Public Choice 41: 7–53 (1983).

A strategic calculus of voting *

THOMAS R. PALFREY

HOWARD ROSENTHAL**

1. Introduction

A universal observation about democratic political processes is that a significant fraction of eligible voters turns out to vote. A controversial and important question in the modern theory of electoral processes is "Why do voters vote?"

Before the advent of modern political economy, voters might have been thought to vote instrumentally, thus affecting the direct outcome of the election. Political economists, however, concluded, erroneously in our view, that the probability that a given voter would be the decisive voter in a large electorate was miniscule, so miniscule in effect that a rational citizen would never find it in his interest to vote on instrumental grounds. Consequently, a variety of explanations for voting has been offered.

* We have benefited from discussion with Peter Coughlin, Terry Levesque, Peter Ordeshook, Steve Salop, and participants in the Conference. This work was supported by National Science Foundation grant DAR-7917576. We thank Glenn Benson for preparing the computer graphics and Kathy Bagwell for processing the manuscript.

** Graduate School of Industrial Administration, Carnegie-Mellon University.

These include citizen duty (socialization), minimax regret, private consumption value, the monitoring of voting and sanctions for nonvoting, and individual voter utility depending on the margin of victory as well as victory itself. We deliberately eschew any of these explanations of voting and indicate that substantial voter turnout may occur even in a totally instrumental, outcome-oriented polity.

We present a game-theoretic model of voter turnout. Our model features voting over two fixed alternatives as in a two-candidate election or in a referendum or initiative vote between a proposal and a status quo. Voters are narrowly self-interested. Their motivation for voting relates solely to the fact that they are better off if their "side" wins. They attach no sense of citizen duty or private consumption value to the act of voting. While, in equilibrium, it is possible that there is substantial voter turnout, multiple equilibria proliferate. As a consequence, with small numbers of voters there are not strong predictions about the size of voter turnout. However, for large electorates we have discovered only two types of equilibria. In one type, turnout approaches zero per cent. In the other, percentage turnout approaches twice the minority side's percentage of the electorate.

The modern theory of turnout began with the observation by Downs (1957), Tullock (1967), Riker and Ordeshook (1968) and others that the act of voting itself is costly. Citizens, then, must weigh the cost of voting against its potential benefits. In decision-theoretic terms, a "rational" voter will vote if and only if the potential benefit of voting exceeds the cots.

A next assumption commonly made is that the benefits are determined by two components: p, the probability that one's vote is decisive (i.e., makes a difference in the outcome), and B, the increase in utility to the voter when his vote is in fact decisive. If the cost of voting is relatively high and the probability of casting a decisive vote is very low, then a rational citizen may choose not to vote even if the citizen has strong preferences between the two alternatives.

This *decision-theory* model can be summarized by the familiar inequalities:

$$\begin{aligned} &\text{Vote} &&\text{if } pB > c \\ &\text{Abstain} &&\text{if } pB < c \end{aligned} \tag{1}$$

The argument continues by observing that if there are many citizens voting, the probability that a citizen is decisive must be incredibly small.[1] Hence the conclusion, often referred to (we feel inappropriately) as the paradox of not voting: significant turnout in elections with many eligible voters is inconsistent with rational behavior.

There have been several proposed explanations for the purported "paradox." One approach was to suppose that citizens derive direct benefits from voting, D, regardless of the outcome. If this direct benefit, sometimes called "citizen duty," is greater than the cost of voting, then voting is not irrational and in fact is rational even if the citizen does not care what the outcome of the election is.

A second approach, adopted by Ferejohn and Fiorina (1974), was to suppose that citizens are not concerned with the probability that they are decisive. Voters vote simply to avoid the possibility that if they abstain but the election ends in a tie or one vote short of a tie, they will suffer substantial regret since their vote would have been decisive.

Both of these avenues of rationalization suffer weaknesses. Clearly, assuming $pB = 0$, many observations are inconsistent with the proposition that a given individual's net cost of voting, $c - D$, is anywhere near constant. The greater turnout in presidential than in off-year elections and the greater turnout in contested than in uncontested elections belie any simple citizen-duty story. Of course, citizen duty could be rescued by arguing that there is a greater sense of duty in presidential and contested elections, but such logic is difficult if not impossible to test.

As to the regret approach, it has been subject to a series of theoretical challenges in an exchange between Ferejohn and Fiorina (1975) and their critics. More importantly, much empirical evidence suggests that probabilities matter. Presidential turnout was low, even among whites, in the old one-party South.[2] Southerners in fact were often more likely to vote in the decisive primaries than in the general elections. Rosenthal and Sen (1973) present extensive evidence for French legislative elections showing that turnout

1. Chamberlain and Rothschild (1981) develop the probability that an election results in a tie when all voters vote and when any randomly chosen voter votes "Yes" with some fixed probability. Acceptance of the idea that the probability of casting a decisive vote is very close to zero also finds its way into the lead article of a major political science journal. See Meehl (1977).

2. Wolfinger and Rosenstone (1980) show persistence of low presidential turnout in the South, even when race, education, and registration laws have been controlled.

varies significantly with a simple ex ante measure of the closeness of the election. Romer and Rosenthal (forthcoming), using observations from over 2,000 school-district referenda in Oregon, show that turnout varies strongly with community size, which, in a *ceteris paribus* decision-theoretic argument, determines the probability of being decisive. A desire to explain these findings and to develop a theory of participation based on utility maximization has led us to a third approach.

In our approach, we investigate how the probability of being decisive, p, is determined. If everyone else votes, p can readily be very small. But if no one else votes, the probability of being decisive would be 1. Clearly, if citizens are rational, the voting probabilities and the turnout decisions are simultaneously determined.[3] Models which consider the turnout question under the assumption that citizens vote and then discover citizens should not vote ignore the simultaneity of the problem. The conclusion from these simple models is not a true paradox, it is a logical fallacy.

In an important paper, Ledyard (1981) observed this fallacy and modelled the simultaneity. In his model, when citizens have "rational expectations" of the probability of being decisive, some rational citizens will vote, and some will not. In Ledyard's model each voter knows the size of the electorate, the spatial positions of the alternatives, and his own preferences. However, a voter's information about other citizens is limited to the knowledge that their spatial preferences are drawn from a continuous probability distribution, the distribution being common knowledge. Ledyard is able to make only limited conclusions about turnout. His major result is that turnout is positive when, for some citizens, B is sufficiently large relative to c. Under these conditions, existence of a symmetric rquilibrium is proved, but the possibility of multiple symmetric equilibria or asymmetric equilibria is not investigated.

We adopt a similar approach in which turnout decisions and the probability of decisiveness are simultaneously determined, but our model differs from his in important respects. First, there are only two types (teams) of citizens, each with identical preferences. Each citizen knows the number of members and the preferences of each team. We assume

3. Downs (1957) takes note of this simultaneity problem or "maze of conjectural variation" (p. 267) as he calls it. The well-known citizen duty story rescues democracy on p. 268.

that the teams have opposite preferences over the alternatives. The alternatives are determined exogenously. Our model is thus a partial equilibrium model, but it applies well to the restricted game where alternatives are fixed and distinct, as is the case for referenda and initiatives. Analysis of this simple model permits us to obtain closed-form solutions of equilibria, to show that the amount of voting can be significant, and to explore the dependence of turnout upon costs, the size of the electorate, the distribution of voters' preferences, and the rule used to resolve a tie vote.

Our specific model is game-theoretic. It is also of interest, simply from a theoretical point of view, because the game we analyze is one of an important class of games which to our knowledge has never been systematically analyzed.[4] This type of game, which we call a *team game*, combines features of asymmetric and symmetric games. There are two (or more) groups of players called teams, and the players on a given team are identical and have common interests. Players on different teams, however, have different interests. Each player chooses a level of action. Cost to the player is increasing in his own action. Payoff to a player is an increasing function of the actions by all players on his team and is a decreasing function of the actions by all players on the opposing teams. Net of cost, all players on the same team receive the same payoff.

A special case of team games, *participation games*, involves binary decisions by all players. Each player has a choice of two actions, which we refer to as *participation* and *nonparticipation*. Participation is costly while nonparticipation is not.

2. Participation games

Let $I = [1, ..., M, M+1, ..., M+N]$ index the players. There are two teams, one, denoted T_1, consisting of the first M players and the other denoted T_2, consisting of the last N players. The action set for any player on any team is $\{0, 1\}$, where 0 is referred to as nonparticipation, and 1 is referred to as participation. An action for player i is denoted s_i. A mixed strategy for a voter is just a probability of voting. We denote these probabilities by

4. Some discussion of games of this sort, which involve elements of both conflict and coordination, appears in Schelling (1960, p. 85).

q_i, for player i in T_1 and r_j for player j in T_2. The payoff for any player i in T_1 is given by $\pi_i = F_1(\sum_{i \epsilon T_1} s_i, \sum_{j \epsilon T_2} s_j) - s_i c_i$. The payoff for any player j in T_2 is given by $\pi_j = F_2(\sum_{i \epsilon T_1} s_i, \sum_{j \epsilon T_2} s_j) - s_j c_j$. Furthermore, F_1 is nondecreasing (nonincreasing) in its first (second) argument and F_2 is nondecreasing (nonincreasing) in its second (first) argument. We here restrict analysis to the situation where all strategy choices must be exercised independently (e.g., simultaneously).

In our analysis of voting as a participation game, the strategy, s_i, is referred to as i's voting decision, and c_i is referred to as i's cost of voting. The payoff functions F_1 and F_2 constitute the voting rules.

We make the following assumptions hereafter:

1. *Identical Cost to All Citizens*

$$c_1 = \dots = c_M = c_{M+1} = \dots = c_{M+N} = c \,\epsilon (0, 1)$$

2. *Symmetric Payoffs to the Teams*

$$F_2(.,.) = 1 - F_1(.,.)$$

We investigate the following two payoff rules:

*R*1. *Coin Toss (Simply Plurality Rule, Ties Broken Randomly)*

$$F_1(\sum_{i \epsilon T_1} s_i, \sum_{j \epsilon T_2} s_j) = \begin{cases} 0 \text{ if } \sum_{i \epsilon T_1} s_i < \sum_{j \epsilon T_2} s_j \\ 1 \text{ if } \sum_{i \epsilon T_1} s_i > \sum_{j \epsilon T_2} s_j \\ 1/2 \text{ if } \sum_{i \epsilon T_1} s_i = \sum_{j \epsilon T_2} s_j \end{cases}$$

R2. Status Quo (Simple Plurality Rule, Ties Broken in Favor of Team 2)

$$F_1\Big(\sum_{i\epsilon T_1} s_i, \sum_{j\epsilon T_2} s_j\Big) = \begin{cases} 0 \text{ if } \sum_{i\epsilon T_1} s_i \leqslant \sum_{j\epsilon T_2} s_j \\ 1 \text{ if } \sum_{i\epsilon T_1} s_i > \sum_{j\epsilon T_2} s_j \end{cases}$$

We interpret the outcome favored by team 2 as the status quo in the sense that it is implemented as the default in case of a tie election.[5]

Our assumptions thus model a two candidate (or proposal) election with members of T_1 each getting a payoff of 1 if its candidate (or proposal) wins 0 and if the other candidate (proposal) wins. The payoffs to voters in T_2 are the opposite.

3. Basic characteristics of equilibrium in participation games: the chickens' dilemma

Having described the formal structure of a participation game, our next task is to investigate equilibrium in the game. Play of the game reflects a basic tension between a motivation to vote in order to obtain the winners' payoff and a motivation to abstain in order to free-ride on the voting of other members of one's own team. These two effects can be succinctly illustrated in the context of two well-known games.

First, we illustrate the competitive element by considering $M = N = 1$ for the coin-toss rule and $c < 1/2$. Since there is only one player on each team, there is no opportunity for free-riding. The bimatrix form of the game is:

		Player 2	
		not vote	vote
Player 1	not vote	1/2 1/2	1 - c 0
	vote	0 1 - c	1/2 - c 1/2 - c

5. Rule $R2$ is found in practice. The status quo wins in Oregon referenda. The oldest candidate wins in tied elections in France. In addition, analysis is usually more tractable for $R2$ than for $R1$.

Readers will quickly recognize this game as a Prisoners' Dilemma. The competitive aspect of the game induces voting. Of course, the game's unique Nash equilibrium is Pareto dominated by just tossing a coin without incurring the deadweight loss of voting.

Second, we show the cooperative aspect by considering $M = 2$, $N = 0$ for the coin-toss, again for $c < 1/2$. Since there is no opposition, we have a pure public goods problem where free riding is the paramount consideration. The bimatrix form is:

		Player 2	
		not vote	vote
Player 2	not vote	$1/2 - c$, $1/2 - c$	1, $1 - c$
	vote	$1 - c$, 1	$1 - c$, $1 - c$

The game is a version of "chicken." There are two pure strategy equilibria, each with one player voting and the other abstaining, and there is a mixed-strategy equilibrium, with both players voting with probability $1 - 2c$.

Chicken demonstrates that mixed strategy equilibria and multiple equilibria are inherent in participation games. It also shows that symmetry is not a compelling criterion in "choosing" among equilibria. Although the mixed strategy equilibrium in chicken is symmetric in the players' strategies while the pure strategy equilibria are not, it is difficult to argue that the symmetric equilibrium is the "more reasonable" one. In fact, the asymmetric equilibria weakly Pareto dominate the symmetric equilibrium. Readers might heed this example when evaluating the various asymmetric equilibria that follow.

The more interesting participation games arise when $M, N > 0$, $M+N > 2$. As they combine the competition and coordination features of the above games, these situations might be labelled "chickens' dilemmas."

By Nash's theorem, every participation game has a Nash equilibrium, possibly in mixed strategies. The study of equilibria begins with a description of the best responses of a player given the strategies used by other players:

Let EV^i_V and EV^i_{NV} denote the expected payoffs to player i from voting and abstaining, respectively, given the strategies of other players.

Then:

Voting ($q_i = 1$) is a best response if $\quad EV^i_V \geqslant EV^i_{NV}$

A mixed strategy $q_i \in (0, 1)$ is a best response if $\quad EV^i_V = EV^i_{NV}$

Abstention ($q_i = 0$) is a best response if $\quad EV^i_V \leqslant EV^i_{NV}$

We denote by $m(n)$ the total number of actual voters in $T_1(T_2)$ and by $m^i(n^j)$ the total number of actual voters in $T_1(T_2)$ other than $i(j)$. The expected payoffs can be expressed as follows.

For T_1 and $R2$ (status quo):

$$EV^i_V = 1 \cdot \text{prob}[m^i + 1 > n] + 0 \cdot \text{prob}[m^i + 1 \leqslant n] - c$$

$$EV^i_{NV} = 1 \cdot \text{prob}[m^i > n] + 0 \cdot \text{prob}[m^i \leqslant n]$$

and $EV^i_V = EV^i_{NV}$ *iff* $c = \text{prob}[m^i = n]$

For T_2 and $R2$:

$$EV^j_V = 1 \cdot \text{prob}[n^j + 1 \geqslant m] + 0 \cdot \text{prob}[n^j + 1 < m] - c$$

$$EV^j_{NV} = 1 \cdot \text{prob}[n^j \geqslant m] + 0 \cdot \text{prob}[n^j < m]$$

and $EV^j_V = EV^j_{NV}$ *iff* $c = \text{prob}[n^i = m - 1]$

For T_1 and $R1$ (coin-toss):

$$EV^i_V = 1 \cdot \text{prob}[m^i + 1 > n] + 1/2\ \text{prob}[m^i + 1 = n] + 0 \cdot \text{prob}[m^i + 1 < n] - c$$

$$EV^i_{NV} = 1 \cdot \text{prob}[m^i > n] + 1/2\ \text{prob}[m^i = n] + 0 \cdot \text{prob}[m^i < n]$$

and $EV^i_V = EV^i_{NV}$ *iff* $2c = \text{prob}[m^i = n] + \text{prob}[m^i = n-1]$

Symmetric expressions exist for T_2 under $R1$.

Using these best-response relationships, we have succeeded in identifying the following equilibria:

(1) All *pure strategy equilibria*. Necessary and sufficient conditions for the existence of pure strategy equilibria, indicated in Section 4, show that pure strategy equilibria arise only in degenerate cases where $c = 0$ or c is very large or in limiting cases where $M = 0$ or $N = 0$ or $M = N$.

The interesting equilibria involve some use of mixed strategies.

(2) *Quasi-symmetric mixed strategy equilibria.*

Definition: An equilibrium is *quasi-symmetric* if

$$i_1, i_2 \in T_1 \Rightarrow q_{i_1} = q_{i_2} \text{ if } q_{i_1} \in (0, 1) \text{ and } j_1, j_2 \in T_2 \Rightarrow r_{j_1} = r_{j_2} \text{ if } r_{j_1} \in (0, 1).$$

In other words, if any member of a team is mixing, everyone on the team is mixing with the same probability. This paper focuses on quasi-symmetric equilibria. They first occur in Section 5 which contains detailed analysis of the "mixed-pure" case where all the members of one team use pure strategies and all the members of the other team use mixed strategies.

Definition: An equilibrium is *totally quasi-symmetric* if the equilibrium is both totally mixed[6] and quasi-symmetric. The equilibrium is denoted as (q, r).

We have been able to analyze some totally quasi-symmetric equilibria. Section 6 covers two cases. First, in the very special case when $M = N$ and $(R1)$, we examine the symmetric equilibrium where $r = q$, i.e., (q, q). Second, in general, we analyze equilibria of the form $r = 1 - q$, i.e., $(q, 1 - q)$. We conjecture that the class of all totally quasi-symmetric equilibria is much larger than those we have been able to investigate.

After presenting the various equilibria, we introduce a numerical example in Section 7. Analysis of how equilibrium changes when M, N, and c change and results concerning expected turnout and plurality are presented in Section 8.

4. Pure strategy voting equilibria

Proposition 1. An exhaustive description of equilibria where all voters use pure strategies is given by:

$R1$. (We omit the laborious but trivial case of $c = 1/2$.)

1. If $c > 1/2$, the only Nash equilibrium has no one voting.

6. Strategies are totally mixed when no player uses a pure strategy.

2. If $c < 1/2$, $M \geqslant 1$, $N = 0$, the set of all pure strategy equilibria contains the M asymmetric equilibria in which exactly one of the players of team T_1 votes and everyone else abstains. Similar equilibria exist for $N \geqslant 1$, $M = 0$.
3. If $c < 1/2$, $M = N \geqslant 1$, there exists a unique pure strategy equilibrium with everyone voting.

$R2$.

4. If $M \geqslant 1, N = 0$, the condition is identical to case 2.
5. If $M = 0, N \geqslant 1$, no voting is the only Nash equilibrium.

Proof: Proof of statements 1-5 is trivial. It remains to show that there are no other totally pure strategy equilibria.

If $c > 1/2$ the proof is obvious for $R1$. Suppose $c < 1/2$ for $R1$, and that there is a Nash equilibrium with exactly m players in T_1 voting and n voters in T_2. We must show that if $M \geqslant 1$, $N \geqslant 1$, and if it is not the case that $m = M = N = n$, a pure strategy Nash equilibria does not exist. There are five possibilities:

(1) $m > n + 1$. In this case no individual's decision can be decisive, so all players have an incentive to abstain.

(2) $m + 1 < n$. Same argument as in (1).

(3) $m = n$. Since $M \neq N$ we know there is some nonvoter. Every nonvoter is decisive and can increase his payoff by $1/2 - c$ by voting. Hence nonvoters will have an incentive to vote.

(4) $m = n - 1$. If there are any nonvoters in T_1, then the argument in (3) applies to them. If there are no nonvoters in T_1, then $m = M \geqslant 1$, so there are some voters in T_1. These voters get $-c$ by voting and 0 by not voting, so they are better off not voting.

(5) $m = n + 1$. Same argument as in (4), applied to T_2. The proof for $R2$ follows an identical line of argument to the above.

The conclusion is that pure strategy equilibria fail to exist except for a few very special cases. We therefore turn to equilibria involving mixed strategies.

5. Mixed-pure strategy equilibria

This section characterizes and demonstrates the general existence of equilibria in which all players on one team vote with an identical probability strictly between 0 and 1, k players on the other team vote with probability 1, and the remaining players vote with probability 0. They are denoted by $(q;\, k)$ and $(k;r)$, depending upon which team uses mixed strategies. These equilibria exist in abundance. Nonetheless, they share a number of common properties, which will be discussed at the end of the paper.

We now develop the "k equilibria" by considering seven exhaustive cases. We begin with rule $R2$ and then turn to rule $R1$.

The status quo rule, pure strategies for T_2

Case 1. $R2$, $k = 0$ for T_2.

We begin with $N = 0$, a generalized chicken game for the status quo rule. Consider the necessary and sufficient condition for q to be a best response for voter i in T_1 when $N = 0$. We must have (for $M > 1$):

$$c = \text{prob}[m^i = 0] = (1 - q)^{M-1} \tag{2}$$

$$q(c) = 1 - c^{\frac{1}{M-1}}$$

Thus, for every c there is a unique symmetric mixed strategy equilibrium in addition to the M pure strategy equilibria with one person voting.

Now when $N > 0$, for $k = 0$ to be an equilibrium we must, in addition to (2), satisfy the necessary and sufficient condition for nonvoting to be a best response for voters in T_2. This condition is:

$$c \geqslant \text{prob}[m = 1] = Mq(1 - q)^{M-1} \tag{3}$$

From these two conditions, we obtain a $c_{min}^{M,0}$ such that an equilibrium of this variety exists for all $c \geqslant c_{min}^{M,0}$. Solving the equations, one obtains

$$c_{min}^{M,0} = (1 - \frac{1}{M})^{M-1}$$

$$q(c_{min}^{M,0}) = \frac{1}{M}$$

$$q(c) = 1 - c^{\frac{1}{M-1}}$$

The $(q;0)$ equilibrium for $N > 0$ is illustrated in Figure 1. For each M in the figure, an equilibrium graph is formed by two distinct curves which intersect at $c_{min}^{M,0}$. The $(q;0)$ graph is the portion of the graph to the right of $c_{min}^{M,0}$. The other portion of the graph is discussed later.

Several interesting properties of this equilibrium can be noted immediately. There is low turnout; and the equilibrium exists only when substantial costs are present. When the equilibrium exists, turnout is *decreasing* in the cost of voting, as would be expected by the conventional decision-theory approach (eq. 1). The equilibrium is independent of N.

As an example, consider $M = 2$, $c = 1/2$. In this case, $q = 1/2$; expected turnout is 1 voter; expected plurality is one in favor of T_1, and T_1 wins with probability 3/4 no matter how large N becomes.

Case 2. $R2, 0 < k \leqslant min[M-1, N]$ for T_2.

For an equilibrium to result in which k members of T_2 vote, we must modify (2) to:

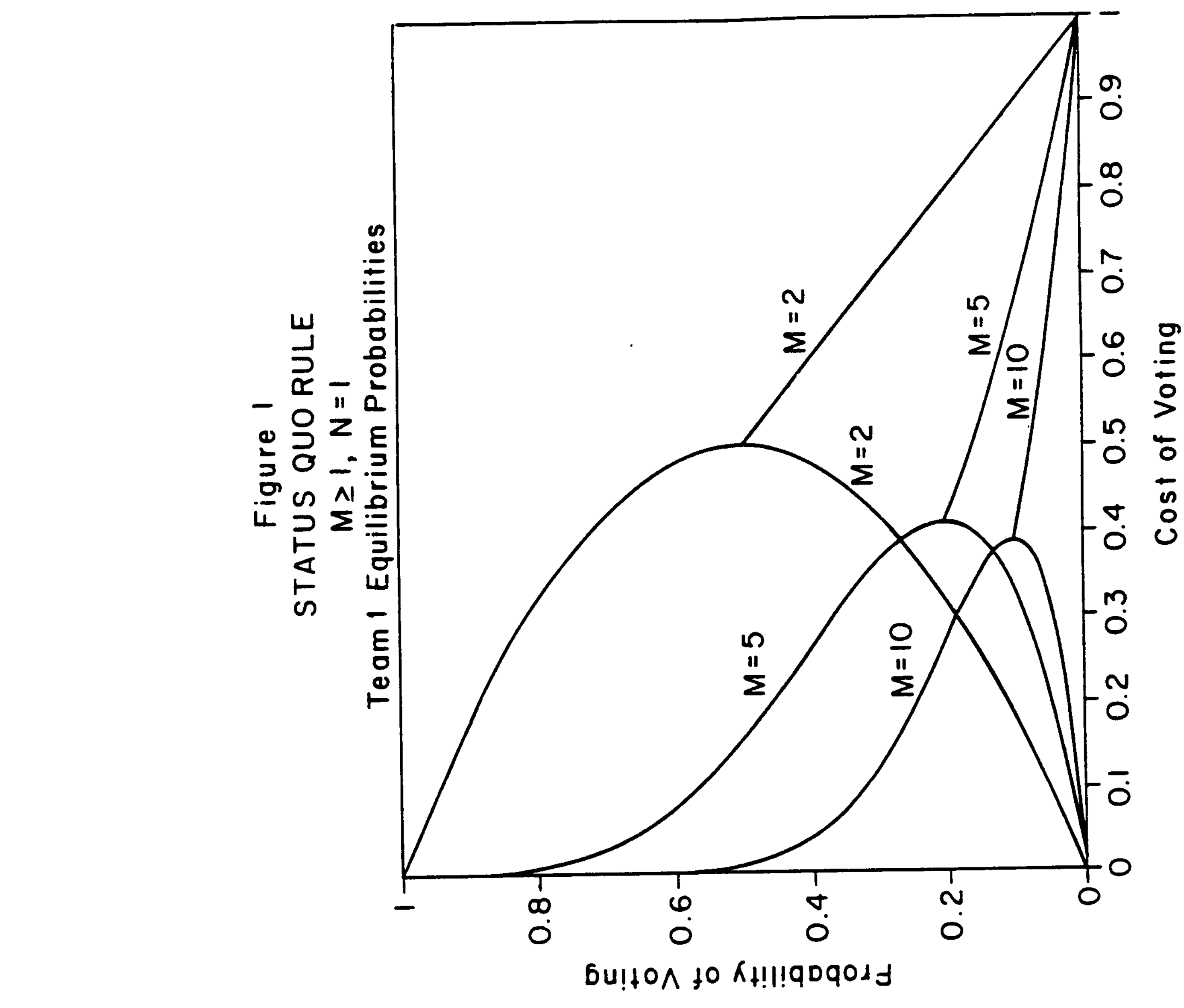
$$c = \text{prob}[m^i = k] = \binom{M-1}{k} q^k (1-q)^{M-k-1} \qquad (4)$$

Non-voters in T_2 must satisfy:

$$c \geqslant \text{prob}[m = k+1] = \binom{M}{k+1} q^k (1-q)^{M-k-1} \qquad (5)$$

An additional equilibrium condition must be satisfied by *voters* in T_2;

Figure I
STATUS QUO RULE
M ≥ I, N = I
Team I Equilibrium Probabilities



$$c \leqslant \text{prob}[m = k] = \binom{M}{k} q^k (1-q)^{M-k} \tag{6}$$

Now since (4) and (5) require $q \leqslant \frac{k+1}{M}$ but (4) and (6) require:

$$q \leqslant \frac{k}{M}, \tag{7}$$

(5) is redundant, and (4), (7) characterize these equilibria.

The $(q;k)$ equilibria for $k \geqslant 1$ are shown in Figure 2, $M = 20$. The illustrations indicate that the equilibrium for $k > 0$ is totally opposite in character to that for $k = 0$. It is easily verified that for each k, c and q are now positively rather than negatively related [i.e., $c'(q) > 0$ if $0 \leqslant q \leqslant \frac{k}{M}$]. We now obtain a $c_{max}^{M,k}$ such that an equilibrium of this variety exists for all $c \leqslant c_{max}^{M,k}$.

Specifically, evaluating (4) at $q = k/M$ shows

$$c_{max}^{M,k} = \binom{M-1}{k} \frac{k^k (M-k)^{M-k-1}}{M^{M-1}}$$

Note that at $c_{max}^{M,k}$ expected turnout is $2k$ and expected plurality is 0, and for $c < c_{max}^{M,k}$ expected turnout is between k and $2k$ and expected plurality is not zero, but always favors T_2. Contrary to the conventional model, expected turnout is an *increasing* function of voting cost.

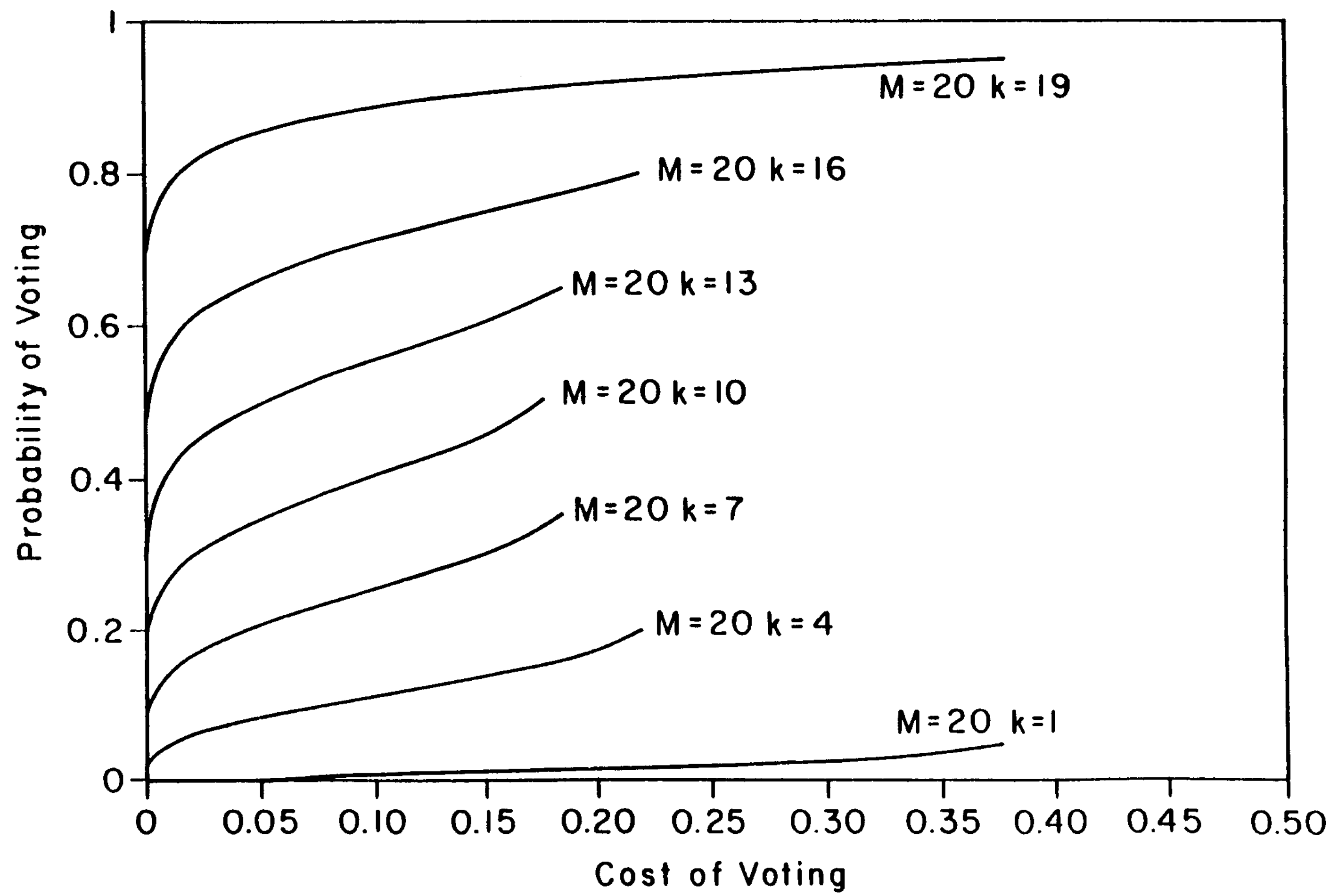
Of particular interest is $k = 1$. One finds readily that $c_{min}^{M,0} = c_{max}^{M,1}$, which establishes the existence of mixed-pure strategy equilibria for all values of c.

Proposition 2. Mixed-pure strategy equilibria exist for all values of $c \in (0,1)$ if $M, N \geqslant 2$.

In the $k = 1$ equilibrium at $c_{max}^{M,1}$, T_2 wins with probability $[2 - 1/M](1 - 1/M)^{M-1}$. This equals .75 for $M = 2$ and has a limit of $2e^{-1} = .736$ as $M \to \infty$. T_2 is even more likely to win for $c < c_{max}$. Thus, in the $k = 1$ equilibrium, T_2 is always more likely to win regardless of its size relative to T_1.

Figure 2

M=20 STATUS QUO RULE q,k



Another interesting case is $k=M-1$. We again readily find that $c^{M,M-1}_{max} = c^{M,1}_{max} = c^{M,0}_{min}$. Together with proposition 2, this establishes:

Proposition 3. For $c \leqslant c^{M,1}_{max}$, $M \geqslant 3, N \geqslant 2$, mixed-pure strategy equilibria exist for at least two values of k.

At $c^{M,M-1}_{max}$, $q = (M-1)/M$ and expected turnout is $2(M-1)$ which can be very substantial! The probability that T_2 wins is $1-(1-1/M)^M$. As M,N enlarge (providing $M-1 \leqslant N$), this probability approaches $1-e^{-1} = .632$. As with the $k = 1$ equilibrium, for all c, T_2 is more likely to win in this equilibrium.

The status quo rule, pure strategies for T_1

Case 3. $R2$, $k = 0$ or 1 for T_1.

Proposition 4. There is no mixed-pure equilibrium with $k = 0$ or 1 for T_1.

Proof: a. If $k = 0$, abstention is a unique best response for all members of T_2.

b. If $k = 1$, r must satisfy

(i) $r \in (0,1)$

(ii) $c = \text{Prob}[n^j = 0] = (1-r)^{N-1}$

(iii) $c \leqslant \text{Prob}[n=0] = (1-r)^N$

These conditions cannot be satisfied for $c \in (0,1)$.

Case 4. $R2$, $2 \leqslant k \leqslant \min\,[M, N]$ for T_1.

The analogous conditions to Case 2 are:

$$c = \binom{N-1}{k-1} r^{k-1} (1-r)^{N-k} \tag{4'}$$

$$c \geqslant \binom{N}{k} r^{k} (1-r)^{N-k} \tag{5'}$$

$$c \leqslant \binom{N}{k-1} \; r^{k-1}(1-r)^{N-k+1} \tag{6'}$$

Once again, (5') is redundant while (4') and (6') yield

$$r \leqslant \frac{k-1}{N} \tag{7'}$$

Equilibria are thus characterized by (4') and (7'). As before, it is easily verified that c and r are positively related [$c'(r) > 0$ when $0 \leqslant r \leqslant \frac{k-1}{N}$] and

$$c_{max}^{N,k} = \binom{N-1}{k-1} \frac{(k-1)^{k-1} \; (N-k+1)^{N-k}}{N^{N-1}}$$

Note that expected turnout at c_{max} is $2k-1$ and expected plurality is 1 (in favor of T_1) at c_{max}, and for $c < c_{max}$ expected turnout is between k and $2k-1$ and expected plurality always favors T_1 by more than 1 vote. Expected turnout is an increasing function of voting cost, for $c \leqslant c_{max}^{M,k}$.

Another result parallel to Case 2 is:

$$c_{max}^{N,2} = c_{max}^{N,N} \; .$$

Results concerning the probability of winning, the existence of multiple equilibria, and turnout also parallel Case 2, except T_1 is now more likely to win than T_2. In particular at $c_{max}^{N,N}$, expected turnout is $2N-1$. Note that when $M=N$, at $c_{max}^{M,M-1}$ and $c_{max}^{N,N}$, nearly everyone votes!

Coin-toss rule

Our mixed-pure strategy results for the coin-toss rule are limited. Without loss of generality, we assume the pure strategies are employed by T_2.

Case 5. $R1$, k=0.

With N = 0, we obtain results similar to Case 1, with expected turnout *decreasing* in cost. In equilibrium,

$$2c \; = \; \text{Prob}[m^i = 0] \; = \; (1-q)^{M-1} \tag{8}$$

$$q(c) = 1 - (2c)^{\frac{1}{M-1}}, \;\; 0 < c \leqslant 1/2$$

One can easily see that each player's probability of voting goes to 0 for any fixed c, as M becomes large. A somewhat more interesting question is how expected total turnout changes as there are more players. Expected turnout is:

$$Mq \; = \; M - M(2c)^{\frac{1}{M-1}}$$

By a simple limiting argument, one can show that

$$\lim_{M \to \infty} [M - M(2c)^{\frac{1}{M-1}}] \; = \; \ell n \left(\frac{1}{2c}\right) \text{ and } \lim_{M \to \infty} (1-q)^M = 2c$$

Consequently, turnout is very small, even for very small c. For example, $\ell n(1/2c) = 13.12$ for $c = 10^{-6}$.

For $N > 0$, we obtain a result similar to Case 3.

Proposition 5. There is no equilibrium for $k = 0, N > 0, R1$.

Proof: Equilibrium would require (8) and

$$2c \; \geqslant \; \text{Prob}[m=1] + \text{Prob}[m=0] \tag{9}$$

$$= mq(1-q)^{M-1} + (1-q)^M$$

Taken together, (8) and (9) imply

$$0 \geqslant (M-1)q$$

which contradicts $q \in (0,1)$. This establishes the desired result.

Case 6. $R1,\ 1 \leqslant k \leqslant M-1$.

The equilibrium conditions that must hold are:

$$2c \;=\; \text{Prob}[m^i = k-1] + \text{Prob}[m^i = k] \tag{8'}$$

$$2c \;\geqslant\; \text{Prob}[m = k+1] + \text{Prob}[m = k] \tag{9'}$$

$$2c \;\leqslant\; \text{Prob}[m = k] + \text{Prob}[m = k-1] \tag{10'}$$

Note that (10') and (9') imply

$$\text{Prob}[m = k-1] \geqslant \text{Prob}[m = k+1]$$

By the properties of the binomial distribution, this condition implies (Feller, 1957, Theorem 1, p. 140).

$$q < \frac{k+1}{M+1} \tag{7''}$$

Moreover, at $q = k/M < \dfrac{k+1}{M+1}$

$$c(k/M) \;=\; c_{max}^{M,k} \;=\; c(k/M) \text{ for } R2$$

These results suggest that $R1$ gives results similar to $R2$, although we have not analyzed (8') - (10') fully.

Case 7. $R1,\ k = M,\ M \leqslant N$.

The equilibrium conditions here are

$$2c \;=\; \text{Prob}[m^i = M-1] = q^{M-1} \tag{8''}$$

$$2c \;\geqslant\; \text{Prob}[m = M] = q^M \tag{9''}$$

$$2c \;\leqslant\; \text{Prob}[m = M] + \text{Prob}[m = M-1] = q^M + Mq^{M-1}(1-q) \tag{10''}$$

Constraints (9") and (10") are nonbinding so this equilibrium exists for all values of $c < 1/2$. Members of T_1 vote with probability $q = (2c)^{\frac{1}{M-1}}$ which approaches 1 as $M \to \infty$.

Furthermore, the election approaches a tie. The probability that the "committed" side, T_2, wins is $1-2^{\frac{1}{M-1}}c^{\frac{N}{M-1}}$ which approaches $1-c$ as M, N become large. Thus T_2 is always more likely to win. Turnout is once again *increasing* in cost and may be substantial.

6. Totally quasi-symmetric equilibria

A. Coin-toss rule

Since we are looking at totally mixed strategy equilibria, $q \in (0,1)$ and $r \in (0,1)$. In order for (q,r) to be an equilibrium, q must be a best response for any player on T_1 and r must be a best response for any player on T_2. A necessary and sufficient condition for this is that in equilibrium each player is indifferent between voting and not voting. Recall that, for T_1, indifference implies

$$2c = \text{prob}[m^i = n] + \text{prob}[m^i = n-1].$$

If other members of T_1 use strategy q and members of T_2 use strategy r, then

$$\text{prob}[m^i=n] = \sum_{k=0}^{min[M-1,\, N]} \binom{M-1}{k}\binom{N}{k} q^k(1-q)^{M-1-k} r^k (1-r)^{N-k}$$

and

$$\text{prob}[m^i=n-1] = \sum_{k=0}^{min[M-1,N-1]} \binom{M-1}{k}\binom{N}{k+1} q^k(1-q)^{M-1-k} r^{k+1}(1-r)^{N-1-k}$$

Therefore, the necessary and sufficient condition for q to be a best response is:

$$\begin{aligned} 2c = &\sum_{k=0}^{min[M-1,N]} \binom{M-1}{k}\binom{N}{k} q^k(1-q)^{M-1-k} r^k(1-r)^{N-k} \\ &+ \sum_{k=0}^{min[M-1,N-1]} \binom{M-1}{k}\binom{N}{k+1} q^k(1-q)^{M-1-k} r^{k+1}(1-r)^{N-1-k} \end{aligned} \tag{11}$$

Similarly, we can derive the necessary and sufficient condition for a member of T_2 to have a best response r, given that members of T_1 use q and all other members of T_2 use r. This condition is:

$$2c = \sum_{k=0}^{min[M,N-1]} \binom{M}{k}\binom{N-1}{k} q^{k+1}(1-q)^{M-1-k} r^k (1-r)^{N-1-k}$$
$$+ \sum_{k=0}^{min[M-1,N-1]} \binom{M}{k+1}\binom{N-1}{k} q^k (1-q)^{M-k} r^k (1-r)^{N-1-k} \qquad (12)$$

Together, (11) and (12) characterize all (q,r) equilibria.

We now examine two special cases of totally mixed strategy equilibria.

Case 8: (q,q) equilibria for $M = N > 1$.

Recall that when $M = N > 1$, there is a pure strategy equilibrium with everyone voting, generalizing the result of the basic Prisoners' Dilemma. However, players may also consider strategies that allow them to free ride on the voting of other members of their team. In particular, we can construct a symmetric, totally mixed equilibrium with $q = r$ or (q,q), for some values[7] of $c < 1/2$. Conditions (11) and (12) are the same, and reduce to

$$2c = \sum_{k=0}^{M-1} \binom{M-1}{k}\binom{M}{k} q^{2k}(1-q)^{2M-2k-1}$$
$$+ \sum_{k=0}^{M-1} \binom{M-1}{k}\binom{M}{k+1} q^{2k+1}(1-q)^{2M-2k-2} \qquad (13)$$

This expression determines the voting cost $c(q)$, associated with every possible (q,q) equilibrium. One might suspect at first that for every $c < 1/2$ (i.e., as long as voting is not a dominated strategy), there is at least one symmetric equilibrium q. This is not the case. For sufficiently small c, there is no (q,q) equilibrium. In other words, these equilibria exist only if c is *large* enough!

7. We again omit the knife-edge case of $c = 1/2$.

As an example, consider $M = N = 2$. Then for $c < 3/8$, there is no symmetric equilibrium. In this case, $c(q)$ is shown as the right curve in Figure 3. The left curve is discussed later. Note that, for $3/8 < c < 1/2$, there are two mixed strategy equilibria. Note further that there is a minimum cost, $c_{min}^{2,2} = 3/8$, at which $q = 1/2$. In general, we can show:

Proposition 6:

(a) $c(1/2) = (1/2)^{2M-1}\binom{2M-1}{M}$

(b) for large M, $c(1/2) \sim \dfrac{2}{\sqrt{\pi(2M-1)}}$

(c) as $M \to \infty$, $c(1/2) \to 0$.

(d) In eq. (13), $c(q)$ has a local minimum at $q = 1/2$.

Proof: The proof uses a well-known conbinatorial identity (Tucker, 1980, p. 64):

$$\sum_{k=0}^{A} \binom{A}{k}\binom{B}{r+k} = \binom{A+B}{A+r}. \tag{14}$$

For details, see the Appendix.

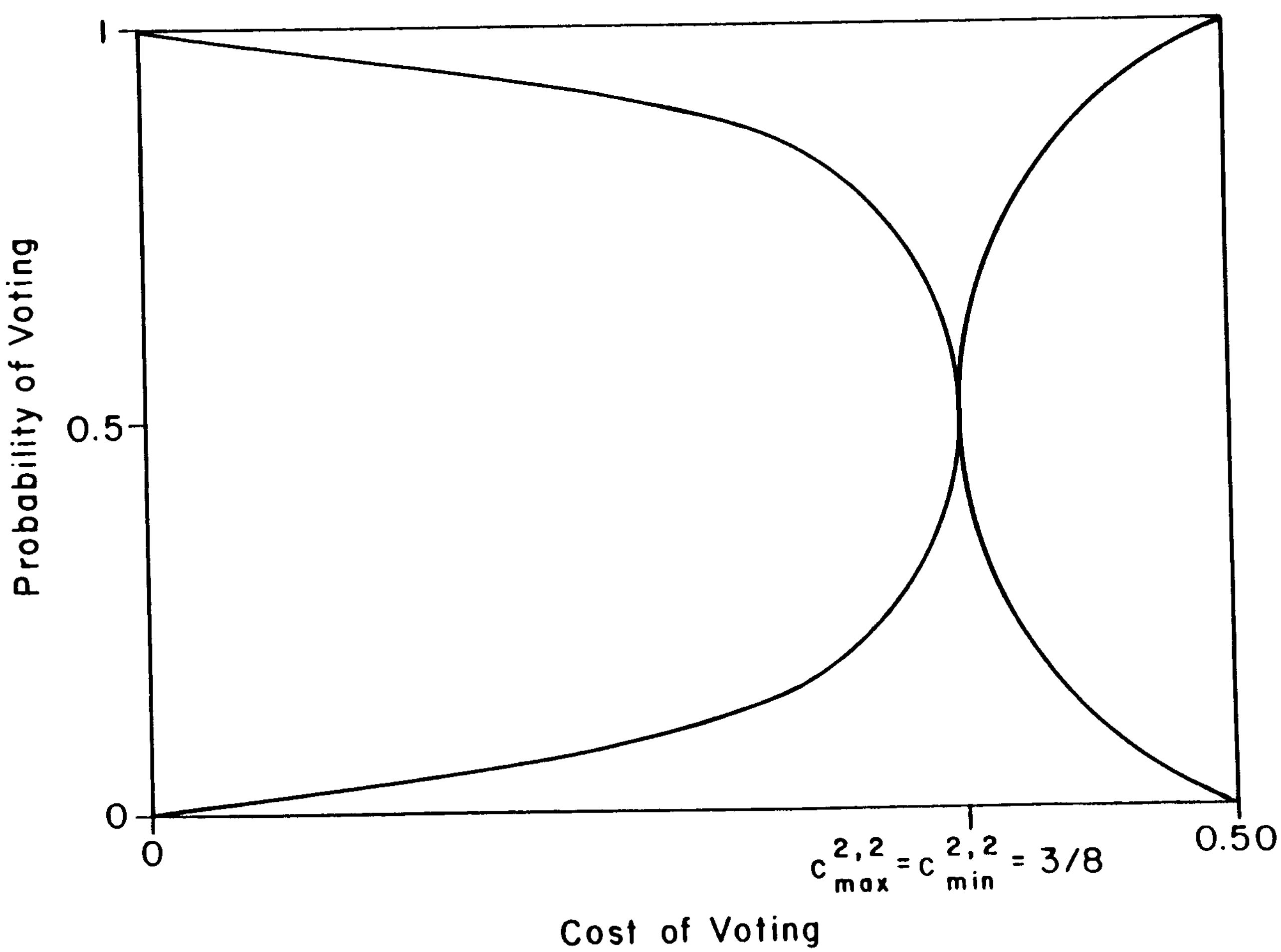
Conjecture: For each integer A such that $M = N = A > 1$, there exists a minimum voting cost, denoted by $c_{min}^{A,A}$ such that:

(1) The unique symmetric equilibria at $c_{min}^{A,A}$ is $q = 1/2$.

(2) $c(q) = c(1-q) \geq c_{min}^{A,A}$ for $q \in (0,1)$.

(3) For every $c \in (0, 1/2)$ and every $\epsilon > 0$, there exists an A_ϵ, such that for all $A \geq A_\epsilon$ there exist exactly two equilibria, $q_A(c)$, $q'_A(c)$, such that $0 < q_A(c) < \epsilon$ and $1 - \epsilon < q'_A(c) < 1$.

Figure 3
COIN TOSS RULE
Mixed Strategy Equilibria for M = N = 2



The last part of this conjecture would be particularly interesting since it implies that with large numbers of voters there are exactly two (q,q) equilibria, one with essentially everyone voting and one with essentially no voting. Numerical calculation supports this conjecture. Figure 4 shows the computed graph of the (q,q) equilibrium correspondence for values of $M = 2, 5, 15, 50, 125$.

Case 9. Equilibria in which $r = 1 - q$, $R1$.

In this case, equations (11) and (12) reduce to

$$2c = q^N(1-q)^{M-1} \left[\sum_{k=0}^{min[M-1,N]} \binom{M-1}{k}\binom{N}{k} + \sum_{k=0}^{min[M-1,N-1]} \binom{M-1}{k}\binom{N}{k+1} \frac{1-q}{q} \right]$$

and

$$2c = q^N(1-q)^{M-1} \left[\sum_{k=0}^{min[M,N-1]} \binom{M}{k}\binom{N-1}{k} \frac{1-q}{q} + \sum_{k=0}^{min[M-1,N-1]} \binom{M}{k+1}\binom{N-1}{k} \right]$$

By using (14) and other identities, one can show that the two equations are equivalent and reduce to:

$$2c = \binom{M+N-1}{M-1} q^N(1-q)^{M-1} + \binom{M+N-1}{M} q^{N-1}(1-q)^M. \tag{15}$$

As before, one can analyze properties of these $(q, 1-q)$ equilibria by examining

$$c(q) = 1/2 \left[\binom{M+N-1}{M-1} q^N(1-q)^{M-1} + \binom{M+N-1}{M} q^{N-1}(1-q)^M \right]$$

Calculating $c'(q)$ gives:

$$c'(q) = \frac{1}{2M} \binom{M+N-1}{M-1} q^{N-2}(1-q)^{M-2}[N(N-1)(1-q)^2 - M(M-1)q^2]$$

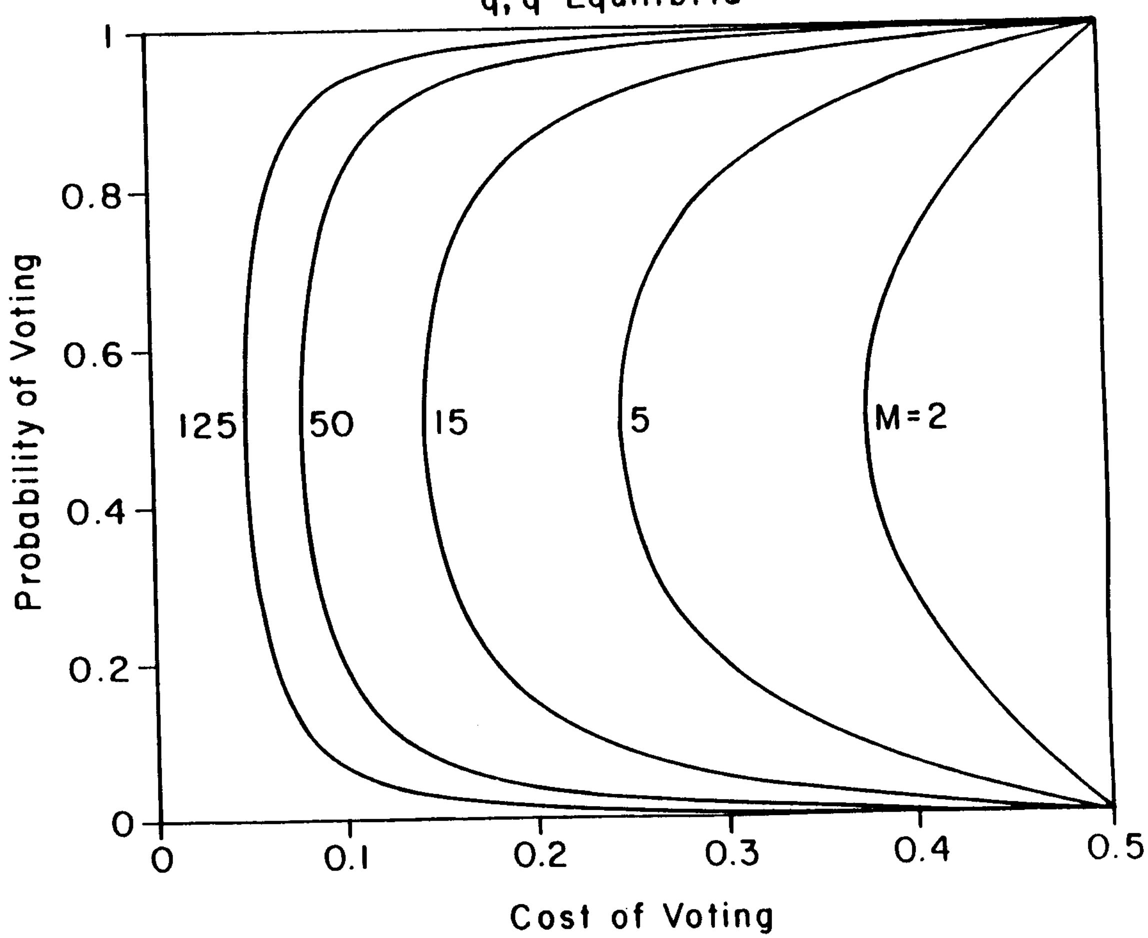
so $$c'(q) \gtreqless 0 \;\; iff \;\; q \lesseqgtr q^* = \frac{\sqrt{N(N-1)}}{\sqrt{M(M-1)} + \sqrt{N(N-1)}} .$$

Figure 4

COIN TOSS RULE

M = N

q, q Equilibria



In addition, $c(0) = 0$, $c(1) = 0$, so that $c(q)$ in general appears as the left curve in Figure 3. The actual curve in that figure represents the equilibrium strategies of one team in the $(q, 1\text{-}q)$ equilibria when $M = N = 2$.

Solving for $c_{max} = c(q^*)$, yields:

$$c_{max}^{M,N} = \frac{(M+N-1)!\,(\sqrt{M(N-1)} + \sqrt{N(M-1)})}{M!N!\,\sqrt{(M-1)(N-1)}} \cdot \frac{[N(N-1)]^{N/2}[M(M-1)]^{M/2}}{2\left[\sqrt{N(N-1)} + \sqrt{M(M-1)}\right]^{M+N-1}}$$

Observe that, as in Figure 3, for all $c < c_{max}$, there are *two* equilibria of this variety, one with $q > q^*$ and one with $q < q^*$. When $M = N$, we have

$$q^* = 1/2$$

$$c_{max}^{M,M} = \binom{2M-1}{M-1}\left(\frac{1}{2}\right)^{2M-1} = c_{min}^{M,M}$$

Assuming our earlier conjecture about the $M = N$ case is true, we have equilibria for all values of $c < 1/2$, with $q = r$ for $c > c_{max}^{M,M}$, $q = 1 - r$ for $c < c_{max}^{M,M}$ and $q = r = 1/2$ for $c = c_{max}^{M,M}$. This is illustrated in Figure 3.

B. Status quo rule

By methods similar to those used in the analysis of the coin-toss rule, one can obtain two necessary and sufficient conditions for (q,r) to be a totally quasi-symmetric equilibrium in the status quo rule game. These are given below:

$$c = \sum_{k=0}^{min[M-1,N]} \binom{M-1}{k}\binom{N}{k} q^k(1-q)^{M-1-k} r^k(1-r)^{N-k} \tag{16}$$

$$c = \sum_{k=0}^{min[M-1,N-1]} \binom{M}{k+1}\binom{N-1}{k} q^{k+1}(1-q)^{M-1-k} r^k(1-r)^{N-1-k} \tag{17}$$

Case 10. Equilibria in which $r = 1 - q$, $R2$.

We concentrate attention on "$r = 1 - q$" equilibria. In this case, (16) and (17) reduce to

$$c = q^N(1-q)^{M-1} \sum_{k=0}^{min[M-1,N]} \binom{M-1}{k}\binom{N}{k} \tag{16'}$$

$$c = q^N(1-q)^{M-1} \sum_{k=0}^{min[M-1,N-1]} \binom{M}{k+1}\binom{N-1}{k} \tag{17'}$$

Again, using combinatorial identities,[8] (16') and (17') are equivalent and reduce to:

$$c = \binom{M+N-1}{N} q^N(1-q)^{M-1} \tag{18}$$

Once again, we can analyze the relationship between c and the equilibrium value(s) of q and r. It can be shown that $c(q)$ is single-peaked as was the case for the coin-toss rule and that

$$q^* = \frac{N}{M+N-1}$$

$$c_{max}^{M,N} = c(q^*) = \binom{M+N-1}{N} \frac{(M-1)^{M-1}N^N}{(M+N-1)^{M+N-1}}$$

At $c_{max}^{M,N}$, expected plurality is $\frac{N}{M+N-1}$ in favor of T_1 and expected turnout is $(2M-1)N/(M+N-1)$. Both sides vote with probability 1/2 if

$$c = (1/2)^{M+N-1} \binom{M+N-1}{N}.$$

8. Note that $\sum_{k=0}^{M-1} \binom{M}{k+1}\binom{N-1}{k} = \frac{M}{N} \sum_{k=0}^{M-1} \binom{M-1}{k}\binom{N}{k+1} = \frac{M}{N}\binom{M+N-1}{M} = \binom{M+N-1}{M-1} = \binom{M+N-1}{N}.$

The $(q,\ 1\text{-}q)$ equilibria are illustrated in Figures 1 and 3. In Figure 1, the $(q,\ 1\text{-}q)$ equilibria are to the left of the intersection. In Figure 4, the curve to the left of the tangency represents the $(q,\ 1\text{-}q)$ equilibrium.

7. A numerical example

At this point, we have found so many equilibria that it may be helpful to recapitulate via a numerical example where $c = 20/81$, $M = N = 3$, and the coin-toss rule is in effect. In this case, we have found 11 equilibria.

The obvious equilibrium is the one with every citizen voting. The parallel mixed strategy equilibrium of (q,q) fails to exist for the assumed cost. By Proposition 6, $c(1/2) = 5/6 > 20/81$; consistent with our earlier conjecture, numerical analysis reveals that (13) has no real roots for $c = 20/81$.

There are, however, two totally mixed quasi-symmetric equilibria found by algebraic manipulation of (15). They are:

$$q = 2/3,\ r = 1/3$$

$$q = 1/3,\ r = 2/3$$

A fourth equilibrium results from solving (8") with $k = 3$, $c = 20/81$, and $M = 3$. The solution is $q = 2\sqrt{10/9}$. There is, of course, another mixed-pure equilibrium with $k = 3$ and $r = 2\sqrt{10/9}$.

With $k = 2$, $c = 20/81$, $M = N = 3$, using (8') - (10') one can show there is an equilibrium with all members of one team voting with probability $1 - \sqrt{41/9}$ and exactly two members of the other team voting. In fact there are six such equilibria, since there are two teams and three ways to have exactly two members of one team voting.

It is easily verified that there are no (k,q) equilibria for either $k = 1$ or $k = 0$.

The fact that we constructed only very special varieties of equilibria suggests that the multiple equilibria problem may be even more pervasive than we show in this paper. We conjecture that generally there are even more equilibria than we have found.

8. Analysis of equilibrium behavior

In this section we examine the effects of the cost of voting, the size of the electorate and the election rule on equilibrium expected turnout, expected plurality, and probability of winning. In addition we make some observations about the efficiency or welfare properties of equilibrium outcomes.

Of the many equilibria, not all share similar properties. When the electorate is small the multiple equilibria problem is particularly severe, and for this reason we are hesitant to make strong claims for small electorates. However, with larger electorates all equilibria seem to be of only two types which have interesting and intuitively plausible properties. For this reason, in this section, we stress asymptotic results that apply to large electorates.

Effect of the size of the electorate on equilibrium turnout

The paradox of not voting basically reduces to asking why does anyone vote when either M or N is large or when $M/(M+N)$ deviates substantially from $1/2$. Since the primary motivation for this paper is to reexamine the "paradox of not voting" in a game-theoretic context, a natural place to begin is to examine limiting properties of the equilibria.

There are a number of different ways one can develop an analysis of limiting behavior. First, keep $M = \alpha N$ constant and allow M, N to grow. Second, fix $M - N = \Delta$ constant and allow M, N to grow. Finally, fix M(or N) and allow the magnitude of Δ to grow. The basic difference of these three approaches is that, when $M+N$ goes to infinity, $M/(M+N)$ approaches $\alpha/(1+\alpha)$, $1/2$, and 0 (or 1), respectively, depending on the relative rates at which M or N grow.

When we first considered pure strategy equilibria, the only equilibrium with substantial voting occurred with $M = N$. There was a full turnout equilibrium that held for any value of M, no matter how large. We first believed this to be a knife-edge case and, for large electorates, turnout as a percentage of the electorate might approach zero as α deviated slightly from 1.0. After all, Chamberlain and Rothschild (1981) had shown conclusively that, for a fixed level of turnout, the probability of being decisive approaches 0 very rapidly when the binomial probability of voting for T_1 deviates even slightly from .5.

Upon further analysis, the full turnout, $M = N$ equilibrium now appears as a limiting rather than a knife-edge case. Note, first, that the conjecture about limiting behavior of the (q,q) equilibrium where $a = 1$ states that there are essentially two equilibria for large electorates, one where nearly everyone votes and one where nearly no one votes.

Thus, when $M = N$, as the electorate gets large, for almost all values of c, there will be exactly two (q,q) equilibria, one close to 100% turnout and one close to 0% turnout. We further conjecture that if $a = 1$ there are similarly two (q,r) equilibria, one with high turnout and one with low turnout.

A second reason for regarding large turnout as a limiting case emerges from an analysis of mixed-pure equilibria. Once again, as M and N get large, there are "essentially" two equilibria. In one type of equilibrium, turnout approaches twice the size of the minority and in the other type turnout approaches 0%. This is stated more formally in the following proposition:

Proposition 7: Let $M > 2$. Then with $(R2)$ for all integer k

$$\lim_{M\to\infty} \lim_{N\to\infty} c_{max}^{M,k} = \lim_{M\to\infty} \lim_{N\to\infty} c_{max}^{M,M\text{-}k} = \frac{(k/e)^k}{k!}$$

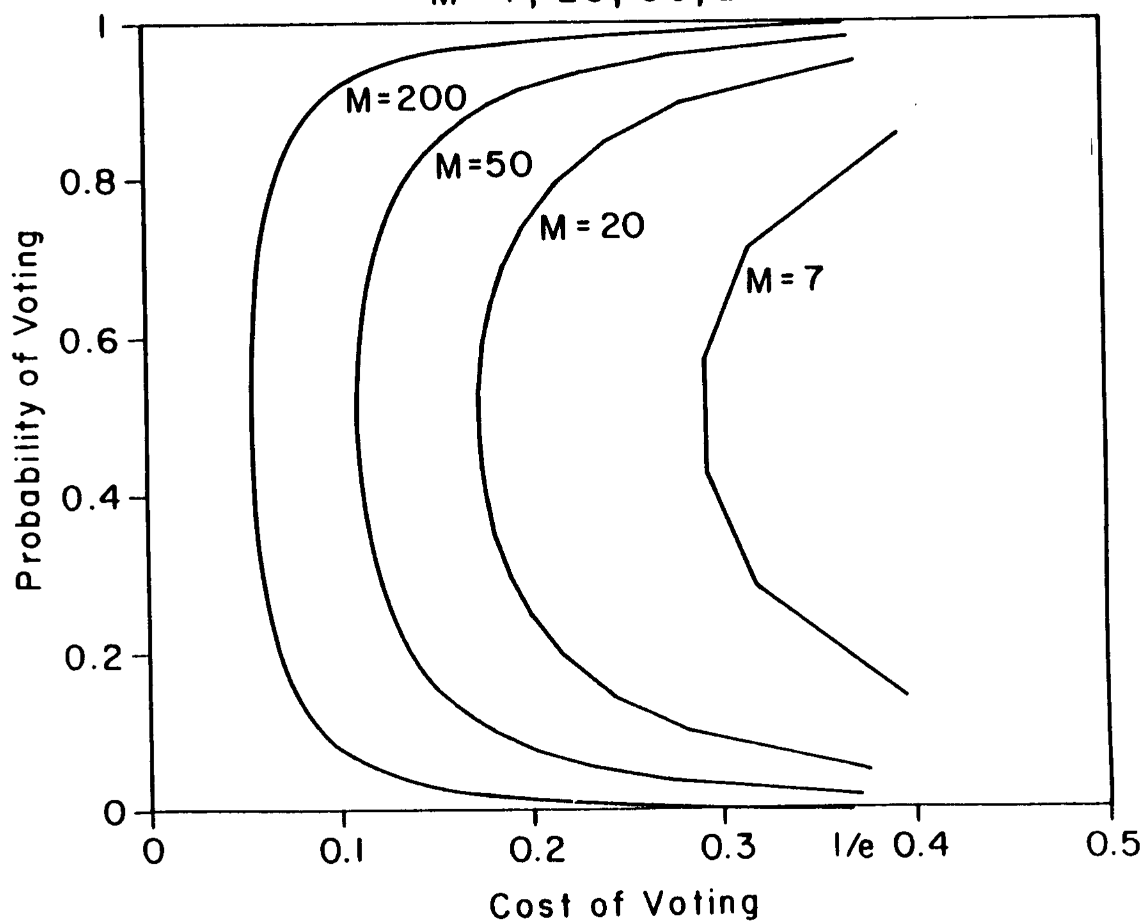
The intuition behind this result is the following. The probability of obtaining k successes in M-1 binomial trials with success probability k/M is $c_{max}^{M,k}$. As $M \to \infty$, $k/M \to 0$, and the result follows directly from the Poisson approximation (Feller, 1957, p. 143). The formal proof is in the Appendix. This convergence process is illustrated in Figure 5. That figure graphs the locus of $\left\{c_{max}^{M,k}\right\}$ points, k = 1, ..., M-1 for M = 7, 20, 50, 200. To make the graph easier to read, a smooth curve has been drawn through each set of (M-1) points.

Figure 5

STATUS QUO RULE

Locus of $\left\{c^{M,k}_{max}\right\}$ $k = 1, \ldots, M-1$

$M = 7, 20, 50, 200$



Corollary: Fix $\beta \in (0,1)$ and $M > 2$. Let $k(M)$ be the least integer greater than or equal to $\beta(M-1)$. Then $\lim_{M\to\infty} \lim_{N\to\infty} c_{max}^{M,k(M)} = 0$.

Proof: Trivial

Corresponding results for $c^{N,k}$ equilibria are easily obtained by reversing the order in which limits are taken. We conjecture a similar result holds for $R1$.

Proposition 7 and the corollary sharply restrict the types of mixed-pure equilibria that exist for large M and N. Indeed, almost all mixed-pure equilibria disappear, in the sense that they are not supported by positive costs of voting in the limit. Those that remain have k "close" to the size of the minority or k "close" to 0. In the latter type of equilibrium, expected turnout converges to 0% of the electorate. In the large k type of equilibria, expected turnout as a proportion of the electorate converges to $\frac{2min(M,N)}{M+N}$.

Summarizing the limiting results so far, we find that all equilibria have either 0% turnout or a percentage equal to twice the minority percentage in the electorate (note that for $M=N$, this is 100%). These results support the notion that turnout should decline as the majority gets large relative to the minority, but they do not support the proposition that voting turnout will become infinitesimal in large jurisdictions.

In contrast to the (q,q) and (q,k) equilibria, the $(q, 1-q)$ equilibria completely disappear in large electorates. Recall that

$$c_{max}^{M,N} = \binom{M+N-1}{N} \frac{(M-1)^{M-1}N^N}{(M+N-1)^{M+N-1}}$$

for the status quo rule. This is just the central term of the binomial distribution with $M+N-1$ trials and success probability $N/(M+N-1)$. Its asymptotic approximate is

$$c_{max} \sim \sqrt{\frac{(M+N-1)}{2\pi(MN-N)}}$$

Now if $M = \alpha N$

$$c_{max} \to \sqrt{\frac{(1+a)}{2\pi a N}} \quad \text{as } N \to \infty$$

while if $M = N + \Delta$

$$c_{max} \to \sqrt{\frac{1}{\pi N}} \quad \text{as } N \to \infty \tag{19}$$

Thus, the $(q,\ 1\text{-}q)$ disappear in large jurisdictions.

When the coin-toss rule is in effect, results are similar to those for the status quo rule. Note that when $M=N$,

$$c_{max/\text{coin-toss}} = c_{max/\text{status quo}} = \binom{2M-1}{M-1}\left(\frac{1}{2}\right)^{2M-1}$$

and for all M, N

$$c\left(\frac{1}{2}\right)_{/\text{coin-toss}} = \frac{1}{2}\left[1 + \frac{N}{M}\right] c\left(\frac{1}{2}\right)_{/\text{status quo}}$$

when a solution exists for $q = 1/2$.

At this point, it is worthwhile to pause to note that much of the past argument that voting is irrational in large electorates has been founded on the belief that the relative size of c had to be on the order of the very small costs implied by (15) and (18), the equilibrium conditions for the $(q,\ 1\text{-}q)$ equilibria. Uncovering the mixed-pure equilibria indicates that, quite the contrary, large turnout can exist even with substantial costs.

Expected plurality and the probability of winning

There are two natural intuitive conjectures about expected plurality and the probability of winning. A naive view is that the majority will win. A somewhat more sophisticated view is that, in trading off the cost of voting against the probability of being decisive, the minority will have incentives to vote more heavily than the majority, forcing expected plurality to zero. Neither of these conjectures holds for all the types of equilibria we have found, but the latter phenomenon appears more pervasive than the former.

A. (q,q) equilibria

In the pure strategy, $M=N$ equilibrium of Proposition 1, everyone votes and the outcome is a tie. In the corresponding (q,q) equilibria, expected plurality remains zero but, due to the mixing, the election ends in a tie only "on average".

B. Mixed-pure equilibria

The mixed strategies, however, also induce what might be termed a bias. Compare the mixed-strategy equilibria for $M=0$ or $N=0$ with the pure strategy equilibria identified in Proposition 1. With large numbers of voters, expected plurality changes from exactly one voter in the pure strategy case to $1-\ln(c)$ or $1-\ln(2c)$. More importantly, the unopposed majority wins only with probability $1-c$ (for the status quo) and $1-2c$ (for the coin-toss).

The effects of mixed strategies can also be seen in the more general mixed-pure equilibria. At the various $c^{M,k}_{max}$ or at $c^{M,0}_{min}$, expected turnout by the mixers is just enough to counterbalance the k pure strategy voters and expected plurality is zero or one vote. But as one moves away from c_{max}, expected plurality increases to k voters and always favors the team that is "committed" to pure strategy voting. This bias in favor of the pure strategy team holds whether that team is the majority or the minority.

Although computation of the probability of winning is complex in the case of mixed-pure equilibria, insight into the bias favoring the pure strategy team is available from considering the asymptotic results for $(q;k)$ at $c_{max} = 1/e$ when $k=1$ or $k=M-1$. Section 5 has already established that the probabilities of winning for the committed team in these two situations are $2e^{-1}$ and $1-e^{-1}$, respectively. In the $k=1$ case, the mixers pay the cost of voting with a probability that approaches zero. Their expected payoff, therefore is $1-2e^{-1} = 0.264$. The nonvoters on the other team have an expected payoff of 0.736 while the single voter gets 0.368. In the $k=M-1$ case, the mixers pay the cost with a probability that approaches one, leading their expected payoff to approach zero. The $M-1$ voters on the other team have an expected payoff of 0.264 and the nonvoters obtain 0.632. Thus, in both cases, even the committed voters obtain a higher expected payoff than the mixers.

We also note, for later reference, that every $k=M-1$ equilibrium at c_{max} is Pareto dominated by several $k=1$ equilibria.

C. **(q, 1-q) equilibria**

Finally, the behavior of plurality is only partly similar in the totally mixed (q, 1-q) equilibria to what it was in the mixed-pure situation. At c_{max}, each team votes in proportion to the other team's strength, and expected plurality is zero. In the neighborhood of c_{max}, the minority always votes more heavily than the majority. But as there are always two equilibria for $c < c_{max}$, there is always one equilibrium with the expected plurality favoring the majority and one with expected plurality favoring the minority. And for c sufficiently low, there are always equilibria with the majority voting more heavily than the minority.

Effect of tne cost of voting

As with the size of pluralities, there is some "common sense" intuition about the cost of voting and turnout that is inconsistent with our results. Common sense (that is, equation (1)), would hold that turnout should be decreasing in c. For the totally mixed equilibria, however, there are always two types of equilibria (except at c_{max} or c_{min}). In one type, turnout is decreasing in cost, but in the other it is increasing.

The mixed-pure equilibria have an unambiguous relationship to voting cost; for $k \geqslant 2$, all equilibria have the voting probability increasing in cost.

That voting can increase with cost seems counterintuitive from the viewpoint of the older decision-theory approach. When the probability that a voter is pivotal is fixed exogenously and held constant, it is apparent that turnout should decline as the cost of voting is increased. But in our model, the equilibrium condition for mixed strategies is that a voter must be indifferent between voting and abstaining. For this to hold, the cost of voting must be equated to the probability that the voter is pivotal (equations 2, 4, 8, 11, 12, 16, and 17). As the cost of voting increases, this probability must also increase, endogenously. For certain equilibria and cost values, all voters on a team must increase their probabilities in order for the probability of being pivotal to increase and offset the increase in cost.

Another counterintuitive result concerning cost is that, asymptotically, turnout is very insensitive to changes in cost. Thus, for (q,q) equilibria under the coin-toss rule, turnout approaches 0% or 100% of the electorate for all costs below 1/2. For $(q;k)$ or $(k;r)$ equilibria under the status quo rule, turnout approaches 0% of the electorate for all costs above e^{-1} and either 0% or twice the minority's percentage for costs at or below e^{-1}. We have already given an explanation for these results: turnout probabilities approach zero or one in large electorates.

The voting rules

As expected, the two voting rules induce strong differences in behavior for M or N small but become essentially identical as M and N grow large together. (In contrast, the Ferejohn and Fiorina (1974) model would hold that for all M, N, voting would occur under the coin-toss rule for $c < 1/4$ but under the status quo rule for $c < 1/2$). The differential effect of being the status quo team can be seen by comparing Figures 1 and 6. In Figure 1, we show the $(q, 1\text{-}q)$ equilibrium when $N=1$; Figure 6 presents the corresponding picture for $M=1$. The right-hand part of Figure 1 occurs when the one status quo voter has an equilibrium strategy of not voting. It can be seen that there are strong differences in equilibrium probabilities as a function of cost.

In contrast to the above two polar cases, Figure 7 shows the status quo equilibria for $M=N$. It can be seen that, for small M, the centering point of c_{max} is distorted from .5 by the status quo rule but that as M, N grow, the $(q, 1\text{-}q)$ approach the centered curves that one would obtain with the coin-toss rule. The figure also shows how the $(q, 1\text{-}q)$ equilibria vanish as M and N become large.

Efficiency of voting equilibria

The welfare measure we use to evaluate efficiency is the sum of expected payoffs to all players. In the coin-toss rule, the most efficient outcome generally occurs if one voter on the majority team votes and all other voters abstain. Clearly this is never an equilibrium unless $M=0$ or $N=0$. In the status quo rule, the efficient outcome is either no one voting (if $N \geqslant M$) or $m=1$, $n=0$ (if $M > N$). This is never an equilibrium unless $M=0$ or $N=0$.

Figure 6
STATUS QUO RULE
$M = 1,\ N \geq 1$
Team 1 (Dashed) and Team 2 (Solid) Equilibrium Probabilities

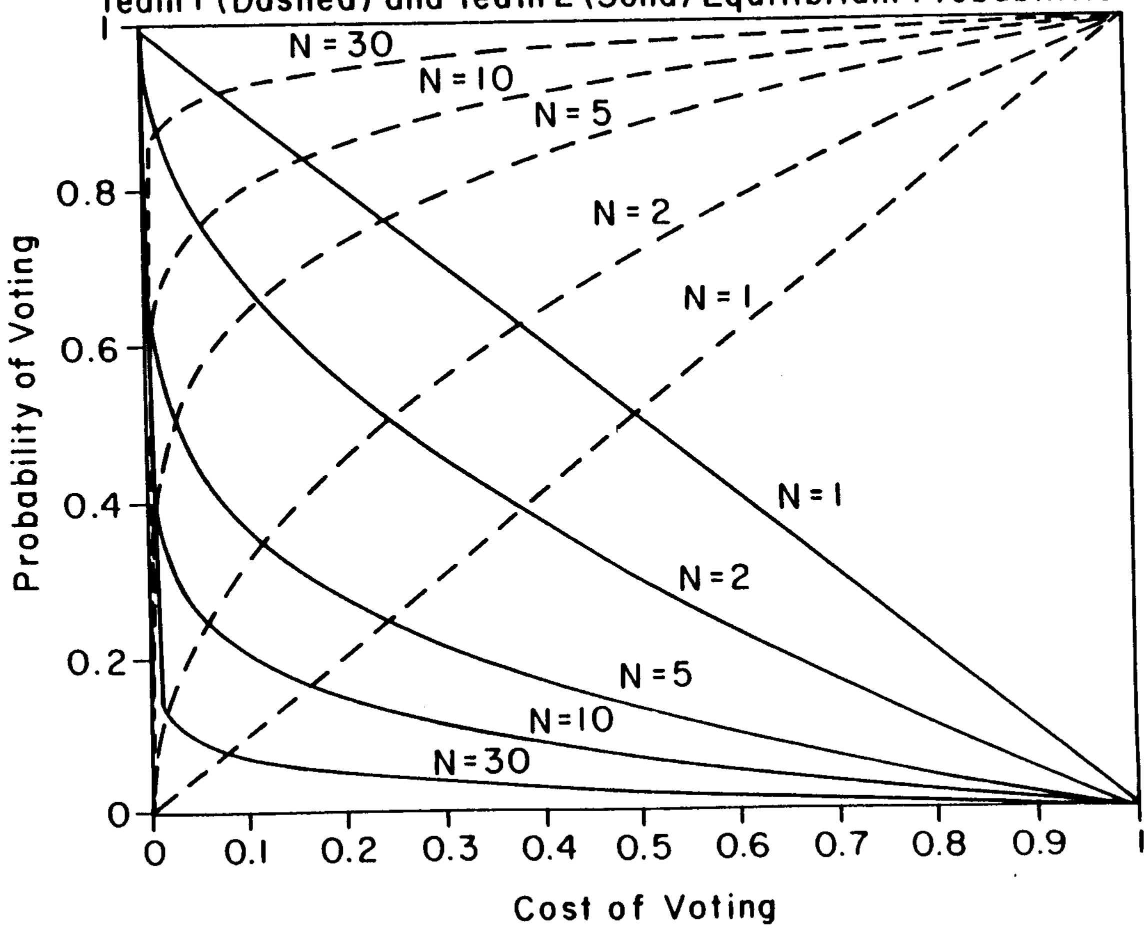
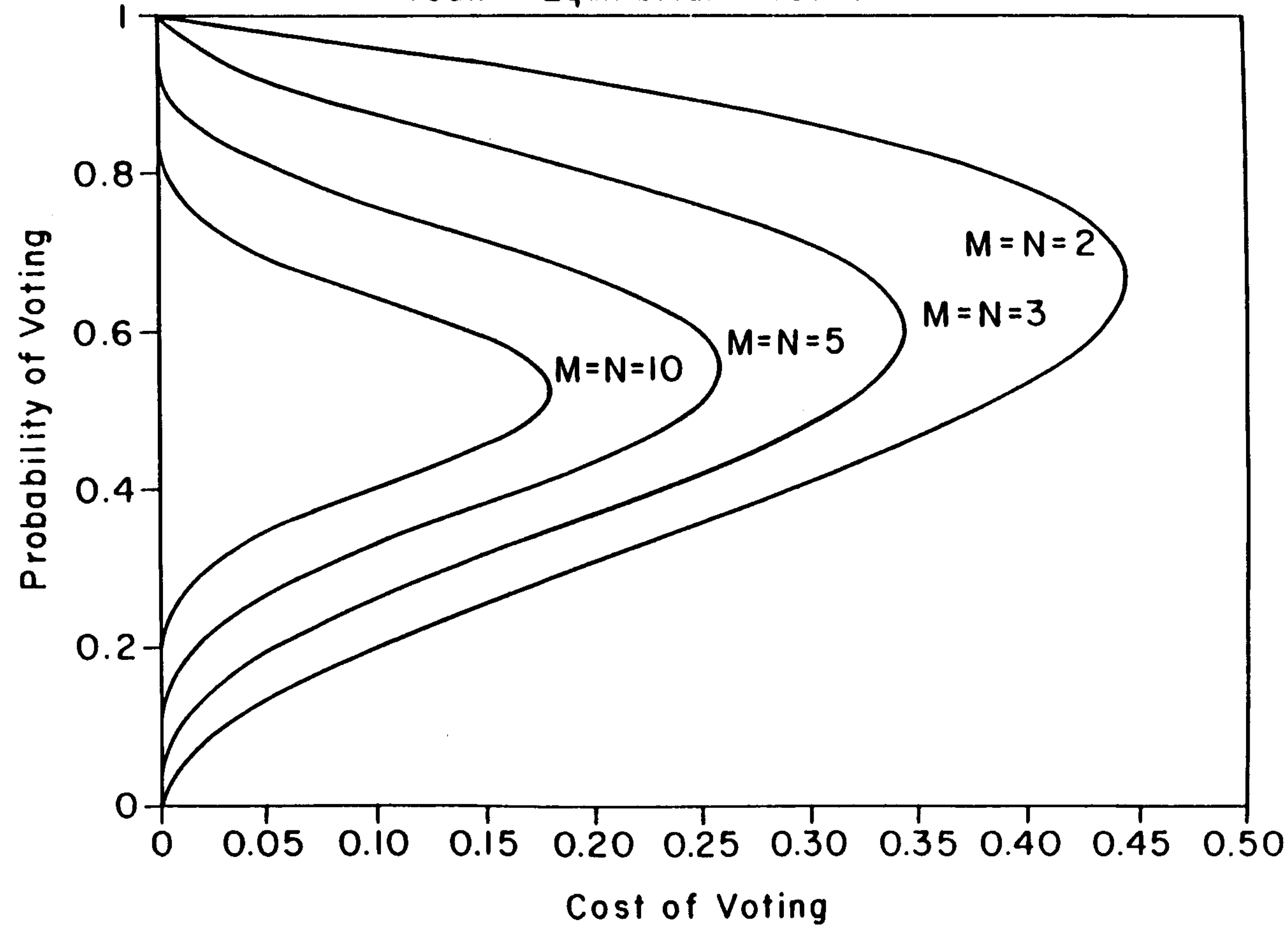


Figure 7
STATUS QUO RULE
M = N
Team 1 Equilibrium Probabilities



Thus equilibria are generally inefficient unless there is "only one team." This suggests that it is the competition between teams, rather than the free-rider problem within a team which is the source of inefficiency of majority voting equilibria. (Recall the Prisoners' Dilemma example, $M=N=1$.) This is not surprising, since the act of voting results in deadweight loss. In fact, as we saw with the $(q;1)$ and $(q;\ M-1)$ equilibria, high turnout equilibria are not only inefficient; they can even be Pareto dominated by low turnout equilibria. Similarly, in the pure strategy $M=N$ equilibria for the coin-toss, everyone votes even though the most efficient outcome has no one at all voting. However, there are generally two (q,q) equilibria for most values of cost, and one of these has almost no one voting. In other words, in our model free-riding (abstention) is generally "good."

There is, however, a type of inefficiency that is caused by the free-rider problem: the minority may win. In our model, majorities will frequently free-ride more than minorities. This is consistent with the stylized fact that majorities tend to be "silent" and minorities active. On the other hand, it means that minorities may frequently win, even if they are very small relative to the population. Consider the mixed-pure $(q;\ 1)$ equilibria with $M >> N$ and c close to $1/e$. In these elections, this equilibrium is extremely inefficient, since the intense minority will win with probability $2(1-e^{-1}) = .732$.

9. Conclusion

There are several major insights which this game theoretic analysis has produced. First, we have shown that equilibria exist with substantial turnout even when both the majority is much larger than the minority and the costs of voting are exceptionally high. For example, in large electorates using the status quo rule, we show mixed-pure equilibria with turnout roughly equal to twice the size of the minority when the cost is nearly equal to 37% (e^{-1} x 100) of the reward (B). Second, with large electorates the many equilibria appear to reduce to just two types, the type just mentioned and a type with almost no turnout. Third, we have shown that turnout may rise as the costs of voting rise. This results when all members of a team "adjust" their turnout probabilities so that the probability of being pivotal increases to match the increased cost of voting. We have also shown that turnout is nearly invariant with costs in large electorates where turnout probabilities approach one or zero. Fourth, the actual split of the vote is likely to be a biased measure of the actual distribution of preferences in the electorate. Because majorities have greater incentives to free-ride, they will turn out less heavily than minorities. Elections can be relatively close, even when one alternative is supported by a substantial majority of the electorate. The probability that the majority will win does not seem to be closely related to the size of the electorate or its size relative to the minority. However, turnout is quite strongly correlated with the relative sizes of the minority and the majority.

Appendix

Proof of Proposition 6. We first prove two additional combinatorial identities.

Lemma

$$\sum_{k=0}^{A} k \binom{A}{k}\binom{B}{k+r} = A\binom{A+B-1}{A+r} \tag{14'}$$

$$\sum_{k=0}^{A} k^2 \binom{A}{k}\binom{B}{k+r} = A\left[(A-1)\binom{A+B-2}{A+r} + \binom{A+B-1}{A+r}\right] \tag{14''}$$

Proof:

$$\sum_{k=0}^{A} k \binom{A}{k}\binom{B}{k+r} = \sum_{k=1}^{A}\left[\frac{A!}{(k-1)!\,(A-k)!}\binom{B}{k+r}\right]$$

$$= A\sum_{j=0}^{A-1}\left[\frac{(A-1)!}{j!(A-j-1)!}\binom{B}{j+r+1}\right]$$

$$= A\binom{A+B-1}{A+r} \quad \text{by (14)}$$

Similarly,

$$\sum_{k=0}^{A} k^2 \binom{A}{k}\binom{B}{k+r} = A\sum_{j=0}^{A}\left[(j+1)\frac{(A-1)!}{j!(A-j-1)!}\binom{B}{j+r+1}\right]$$

The result (14") then follows by applying (14) and (14').

We now turn to the proposition.

(a) With $q = 1/2$, we use (14) to simplify (13) to

$$2c = (1/2)^{2M-1} \sum_{k=0}^{M-1} \binom{M-1}{k}\binom{M}{k} + \sum_{k=0}^{M-1} \binom{M-1}{k}\binom{M}{k+1}$$

$$= (1/2)^{2M-1} \binom{2M-1}{M-1} + \binom{2M-1}{M}$$

hence

$$c = (1/2)^{2M-1} \binom{2M-1}{M} \text{ , since } \binom{2M-1}{M-1} = \binom{2M-1}{M} .$$

(b) Part (b) follows from recognizing that $c(1/2)$ is the central term of a binomial distribution. The approximation is given by Feller (1957, p. 140).

(c) Part (c) follows directly from part (b).

(d) $\frac{dc}{dq}$, evaluated at $q = 1/2$ is given by:

(i) $$\sum_{k=0}^{M-1} (12-8M+16k) \binom{M-1}{k}\binom{M}{k+1} + \sum_{k=0}^{M-1} (4-8M+16k) \binom{M-1}{k}\binom{M}{k}$$

(ii) $$= \Big[12-8M+4-8M+[16(M-1)(M+M-1)/(2M-1)]\Big]\ 2^{-2M-1} \binom{2M-1}{M}$$

(iii) $$= 0$$

$\frac{d^2c}{dq^2}$, evaluated at $q=1/2$ is given by:

(iv) $$\sum_{k=0}^{M-1} (32M^2 - 112M - 128Mk + 192k + 128k^2 + 80) \binom{M-1}{k}\binom{M}{k+1}$$

$$+ \sum_{k=0}^{M-1} (32M^2 - 48M - 128Mk + 64k + 128k^2 + 16) \binom{M-1}{k} \binom{M}{k}$$

$$\text{(v)} \qquad = \Bigg[64M^2 - 160M + 96 + (M-1) \frac{(192-128M)(M-1) + (64-128M)M}{2M-1}$$

$$+ \; 128\,(M-1) \left[\frac{(M-2)[(M-1)(M-2) + M(M-1)]}{(2M-1)(2M-2)} + \frac{M + M-1}{2M-1} \right] \Bigg] \binom{2M-1}{M} \Big/ 2^{2M+1}$$

$$\text{(vi)} \qquad = \frac{(16M - 16)}{(2M-1)} \binom{2M-1}{M} \Big/ 2^{2M+1}$$

$$\text{(vii)} \qquad = \frac{\binom{2M-2}{M}}{2^{2(M-2)}} > 0 \quad \text{for } M \geqslant 2$$

Steps (i), (iii), (iv), (vi), were produced by MACSYMA.

Steps (ii), and (v) resulted from using (14') and (14") to eliminate the summations.

Step (vii) is direct.

Proof of Proposition 7.

First recall that for $M < N$

$$c^{M,k}_{max} \binom{M-1}{k} \frac{k^k}{} \frac{(M-k)^{M-k-1}}{M^{M-1}} \qquad k = 1, \ldots, M-1$$

$$= 0 \qquad\qquad k = M, M+1, \ldots$$

Fix M. Then $\lim\limits_{N \to \infty} c^{M,k}_{max} = c^{M,k}_{max}$ since $c^{M,k}_{max}$ is independent of N for $N > M$.

Simple algebraic manipulation shows that

$$\binom{M-1}{k} \frac{k^k(M-k)^{M-k-1}}{M^{M-1}} = \frac{k^k}{k!} \left(1-\frac{k}{M}\right)^{M-k-1} \prod_{j=1}^{k} \left(1-\frac{j}{M}\right)$$

Since $\lim\limits_{M\to\infty} \left(1 - \frac{k}{M}\right)^{M-k-1} = e^{-k}$

and $\lim\limits_{M\to\infty} \prod_{j=1}^{k} \left(1 - \frac{j}{M}\right) = 1$

we have

$$\lim_{M\to\infty} \lim_{M\to\infty} c_{max}^{M,k} = \frac{(k/e)^k}{k!}$$

Finally, observe that $c_{max}^{M,M-k} = \binom{M-1}{M-k} \frac{(M-k)^{M-k} k^{k-1}}{M^{M-1}}$

$$= \binom{M-1}{M-k-1} \frac{(M-k)^{M-k-1}\, k^k}{M^{M-1}}$$

$$= \binom{M-1}{k} \frac{k^k(M-k)^{M-k-1}}{M^{M-1}}$$

$$= c_{max}^{M,k}$$

Q.E.D.

REFERENCES

Chamberlain, G. and Rothschild, M.

(1981) A Note on the Probability of Casting a Decisive Vote. *Journal of Economic Theory,* **25**: 152-162.

Downs, A.

(1957) *An Economic Theory of Democracy,* New York: Harper and Row.

Feller, W.

(1957) *An Introduction to Probability Theory and Its Applications.* Vol. 1, 2nd Edition, New York: Wiley.

Ferejohn, J. and Fiorina, M.

(1974) The Paradox of Not Voting: A Decision Theoretic Analysis, *American Political Science Review,* **68**: 525-536.

____

(1975) Closeness Counts only in Horseshoes and Dancing, *American Political Science Review,* **69**: 920-925.

Ledyard, J.

(1981) The Paradox of Voting and Candidate Competition: A General Equilibrium Analysis. G. Horwich and J. Quirk, eds., *Essays in Contemporary Fields of Economics,* Purdue University Press.

Meehl, P.

(1977) The Selfish Voter Paradox and the Thrown-Away Vote Argument, *American Political Science Review,* **71**: 11-30.

Riker, W. and Ordeshook, P.

(1968) A Theory of the Calculus of Voting, *American Political Science Review,* **62**: 25-42.

Romer, T. and Rosenthal, H.

(forthcoming) Voting and Spending: Some Empirical Relationships in the Political Economy of Local Public Finance. P. Mieszkowski and G. Zodrow, eds., *Provision of Local Public Services: The Tiebout Hypothesis After 25 Years,* Academic Press.

Rosenthal, H. and Sen, S.

(1973) Electoral Participation in the French Fifth Republic, *American Political Science Review,* 67: 29-54.

Schelling, T.

(1960) *The Strategy of Conflict,* Harvard University Press.

Tucker, A.

(1980) *Applied Combinatorics,* New York: Wiley.

Tullock, G.

(1967) *Towards a Mathematics of Politics,* University of Michigan Press.

Wolfinger, R. and Rosenstone, S.

(1980) *Who Votes?,* Yale University Press.