue maximizing policy is more "conservative". Denote by $y_i^e(t)$ ($y_i^o(t)$) the efficient (revenue-maximizing) cut-off curve for the object with the i'th highest characteristic.

Theorem 9 *Assume that*

1. $\lambda(t) = \lambda$
2. *the distribution of values F is IFR.*

Then, for any $t \in [0, T]$, $y_i^e(t) < y_i^o(t)$, $i = 1, 2, ..n$.

Proof. See Appendix. ■

The next example illustrates the Theorem:

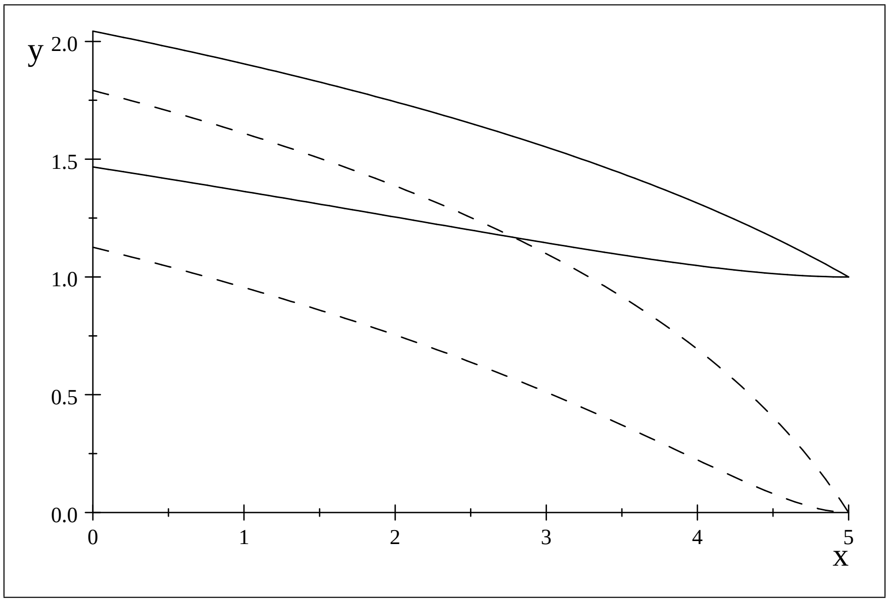
Example 8 *Assume that there are two objects, that the arrival process is homogenous with rate $\lambda(t) = \lambda = 1$, and that the distribution of agents' types is exponential, i.e., $F(x) = 1 - e^{-x}$. The efficient policy is described by the cutoff functions*

$$\begin{aligned} y_1^e(t) &= \ln(1 + T - t) \\ y_2^e(t) &= \ln\left(1 + \frac{(T-t)^2}{2(1+T-t)}\right) \end{aligned}$$

while the revenue-maximizing policy is given by

$$\begin{aligned} y_1^o(t) &= \ln(e + T - t) \\ y_2^o(t) &= \ln\left(\frac{1}{2}(e + T - t) + \frac{1}{2}\frac{e^2}{(e+T-t)}\right) \end{aligned}$$

The following figure plots the solutions for $T = 5$. The solid lines represent the revenue maximizing cut-off curves and the dashed lines represent the efficient cut-off curves.



Our last example shows that the efficient and revenue-maximizing cutoff curves may coincide if the IFR assumption is not satisfied. In particular, an increasing virtual valuation is not sufficient for the result.

Example 9 *(McAfee and te Velde [19]) Assume that there is one object , that the arrival process is Poisson with rate* $\lambda(t) = 1$*, and that values are distributed according to a Pareto distribution, that is* $F(x) = 1 - x^{-\epsilon}$ *for* $x \geq 1$, $\epsilon > 1$.

The failure rate is given by $\frac{f(x)}{1-F(x)} = \frac{\epsilon}{x}$ *which is of course decreasing. McAfee and te Velde showed for this case that the efficient and revenue-maximizing cutoff curves coincide:*

$$y_1^e(t) = y_1^o(t) = \left[\frac{\epsilon}{\epsilon - 1}(T-t)\right]^{\frac{1}{\epsilon}}.$$

Note also that the virtual valuation is given by $x - \frac{1-F(x)}{f(x)} = x(1 - \frac{1}{\epsilon})$ *, which is increasing in* x *since* $\epsilon > 1$*. This shows that an increasing virtual valuation is not sufficient for the result of Theorem 9.*

References

[1] Albright S.C. (1974): "Optimal Sequential Assignments with Random Arrival Times", *Management Science* **21** (1), 60-67.

[2] Arnold, M.A. and Lippman, S.A. (2001): "The Analytics of Search with Posted Prices", *Economic Theory* **17**, 444-466.

[3] Athey, S. and Segal, Y (2007): "An Efficient Dynamic Mechanism", discussion paper, Stanford University.

[4] Bergemann, D. and Välimäki, J. (2006): "Efficient Dynamic Auctions", *Cowles Foundation Discussion Paper No. 1584*

[5] Bitran, G.R. and Mondschein, S.V. (1997): "Periodic Pricing of Seasonal Products in Retailing", *Management Science* **43** (1), 64-79.

[6] Bitran, G.R. and Caldentey, R. (2003): "An Overview of Pricing Models for Revenue Management", *Manufacture & Services Operations Management* **5**(3), 203-229

[7] De La Cal, J. and Carcamo, J. (2006): "Stochastic Orders and Majorization of Mean Order Statistics", *Journal of Applied Probability* **43**, 704-712.

[8] Derman, C., Lieberman, G.J., and Ross, S.M. (1972): "A Sequential Stochastic Assignment Process", *Management Science* **18** (7), 349-355.

[9] Das Varma, G., and Vettas, N. (2001): "Optimal Dynamic Pricing with Inventories", *Economic Letters* 72 (3), 335-340.

[10] Dolan, R.J. (1978): "Incentive Mechanisms for Priority Queueing Problems", *The Bell Journal of Economics* **9** (2), 421-436.

[11] Elfving, G. (1967): "A Persistency Problem Connected with a Point Process", *Journal of Applied Probability* **4**, 77-89.

[12] Elmaghraby, W. and Keskinocak, P. (2003): "Dynamic Pricing in the Presence of Inventory Considerations: Research Overview, Current Practices, and Future Directions", *Management Science* **49** (10), 1287-1309.

[13] Gallego, G. and van Ryzin, G. (1994): "Optimal Dynamic Pricing of Inventories with Stochastic Demand over Finite Horizons", *Management Science* **40** (8), 999-1020.

[14] Gallego, G. and van Ryzin, G. (1997): "A Multiproduct Dynamic pricing Problem and its Applications to Network Yield Management", *Operations Research* **45** (1), 24-41.

[15] Gallien, J. (2006): "Dynamic Mechanism Design for Online Commerce", *Operations Research* **54** (2), 291-310.

[16] Gihman, I.I. and Skorohod, A.V. (1979): *Controlled Stochastic Processes*, Springer: New York

[17] Hardy, G., Littlewood, J.E., and Polya, G. (1934): *Inequalities*, Cambridge University Press: Cambridge.

[18] Kincaid, W.M., and Darling, D. (1963): "An Inventory Pricing Problem", *Journal of Mathematical Analysis and Applications* **7**, 183-208.

[19] McAfee, P. and te Velde, V. (2007a): "Dynamic Pricing with Constant Demand Elasticity", *Production and Operations Management,* Special Issue on Revenue Management and Dynamic Pricing, forthcoming.

[20] McAfee, P. and te Velde, V. (2007b): "Dynamic Pricing in the Airline industry", *Handbook on Economics and Information Systems*, Ed: T.J. Hendershott, Elsevier Handbooks in Information Systems, Volume 1.

[21] Parkes, D.C., and Singh, S. (2003): "An MDP-Based Approach to Online Mechanism Design", *Proceedings of 17th Annual Conference on Neural Information Processing Systems* (NIPS 03)

[22] Pecaric, J.E., Proschan, F., and Tong, Y.L. (1992): *Convex Functions, Partial Orderings, and Statistical Applications,* Academic Press: Boston.

[23] Puterman, M.L. (2005): *Markov Decision Processes*, Wiley: Hoboken, NJ

[24] Ross, S.M. (1983): *Stochastic Processes*, Wiley: New York.

[25] Talluri, K.T., and Van Ryzin, G. (2004): *The Theory and Practice of Revenue Managment,* Springer: New York.

7 Appendix

Proof of Lemma 1. $\Longleftarrow$ Assume that $\sum_{i=1}^{n} p_i\alpha_{(i)} \leq \sum_{i=1}^{n} p_i\beta_{(i)}$ for any $p_n \leq p_{n-1} \leq ... \leq p_1$. For each $k = 1, 2, .., n-1$ consider $p^k = (p_1^k, p_2^k, .., p_n^k)$ where $p_i^k = 1$ for $i = 1, 2, ..k$, and $p_i^k = 0$ for $i = k+1, k+2, ..n$. Then, for each k we obtain

$$\sum_{i=1}^{n} p_i^k \alpha_{(i)} \leq \sum_{i=1}^{n} p_i^k \beta_{(i)} \Leftrightarrow \sum_{i=1}^{k} \alpha_{(i)} \leq \sum_{i=1}^{k} \beta_{(i)}$$

and thus $\alpha \prec \beta$.

$\Longrightarrow$ Assume $\alpha \prec \beta$ and let $p_n \leq p_{n-1} \leq ... \leq p_1$. Then we have the following chain:

$$\begin{aligned}
\sum_{i=1}^{n} p_i[\beta_{(i)} - \alpha_{(i)}] &= p_n \sum_{i=1}^{n}[\beta_{(i)} - \alpha_{(i)}] + \sum_{i=1}^{n-1}(p_i - p_n)[\beta_{(i)} - \alpha_{(i)}] \\
&= p_n \sum_{i=1}^{n}[\beta_{(i)} - \alpha_{(i)}] + (p_{n-1} - p_n) \sum_{i=1}^{n-1}[\beta_{(i)} - \alpha_{(i)}] \\
&\quad + \sum_{i=1}^{n-2}(p_i - p_{n-1})[\beta_{(i)} - \alpha_{(i)}] \\
&= ... \\
&= p_n \sum_{i=1}^{n}[\beta_{(i)} - \alpha_{(i)}] + \sum_{j=1}^{n-1}(p_j - p_{j+1})(\sum_{i=1}^{j}[\beta_{(i)} - \alpha_{(i)}]) \geq 0
\end{aligned}$$

The last inequality follows since: 1. $\sum_{i=1}^{n}[\beta_{(i)}-\alpha_{(i)}]=\sum_{i=1}^{n}\beta_i-\sum_{i=1}^{n}\alpha_i=0$ by definition; 2. $\forall j$, $p_j-p_{j-1}\geq 0$ by definition; 3. $\forall j$, $\left(\sum_{i=1}^{j}[\beta_{(i)}-\alpha_{(i)}]\right)\geq 0$ by majorization. ■

Proof of Theorem 2. By Theorem 1 we know that

$$\begin{aligned}\frac{d(\sum_{i=1}^{k} y_i^F(t))}{dt} &= -\lambda(t)\int_{y_k^F(t)}^{\infty}(1-F(x))dx \\ \frac{d(\sum_{i=1}^{k} y_i^G(t))}{dt} &= -\lambda(t)\int_{y_k^G(t)}^{\infty}(1-G(x))dx\end{aligned}$$

Define first: $H_F(s)=\int_s^{\infty}(1-F(x))dx$ and $H_G(s)=\int_s^{\infty}(1-G(x))dx$. These are both positive, decreasing functions with $H_F(0)=H_G(0)=\mu$.

By SSD, for any $s\geq 0$ it holds that

$$\begin{aligned}\int_0^s F(x))dx &\leq \int_0^s G(x))dx \Leftrightarrow \int_0^s (1-F(x)))\,dx \geq \int_0^s (1-G(x)))\,dx \\ &\Leftrightarrow \int_s^{\infty}(1-F(x))dx \leq \int_s^{\infty}(1-G(x))dx \Leftrightarrow H_F(s)\leq H_G(s)\end{aligned}$$

where the second line follows because

$$\int_0^{\infty}(1-F(x))dx=\mu_F=\mu_G=\int_0^{\infty}(1-G(x))dx.$$

Thus, the curve H_F is always below H_G.

Consider now $y_1^F(t)$ and $y_1^G(t)$. These are, respectively, the solutions to the differential equations :

$$y'=-\lambda(t)H_F(y) \text{ and } y'=-\lambda(t)H_G(y)$$

with boundary condition $y(T)=0$. Integrating the above equations from t to T, and using the boundary condition, we get the integral equations:

$$\begin{aligned}y(T)-y(t) &= -\int_t^T \lambda(s)H_F(y(s))ds \Leftrightarrow y(t)=\int_t^T \lambda(s)H_F(y(s))ds \text{ and} \\ y(T)-y(t) &= -\int_t^T \lambda(s)H_G(y(s))ds \Leftrightarrow y(t)=\int_t^T \lambda(s)H_G(y(s))ds\end{aligned}$$

Because H_F is always below H_G and because these are decreasing functions, we obtain $y_1^F(t)\leq y_1^G(t)$.

Consider now $y_1^F(t)+y_2^F(t)$ and $y_1^G(t)+y_2^G(t)$. These functions satisfy the differential equations:

$$y'=-\lambda(t)\int_{y-y_1^F(t)}^{\infty}(1-F(x))dx \text{ and } y'=-\lambda(t)\int_{y-y_1^G(t)}^{\infty}(1-F(x))dx$$

with boundary condition $y'(T) = 0$. Integrating from t to T yields the equations:

$$\begin{aligned} y(t) &= \int_t^T \lambda(s) \left[\int_{y(s)-y_1^F(s)}^{\infty} (1 - F(x))dx \right] ds = \int_t^T \lambda(s) H_F[(y(s) - y_1^F(s)] \, ds \\ y(t) &= \int_t^T \lambda(s) \left[\int_{y(s)-y_1^G(s)}^{\infty} (1 - G(x))dx \right] ds = \int_t^T \lambda(s) H_G[(y(s) - y_1^G(s)] \, ds \end{aligned}$$

We have :

$$\forall t, \; H_F[(y(t) - y_1^F(t)] \leq H_F[(y(t) - y_1^G(t)] \leq H_G[(y(t) - y_1^G(t)]$$

where the first inequality follows because $y_1^F(t) \leq y_1^G(t) \Leftrightarrow y(t) - y_1^F(t) \geq y(t) - y_1^G(t)$ and because the function H_F is decreasing, and the second inequality follows because H_F is always below H_G. This yields $y_1^F(t) + y_2^F(t) \leq y_1^G(t) + y_2^G(t)$, as required. The rest of the proof follows analogously.

2. The expected welfare terms from time t on if there are k objects left are given by $\sum_{i=1}^k p_{(i)} y_i^F(t)$ and by $\sum_{i=1}^k p_{(i)} y_i^G(t)$, respectively. By point 1, we know that for each k and for each t, $y^{kF}(t) = (y_1^F(t), y_2^F(t), ..y_k^F(t)) \prec_w (y_1^G(t), y_2^G(t), ..y_k^G(t)) := y^{kG}(t)$. By Result 12.5 (b) in Pecaric et. al [22], for each k and each t there exists a $k-$vector $z(t)$ such that $z(t) \prec y^{kG}(t)$ and such that $z_i(t) \geq y_i^{kF}$, $\forall i$. We obtain then:

$$\forall k, t \; \sum_{i=1}^k p_{(i)} y_i^F(t) \leq \sum_{i=1}^k p_{(i)} z_i(t) \leq \sum_{i=1}^k p_{(i)} y_i^G(t)$$

where the last inequality follows from Lemma 1 ■

Proof of Theorem 5. 1. Define first $H_F(s) = \frac{\widetilde{B}(\alpha)}{1-\widetilde{B}(\alpha)} \int_s^\infty (1 - F(x))dx$ and $H_G(s) = \frac{\widetilde{B}(\alpha)}{1-\widetilde{B}(\alpha)} \int_s^\infty (1 - G(x))dx$. These are both decreasing functions and

$$H_F(0) = H_G(0) = \frac{\widetilde{B}(\alpha)}{1 - \widetilde{B}(\alpha)} \mu$$

Consider now y_1^F and y_1^G. These are, respectively, the solutions to the equations:

$$s = H_F(s) \;\text{ and }\; s = H_G(s)$$

By SSD, for any $s \geq 0$ it holds that

$$\begin{aligned} \int_0^s F(x))dx &\leq \int_0^s G(x))dx \Leftrightarrow \int_0^s (1 - F(x)))\, dx \geq \int_0^s (1 - G(x)))\, dx \\ &\Leftrightarrow \int_s^\infty (1 - F(x))dx \leq \int_s^\infty (1 - G(x))dx \Leftrightarrow H_F(s) \leq H_G(s) \end{aligned}$$

Thus, the decreasing curve $H_F(s)$ is always below the decreasing curve $H_G(s)$ and we obtain $y_F^1 \leq y_G^1$. Consider now y_2^F and y_2^G which are defined by the equations:

$$
\begin{aligned}
y_2^F + y_1^F &= \frac{\widetilde{B}(\alpha)}{1-\widetilde{B}(\alpha)} \int_{y_2^F}^{\infty} (1 - F(x))dx \\
y_2^G + \ y_1^G &= \frac{\widetilde{B}(\alpha)}{1-\widetilde{B}(\alpha)} \int_{y_2^G}^{\infty} (1 - G(x))dx
\end{aligned}
$$

Equivalently, $y_2^F + y_2^F$ and $y_2^G + y_1^G$ are, respectively, the solutions of:

$$s = H_F(s - y_1^F) \text{ and } s = H_G(s - y_1^G)$$

Recalling that $y_1^F \leq y_1^G$, we obtain $s - y_1^F \geq s - y_1^G$, $\forall s$. This yields:

$$H_F(s - y_1^F) \leq H_F(s - y_1^G) \leq H_G(s - y_1^G)$$

where the first inequality follows because the function H_F is decreasing, and the second inequality follows by SSD. Thus, the curve $H_F(s - y_1^G)$ is always below the curve $H_G(s - y_1^G)$ and the result follows as above. The rest of the proof is completely analogous.

2. The expected welfare terms from time t on if k objects left are given by $e^{-\alpha t}\left[\sum_{i=1}^{k} p_{(i)} y_i^F\right]$ and by $e^{-\alpha t}\left[\sum_{i=1}^{k} p_{(i)} y_i^G\right]$, respectively. The proof proceeds exactly as that of Theorem 2-2. ■

Proof of Theorem 6. **1.** Let $H_B(s) = \frac{\widetilde{B}(\alpha)}{1-\widetilde{B}(\alpha)} \int_s^{\infty} (1 - F(x))dx$, $H_E(s) = \frac{\widetilde{E}(\alpha)}{1-\widetilde{E}(\alpha)} \int_s^{\infty} (1 - F(x))dx$ where $\widetilde{B}$ and $\widetilde{E}$ are the respective Laplace transforms. By the definition of the Laplace transform, and by the assumption $B \geq_{Lt} E$, we know that $\widetilde{B}(\alpha) \leq \widetilde{E}(\alpha)$. This yields:

$$\widetilde{B}(\alpha) \leq \widetilde{E}(\alpha) \Leftrightarrow \frac{\widetilde{B}(\alpha)}{1-\widetilde{B}(\alpha)} \leq \frac{\widetilde{E}(\alpha)}{1-\widetilde{E}(\alpha)} \Leftrightarrow H_B(s) \leq H_E(s)$$

The first equivalence follows because the function $\frac{x}{1-x}$ is increasing on the interval $[0, 1)$ with $\lim_{x\to 1} \frac{x}{1-x} = \infty$, and because Laplace transforms take values in the interval $[0, 1]$.

Thus, we obtained that the decreasing function $H_B(s)$ is always below the decreasing function $H_E(s)$. Consider first y_1^B and y_1^E. These are, respectively, the solutions to the equations

$$s = H_B(s) \quad \text{and} \quad s = H_E(s)$$

The rest of the proof continues analogously to the proof of Theorem 5-1.

2. This follows analogously to the proof of Theorem 2-2. ■

Proof of Theorem 7. If only one object with characteristic p_i is available at time t, then the expected revenue is given by

$$p_i \int_t^T y_{1,p_i}(s) h_{1,p_i}(s) ds$$

where $h_{1,p_i}(s)$ represents the density of the waiting time till the first arrival of an agent with a value that is at least $y_{1,p_i}(s)$. Note that this density is equal to the density of the first arrival in a non-homogenous Poisson process with rate $\lambda(s)(1-F(y_{1,p_i}(s))$. The density of the time of the $n-$th arrival in a non-homogenous Poisson process with rate $\delta(s)$ is given by (see Ross [24])

$$g_n(s)=\delta(s)e^{-m(s)}\frac{m(s)^{n-1}}{(n-1)!}\ ,\ \text{where } m(s)=\int_t^s \delta(z)dz \tag{11}$$

Thus, in our case, we obtain

$$h_{1,p_i}(s)=\lambda(s)(1-F(y_{1,p_i}(s))e^{-\int_t^s \lambda(z)[1-F(y(z))]dz} \text{ for } t\le s\le T$$

and (3) follows.

If two objects are still available, the expected revenue is given by

$$\int_0^T \left[q_{2,2}(t)+R\left(p_1,t\right)\right]h_{2,2}(t)dt+\int_0^T \left[q_{1,2}(t)+R\left(p_2,t\right)\right]h_{1,2}(t)dt \tag{12}$$

Here $h_{1,2}(t)$ represents the density of the waiting time till the first arrival of an agent with a value that is at least $y_{1,2}\left(t\right)$ if no arrival of an agent with value in the interval $[y_{2,2}\left(t\right),y_{1,2}\left(t\right))$ has occurred. Similarly, $h_{2,2}(t)$ represents the density of the waiting time till the first arrival of an agent with a value in the interval $[y_{2,2}\left(t\right),y_{1,2}\left(t\right))$ if no arrival of an agent with value in the interval $[y_{1,2}\left(t\right),\infty)$ has occurred. Since the arrival processes of agents with types in the intervals $[y_{2,2}\left(t\right),y_{1,2}\left(t\right))$ and $[y_{1,2}\left(t\right),\infty)$, respectively, are **independent** non-homogenous Poisson processes (see Proposition 2.3.2 in Ross [24]), using (11) we obtain

$$\begin{aligned} h_{1,2}(t) &= \lambda\left(1-F(y_{1,2}\left(t\right))\right)e^{-\int_0^t \lambda[1-F(y_{1,2}(s))]ds}e^{-\int_0^t \lambda[F(y_{1,2}(s))-F(y_{2,2}(s))]ds} \\ &= \lambda\left(1-F(y_{1,2}\left(t\right))\right)e^{-\int_0^t \lambda[1-F(y_{2,2}(s))]ds} \end{aligned}$$

and

$$\begin{aligned} h_{2,2}(t) &= \lambda\left(F(y_{1,2}\left(t\right))-F(y_{2,2}\left(t\right))\right)e^{-\int_0^t \lambda[F(y_{1,2}(s))-F(y_{2,2}(s))+1-F(y_{1,2}(s))]ds} \\ &= \lambda\left(F(y_{1,2}\left(t\right))-F(y_{2,2}\left(t\right))\right)e^{-\int_0^t \lambda[1-F(y_{2,2}(s))]ds} \end{aligned}$$

Finally, recall that incentive compatibility implies that

$$q_{2,2}(t)=p_2y_{2,2}(t) \text{ and } q_{1,2}(t)=p_2y_{2,2}(t)+\left(p_1-p_2\right)y_{1,2}(t),$$

Plugging the expressions for $q_{1,2}(t), q_{2,2}(t), h_{1,2}(t)$ and $h_{2,2}(t)$ into the expression for expected revenue (12) yields the required formula. ■

Proof of Claim 1. If only the object with characteristic p_j is available, it follows from Theorem 7 that the expected revenue at time t is given by

$$R\left(p_j,t\right)=p_j\int_t^T y_{1,p_j}\left(s\right)\lambda(1-F(y_{1,p_j}\left(s\right))e^{-\int_t^s \lambda[1-F(y_{1,p_j}(z))]dz}ds.$$

Let $H(s) = \int_t^s \lambda[1 - F(y_{1,p_j}(z))]dz$. Then, we obtain

$$R(p_j, t) = p_j \int_t^T F^{-1}\left[1 - \frac{H'(s)}{\lambda}\right] H'(s) e^{-H(s)} ds.$$

This expression for revenue is appropriate for using a variational argument with respect to the function H. The corresponding necessary condition for the variational problem (i.e., the Euler-Lagrange equation) is

$$-\left(H'(s)\right)^2 + 2H''(s) + \frac{H'(s)H''(s)f'\left(F^{-1}\left(1 - \frac{H'(s)}{\lambda}\right)\right)}{\left(f\left(1 - \frac{H'(s)}{\lambda}\right)\right)^2} = 0$$

Plugging back the expression for $H(s)$ gives

$$-\lambda[1 - F(y_{1,p_j}(s))]^2 - 2f(y_{1,p_j}(s))y'_{1,p_j}(s) - \frac{[1 - F(y_{1,p_j}(s))]f'(y_{1,p_j}(s))y'_{1,p_j}(s)}{f(y_{1,p_j}(s))} = 0$$

This implies that for any $s \in [0, T]$, the solution $y_{1,p_j}(s)$ should satisfy

$$-y'_{1,p_j}(s) - y'_{1,p_j}(s)\left(1 + \frac{\left(1 - F(y_{1,p_j}(s))\right)f'(y_{1,p_j}(s))}{\left(f(y_{1,p_j}(s))\right)^2}\right) = \lambda\frac{\left(1 - F(y_{1,p_j}(s))\right)^2}{f(y_{1,p_j}(s))} \tag{13}$$

Since for any t , and for any differentiable $y(t)$ it holds that

$$-y'(t)\left(1 + \frac{\left(1 - F(y(t))\right)f'(y(t))}{\left(f(y(t))\right)^2}\right) = \frac{d}{dt}\left(\frac{1 - F(y(t))}{f(y(t))}\right),$$

we can rewrite the necessary condition as

$$y'_{1,p_j}(s) + \lambda\frac{\left(1 - F(y_{1,p_j}(s))\right)^2}{f(y_{1,p_j}(s))} = \frac{d}{ds}\left(\frac{1 - F(y_{1,p_j}(s))}{f(y_{1,p_j}(s))}\right)$$

Taking now the integral between t and T

$$\int_t^T y'_{1,p_j}(s)\,ds + \lambda\int_t^T \frac{\left(1 - F(y_{1,p_j}(s))\right)^2}{f(y_{1,p_j}(s))}ds = \int_t^T \frac{d}{ds}\left(\frac{1 - F(y_{1,p_j}(s))}{f(y_{1,p_j}(s))}\right)ds$$

yields

$$\begin{aligned} & y_{1,p_j}(T) - y_{1,p_j}(t) + \lambda\int_t^T \frac{\left(1 - F(y_{1,p_j}(s))\right)^2}{f(y_{1,p_j}(s))}ds \\ = \quad & \frac{1 - F(y_{1,p_j}(T))}{f(y_{1,p_j}(T))} - \frac{1 - F(y_{1,p_j}(t))}{f(y_{1,p_j}(t))}. \end{aligned}$$

Together with the boundary condition

$$y_{1,p_j}(T) - \frac{1 - F(y_{1,p_j}(T))}{f(y_{1,p_j}(T))} = 0$$

we get (4). The assumptions of increasing virtual valuation and finite density insure that a solution to (4) exists for any t.

To complete the proof and obtain the expression for revenue (5) note that the expected revenue is given by $R(p_j, t) = p_j R(1,t)$ where

$$R(1,t) = \int_t^T y_1(s)\,\lambda(1 - F(y_1(s))e^{-\int_t^s \lambda[1-F(y_1(z))]dz}ds$$

Differentiating the above with respect to t gives

$$R'(1,t) = \lambda(1 - F(y_1(t))\,(R(1,t) - y_1(t))$$

It is then straightforward to verify that the function $\int_t^T \frac{[1-F(y_1(s))]^2}{f(y_1(s))}ds$ satisfies the above differential equation with the boundary condition $R(1,T) = 0$. ■

Proof of Claim 2. **I.** We consider first the case where $p_1 > p_2$. That is, the seller needs to specify two different prices and hence two different cutoff curves, $y_{1,2}(t)$ and $y_{2,2}(t)$. We can re-write the expected revenue given by Theorem 7 as

$$\int_0^T \left(p_1 F^{-1}\left(1 - \frac{H'(t)}{\lambda}\right) + p_2 R(1,t)\right) H'(t)e^{-H(t)}dt$$
$$+ (p_2 - p_1)\int_0^T \left[F^{-1}\left(1 - \frac{G'(t)}{\lambda}\right) - R(1,t)\right] G'(t)e^{-H(t)}dt$$

where

$$\int_0^t \lambda[1 - F(y_{2,2}(s))]\,ds \quad : \quad = H(t)$$
$$\int_0^t \lambda[1 - F(y_{1,2}(s))]\,ds \quad : \quad = G(t).$$

The necessary conditions for the variational problem (i.e., the Euler-Lagrange equation) with respect to the functions $H(t)$ and $G(t)$, respectively, are:

$$-(p_2 - p_1)\,G'(t)\left[F^{-1}\left(1 - \frac{G'(t)}{\lambda}\right) - R(1,t)\right] - p_1\frac{\frac{1}{\lambda}(H'(t))^2}{f\left(F^{-1}\left(1 - \frac{H'(t)}{\lambda}\right)\right)}$$
$$-2p_1\frac{\frac{1}{\lambda}H''(t)}{f\left(F^{-1}\left(1 - \frac{H'(t)}{\lambda}\right)\right)} + p_2 R'(1,t) - p_1\frac{\frac{1}{\lambda^2}H'(t)H''(t)f'\left(F^{-1}\left(1 - \frac{H'(t)}{\lambda}\right)\right)}{\left[f\left(F^{-1}\left(1 - \frac{H'(t)}{\lambda}\right)\right)\right]^3} = 0$$

and

$$-\frac{2\frac{1}{\lambda}G''(t)}{f\left(F^{-1}\left(1 - \frac{G'(t)}{\lambda}\right)\right)} - R'(1,t) - \frac{\frac{1}{\lambda^2}G'(t)G''(t)f'\left(F^{-1}\left(1 - \frac{G'(t)}{\lambda}\right)\right)}{\left[f\left(F^{-1}\left(1 - \frac{G'(t)}{\lambda}\right)\right)\right]^3}$$
$$-H'(t)\left[-\frac{\frac{1}{\lambda}G'(t)}{f\left(F^{-1}\left(1 - \frac{G'(t)}{\lambda}\right)\right)} + F^{-1}\left(1 - \frac{G'(t)}{\lambda}\right) - R(1,t)\right] = 0.$$

Plugging the expressions for $H(t)$ and $G(t)$ allows us to write the necessary conditions in the following way:

$$
\begin{aligned}
&-(p_2 - p_1)\lambda\left[1 - F(y_{1,2}(t))\right](y_{1,2}(t) - R(1,t)) - p_1 \frac{\lambda\left[1 - F(y_{2,2}(t))\right]^2}{f(y_{2,2}(t))} \\
&-2p_1 y_{2,2}'(t) - p_2 R'(1,t) - p_1 \frac{y_{2,2}'(t)\left[1 - F(y_{2,2}(t))\right] f'(y_{2,2}(t))}{\left[f(y_{2,2}(t))\right]^2} = 0
\end{aligned} \tag{14}
$$

and

$$
\begin{aligned}
&\left[1 - F(y_{2,2}(t))\right]\left[\frac{1 - F(y_{1,2}(t))}{f(y_{1,2}(t))} - y_{1,2}(t) + R(1,t)\right] - 2y_{1,2}'(t) \\
&-R'(1,t) + \frac{y_{1,2}'(t)\left[1 - F(y_{1,2}(t))\right] f'(y_{1,2}(t))}{\left[f(y_{1,2}(t))\right]^2} = 0
\end{aligned} \tag{15}
$$

Next, we show that a solution to the system of differential equations 14 and 15 is given by $y_{1,2}(t) = y_1(t)$ and $y_{2,2}(t) = y_2(t)$ where $y_1(t)$ and $y_2(t)$ solve the system of equations:

$$
y_1(t) = \frac{1 - F(y_1(t))}{f(y_1(t))} + \lambda \int_t^T \frac{\left[1 - F(y_1(s))\right]^2}{f(y_1(s))} ds \tag{16}
$$

$$
y_2(t) = \frac{1 - F(y_2(t))}{f(y_2(t))} + \lambda \int_t^T \frac{\left[1 - F(y_2(s))\right]^2}{f(y_2(s))} ds - R(t), \tag{17}
$$

Again, assumptions of increasing virtual value and finite density guarantee the existence of solutions for (16) and (17). Differentiation of (16) with respect to t gives

$$
2y_1'(t) = -y_1'(t)\frac{\left[1 - F(y_1(t))\right] f'(y_1(t))}{\left[f(y_1(t))\right]^2} - \lambda \frac{\left[1 - F(y_1(t))\right]^2}{f(y_1(t))}.
$$

Plugging the above expression into (15), and using the fact that

$$
R'(1,t) = -y_1(t)\lambda\left(1 - F(y_1(t))\right) + \lambda\left(1 - F(y_1(t))\right) R(1,t) \tag{18}
$$

yields

$$
\left[\lambda \int_t^T \frac{\left[1 - F(y_1(s))\right]^2}{f(y_1(s))} ds - R(1,t)\right]\left[\lambda\left(1 - F(y_1(t))\right) - \left(1 - F(y_{2,2}(t))\right)\right] = 0
$$

where last equality follows from Claim 1. Thus, we have showed that $y_{1,2}(t) = y_1(t)$ solves (15) for any $y_{2,2}(t)$. We still need to show that $y_{1,2}(t) = y_1(t)$ and $y_{2,2}(t) = y_2(t)$ solve equation 14. Differentiation of (17) with respect to t gives

$$
2y_2'(t) = -y_2'(t)\frac{\left[1 - F(y_2(t))\right] f'(y_2(t))}{\left[f(y_2(t))\right]^2} - \lambda \frac{\left[1 - F(y_2(t))\right]^2}{f(y_2(t))} - R'(1,t).
$$

Plugging this equality into (14), we have to show that

$$-\left(p_2-p_1\right)\lambda\left[1-F(y_{1,2}\left(t\right))\right]\left(y_{1,2}\left(t\right)-R(1,t)\right)-(p_2-p_1)R'(1,t)=0.$$

For $y_{1,2}\left(t\right)=y_1(t)$, this equality holds by (18).

II. We now consider the case with $p_1=p_2=p$. Since $R\left(p,t\right)=pR(1,t)$, Theorem 7 implies that we can rewrite the expected revenue as

$$p\int_0^T\left(y_{2,2}\left(t\right)+R\left(1,t\right)\right)\lambda\left(1-F(y_{2,2}\left(t\right))\right)e^{-\int_0^t\lambda(1-F(y_{2,2}(s)))ds}dt.$$

The proof that the revenue maximizing cutoff curves are given by $y_1(t)$ and $y_2(t)$ as above is analogous to the above case, and we omit it here.

Theorem 7 implies then that

$$R(\{1,1\},t)=\int_t^T\left(y_2(s)+R\left(1,s\right)\right)\lambda\left(1-F\left(y_2(s)\right)\right)e^{-\int_t^s\lambda(1-F(y_2(z)))dz}ds.$$

Differentiation with respect to t yields

$$R'(\{1,1\},t)=\lambda\left(1-F\left(y_2(t)\right)\right)\left(R(\{1,1\},t)-y_2(t)-R\left(1,t\right)\right). \tag{19}$$

Recall that $y_2(t)$ solves

$$y_2(t)+R\left(1,t\right)=\frac{1-F\left(y_2(t)\right)}{f\left(y_2(t)\right)}+\lambda\int_t^T\frac{\left[1-F\left(y_2(s)\right)\right]^2}{f\left(y_2(s)\right)}ds \tag{20}$$

Using equation (20), it is easy to verify that $R(\{1,1\},t)$ given by equation (7) satisfies differential equation (19) with the boundary condition $R(\{1,1\},T)=0$.
■

Proof of Theorem 9. We start with the proof for $i=1$. From Theorem 8 we know that

$$y_1^o(t)=\frac{1-F\left(y_1^o(t)\right)}{f\left(y_1^o(t)\right)}+\lambda\int_t^T\frac{\left[1-F\left(y_1^o(s)\right)\right]^2}{f\left(y_1^o(s)\right)}ds$$

while from Theorem 1 we know that

$$-y_1^{e\prime}\left(s\right)=\lambda\int_{y_1^e(s)}^{\infty}(1-F(x))dx$$

and that $y_1^e(T)=0$. Integrating both sides of the above differential equation between t and T and using the boundary condition, yields:

$$y_1^e\left(t\right)=\lambda\int_t^T\left[\int_{y_1^e(s)}^{\infty}(1-F(x))dx\right]ds.$$

First, we will argue that

$$\frac{1-F\left(y_1^o(t)\right)}{f\left(y_1^o(t)\right)}>0\text{ for any }t\in\left[0,T\right]. \tag{21}$$

Assume that there exists t^* such that

$$\frac{1 - F\left(y_1^o(t^*)\right)}{f\left(y_1^o(t^*)\right)} = 0 \tag{22}$$

Then since $f(x) < \infty$ for any x, (22) implies that $F\left(y_1^o(t^*)\right) = 1$. That is, the probability that an agent who arrives at time t has a type above $y_1^o(t^*)$ is zero. By Theorem 8, we can assume that $p = 1$. This yields

$$0 < R(1, t^*) < y_1^o(t^*). \tag{23}$$

Consider then decreasing $y_1^o(t^*)$ to $R(1, t^*) + \epsilon$ where $y_1^o(t^*) - R(1, t^*) > \epsilon > 0$ (inequality (23) implies that such ϵ exists). This change matters only if at t^* some agent arrives. But, in this case the proposed change increases the revenue, since the object can be sold to that agent at the price $R(1, t^*) + \epsilon$, while prior tot the change the change the probability of a sale was zero. This yields a contradiction that $y_1^o(t^*)$ was chosen optimally.

In order to complete the proof for the one object case, it is enough (given inequality 21) to show that

$$\forall y, \ \frac{(1 - F(y))^2}{f(y)} \geq \int_y^\infty (1 - F(x))dx$$

This follows from

$$\int_y^\infty (1 - F(x))dx = \int_y^\infty \frac{1 - F(x)}{f(x)} f(x)\, dx \leq \frac{1 - F(y)}{f(y)} (1 - F(y))$$

where the last inequality follows by the IFR assumption.

We now proceed to the proof for two objects. After plugging in the expression for $R(1, t)$, we know from Theorem 8 that $y_2^o(t)$ solves

$$y_2^o(t) = \frac{1 - F\left(y_2^o(t)\right)}{f\left(y_2^o(t)\right)} + \lambda \int_t^T \left[\frac{\left[1 - F\left(y_2^o(s)\right)\right]^2}{f\left(y_2^o(s)\right)} - \frac{\left[1 - F\left(y_1^o(s)\right)\right]^2}{f\left(y_1^o(s)\right)} \right] ds$$

By Theorem 1 we know that

$$-y_2^{e\prime}(s) = \lambda \int_{y_2^e(s)}^{y_1^e(s)} (1 - F(x))dx$$

Integrating again both sides between t and T yields

$$y_2^e(t) = \lambda \int_t^T \left[\int_{y_2^e(s)}^{y_1^e(s)} (1 - F(x))dx \right] ds.$$

By Theorem 1 we also know that $y_2^e(t) < y_1^e(t)$. Together with the result for the one object case (see proof above) we obtain that $y_2^e(t) < y_1^o(t)$.

Let $y_2^e(t)$ be the solution to

$$y(t) = H(y(t))$$

and let $y_2^o(t)$ be the solution to

$$y\left(t\right) = G(y\left(t\right)).$$

We are now going to show that $G\left(y\left(t\right)\right) > H(y\left(t\right))$ for any $y\left(t\right) < y_1^o(t)$. Together with $y_2^e\left(t\right) < y_1^o(t)$, this will complete the proof. Note that

$$\begin{aligned} H(y\left(t\right)) &= \int_t^T \left[\int_{y(s)}^{y_1^e(s)} (1-F(x))dx\right] ds \qquad (24) \\ &= \int_t^T \left[\int_{y(s)}^{y_1^e(s)} \frac{1-F(x)}{f\left(x\right)} f\left(x\right) dx\right] ds \\ &\leq \int_t^T \frac{1-F\left(y\left(s\right)\right)}{f\left(y\left(s\right)\right)} \left(\left[1-F\left(y\left(s\right)\right)\right]-\left[1-F\left(y_1^e\left(s\right)\right)\right]\right) ds \\ &= \int_t^T \left(\frac{\left(1-F\left(y\left(s\right)\right)\right)^2}{f\left(y\left(s\right)\right)} - \frac{\left(1-F\left(y\left(s\right)\right)\right)\left(1-F\left(y_1^e\left(s\right)\right)\right)}{f\left(y\left(s\right)\right)}\right) ds \\ &\leq \int_t^T \left(\frac{\left(1-F\left(y\left(s\right)\right)\right)^2}{f\left(y\left(s\right)\right)} - \frac{\left(1-F\left(y_1^o(s)\right)\right)^2}{f\left(y_1^o(s)\right)}\right) ds < G(y\left(t\right)) \end{aligned}$$

The third line follows from IFR assumption, and the fourth line follows from IFR together with the assumption $y\left(t\right) < y_1^o(t)$ and $y_1^e\left(s\right) < y_1^o(t)$. The last inequality follows from the same argument as in the one object case since $\frac{1-F(y_2^o(t))}{f\left(y_2^o(t)\right)} > 0$. In addition, note that the IFR assumption implies that G is a decreasing function.

Assume now, by contradiction, that there exists some t such that $y_2^e\left(t\right) > y_2^o\left(t\right)$. Then

$$y_2^e\left(t\right) = H\left(y_2^e\left(t\right)\right) < G\left(y_2^e\left(t\right)\right) < G\left(y_2^o\left(t\right)\right) = y_2^o\left(t\right),$$

where the first inequality follows from (24), while monotonicity of G implies the second inequality. Therefore, we got that $y_2^e\left(t\right) < y_2^o\left(t\right)$, which is a contradiction. The proof for $n > 2$ follows analogously. ■