

## Efficient Equilibrium Selection in Evolutionary Games with Random Matching\*

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This paper investigates the sensitivity of recent evolutionary models of learning to the specification of the matching mechanism. We study a literally random matching mechanism, combined with a process of strategy adjustment based on the realized average performance of each strategy. In the key class of symmetric  $2 \times 2$  coordination games, the Pareto-efficient equilibrium, *per se*, is selected, rather than the risk-dominant equilibrium, as the probability of experimentation (or “mutation”) goes to zero. Furthermore, convergence to the equilibrium is relatively fast. We extend these results, for example, to games of common interest. *Journal of Economic Literature* Classification Numbers: C72, C73. © 1996 Academic Press, Inc.

### 1. INTRODUCTION

Kandori, Mailath, and Rob [9, hereafter KMR], building on work by Foster and Young [6], consider experimentation (or “mutation”) in a learning model as an intriguing basis for equilibrium selection. In particular, KMR show that, in a central class of symmetric  $2 \times 2$  coordination games, the risk-dominant equilibrium has the larger of the two basins of attraction and, therefore, is selected in the limit of the invariant distribution as the mutation rate tends to zero.

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Such a risk-dominant equilibrium may, of course, be Pareto-dominated to an arbitrary extent by the other equilibrium. (Risk-dominance is as defined by Harsanyi and Selten [8].) A practical limitation of the KMR procedure is that, for a small mutation rate, it readily generates an extremely slow speed of convergence to the invariant distribution. (Ellison [5] considers this issue in detail.)

The present paper explores the sensitivity of the above two key properties of the KMR model to the exact specification of the mechanism which matches individuals in each stage of play. The KMR mechanism might be interpreted as a round robin tournament, so each player confronts each other player exactly once. This tournament mechanism is equivalent to pairing the players randomly if players receive the *expected* value from this. Random pairing is, of course, a familiar assumption in evolutionary game theory, but there the large numbers of individuals is taken to justify the use of the expectation. (See Maynard Smith [11, Chap. 2].) Thus, it is of interest to investigate the implications of a literally random pairing mechanism.<sup>1</sup> This seems as inherently plausible an interaction structure as a round robin tournament.

The adoption of a genuinely random pairing mechanism necessitates an appropriate extension of the KMR adjustment process. An underlying philosophy akin to the most naive of the interpretations offered by KMR is adopted here. It is assumed that individuals are aware of the payoffs just generated by each of the strategies and tend to adopt that strategy which led to the highest average payoff. They tend to emulate such success blindly, on the basis of a naive view that this will persist into the next period. In particular, they do not recognize that success may be based on lucky pairings. Individuals are still assumed to experiment occasionally by choosing an arbitrary new strategy.

In the model just outlined the conclusions of the KMR approach are substantially modified. In particular, in the key class of  $2 \times 2$  coordination games, the Pareto-superior equilibrium is selected, *per se*, even if it is not also risk-dominant. Furthermore, convergence to the invariant distribution is relatively fast.

Section 2 presents an example of a coordination game in which the Pareto-superior equilibrium differs from the risk-dominant equilibrium. The simplest fast version of the present adjustment process is considered, together with a single round of matching per period. As long as the number of players is large enough, the Pareto-superior equilibrium is assigned probability one in the limit of the invariant distribution as the mutation rate tends to zero.

<sup>1</sup> In general, this pairing is repeated a number of times in each period.

	2	$s_1$	$s_2$
1			
	$s_1$	$\sqrt{3}, \sqrt{3}$	0, 1
	$s_2$	1, 0	1, 1

FIG. 1. Example 1.

Section 3 formalizes this argument for the entire class of symmetric  $2 \times 2$  coordination games, under a general random matching mechanism and a general random adjustment process. By allowing a variable number of rounds every period, we obtain a sharp contrast between the present model and that of KMR. The qualitative selection results here are independent of the number of rounds in each period. Therefore, if we let the number of rounds tend to infinity *after* taking the limit on the mutation rate, the Pareto-dominant equilibrium is still selected. However, if this limit is taken *before* that on the mutation rate, the model becomes equivalent to that of KMR, selecting the risk-dominant equilibrium, whether or not this is Pareto-efficient.

Section 4 extends the coverage of the present model to other symmetric  $2 \times 2$  games. The most problematic type of game has a single symmetric equilibrium involving mixed strategies. In order to select this equilibrium, the speed of adjustment is limited, in a fashion analogous to KMR. In the present model, as the population size tends to infinity, the limiting invariant distribution converges in probability to the polymorphic strategy profile matching the mixed strategy equilibrium, provided that the number of individuals permitted to revise each period is bounded.<sup>2</sup> Overall, therefore, there is a reasonable sense in which the present model selects the Pareto-efficient symmetric equilibrium in any symmetric  $2 \times 2$  game.

Section 5 extends the analysis to common interest games, essentially as defined by Aumann and Sorin [1]. These permit asymmetry between players and allow each player to have an arbitrary number of strategies. There is required to be an equilibrium whose payoffs Pareto-dominate those at all other strategy pairs. It is shown that this equilibrium is selected and that convergence is fast under a modified version of the present adjustment process. Finally, in Section 6, we review other related literature.

## 2. AN EXAMPLE

Consider the bilateral symmetric game in Fig. 1. Under the KMR tournament mechanism, the risk-dominant equilibrium  $(s_2, s_2)$  is

<sup>2</sup> A polymorphic profile is such that, although each individual plays a pure strategy, both pure strategies are represented in the population.

selected in the long run, despite the higher payoff available at the equilibrium  $(s_1, s_1)$ . Consider the following alternative model with a genuinely random pairing mechanism. There are a fixed even number,  $N$ , say, of individuals who are randomly matched in pairs, once in every period, in order to play the above game. At the start of period  $t = 1, 2, \dots$ , let  $z_t > 0$  be the number of type 1 individuals who play  $s_1$  and  $N - z_t > 0$  the number of type 2's who play  $s_2$ . Suppose each of the possible ways of pairing up all individuals is chosen with equal probability. This induces *random* average payoffs for the type 1's,  $\pi_1(z_t, \tilde{p}_t)$ , and for the type 2's,  $\pi_2(z_t, \tilde{p}_t)$ , given by

$$\pi_1(z_t, \tilde{p}_t) = \sqrt{3} (z_t - \tilde{p}_t) / z_t \quad \text{and} \quad \pi_2(z_t, \tilde{p}_t) = 1,$$

where  $\tilde{p}_t$  is the induced *random* number of "cross-pairings" between type 1 and type 2. Let  $B(z_t, p_t)$  represent the number of individuals who are of type 1 after they revise their strategy in period  $t$ , where  $p_t$  denotes the realized number of cross-pairings in this period. The analog of the fastest type of adjustment process considered by KMR satisfies

$$B(z_t, p_t) = \begin{cases} N, & \text{if } \pi_1(z_t, p_t) > \pi_2(z_t, p_t), \\ 0, & \text{if } \pi_1(z_t, p_t) < \pi_2(z_t, p_t). \end{cases} \quad (1)$$

Note that the payoff  $\sqrt{3}$  implies that tied payoffs cannot occur in this example. When all agents play the same strategy, set  $B(N, 0) = N$  and  $B(0, 0) = 0$ . (A rationale for a general version of this adjustment process is offered in the next section.) Finally, assume that, after strategy revision has taken place in each period  $t$ , each individual independently changes her strategy with a small probability  $\varepsilon > 0$ .

To simplify the discussion of the present example, consider the state of the system at time  $t = 1, 2, \dots$ , immediately following the process of strategy revision.<sup>3</sup> The only such states which can be observed are  $w_t = 0$  or  $w_t = N$ , so the system is a two-state Markov chain. Whatever the value of  $N \geq 2$ , for all small enough  $\varepsilon$ , we have

$$q(0, N) \equiv \text{Prob}\{w_{t+1} = N \mid w_t = 0\} \geq \beta \varepsilon^2, \quad \text{for some } \beta > 0, \quad (2)$$

since, if two type 2 individuals mutate to type 1's, and these two individuals happen to be paired by the next random pairing, they will obtain the highest possible payoff of  $\sqrt{3}$ . On the other hand, if  $N \geq 8$ , and there are no more than two type 2's, so that  $N - z_t \leq 2$ , then it follows readily that  $\pi_1(z_t, p_t) > \pi_2(z_t, p_t)$ , for all possible realizations  $p_t$  of the random

<sup>3</sup> In the general treatment of the present paper, the state of the system is taken to be that at the start of each period. If an equilibrium is selected in the sense of the present example, it is selected in this second sense.

variable  $\tilde{p}_t$ . Hence strictly more than two mutations are needed to induce a transition from  $w_t = N$  to  $w_{t+1} = 0$ . Thus, if

$$q(N, 0) \equiv \text{Prob}\{w_{t+1} = 0 \mid w_t = N\}, \quad \text{then } q(N, 0)/\varepsilon^2 \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0.$$

For such a two-state Markov chain, the invariant distribution  $\mu: \{0, N\} \rightarrow (0, 1)$  is given by:

$$\mu(N) = q(0, N) / [q(0, N) + q(N, 0)] \quad \text{and} \quad \mu(0) = 1 - \mu(N).$$

It follows that  $\mu(N) \rightarrow 1$  as  $\varepsilon \rightarrow 0$ . In other words, the invariant distribution of the process is arbitrarily concentrated on the efficient equilibrium for any sufficiently small mutation rate.

Given that  $z = N$  is selected in this fashion, a natural way of viewing the speed of adjustment issue can be introduced as follows. Suppose that the system is initially in the "wrong" equilibrium with  $z = 0$ . What is the expected time until it first attains the "right" equilibrium, that is, until  $z = N$ ? Given that the transition probability is  $q(0, N)$  in each period this expected time is  $1/q(0, N)$ . Thus the magnitude of  $q(0, N)$  is relevant not only because it overwhelms  $q(N, 0)$  in the limit, but also in the absolute sense that it determines this *expected first passage time*<sup>4</sup> from  $z = 0$  to  $z = N$ . In the present example, this expected time is no greater than  $1/(\beta \varepsilon^2)$ . This contrasts with the KMR model which, for this example, involves a transition probability of an order in  $\varepsilon$  which is, approximately, a *given fraction* of  $N$ . Hence the expected first passage time in KMR may increase much more rapidly as  $\varepsilon \rightarrow 0$ .<sup>5</sup>

What if the number of players,  $N$ , is less than 8 in the present example? If there are four or six players altogether, both equilibria are assigned positive probability in this construction. Only if there are just two players will the risk-dominant equilibrium be selected. Now the off-diagonal payoffs result when there is a single individual of each type. Since these payoffs favor type 2, the transition from  $w = N = 2$  to  $w = 0$  requires only a single mutation whereas the reverse transition requires two. Therefore,

<sup>4</sup> First passage times for such *regular* Markov chains are discussed, for example, in Kemeny and Snell [10, Chap. IV].

<sup>5</sup> For the more general games considered here, results will be given concerning the *maximum* expected first passage time from *any*  $z \in \Omega \sim \{N\}$  to  $z = N$ . This direct approach to the issue of the speed of convergence yields equivalent results to those of Ellison [5] for any asymmetric  $2 \times 2$  coordination game under the KMR matching mechanism and rapid adjustment. That is, [5, Theorem 2, p. 1059] shows that the exponential rate of convergence of the distribution of the stochastic process to its invariant distribution is determined by the minimum number of mutations it takes to go from the "wrong" equilibrium to the basin of attraction of the "right" equilibrium. This number of mutations also determines the maximum expected first passage time.

	2	$s_1$	$s_2$
1	$s_1$	$a, a$	$b, c$
	$s_2$	$c, b$	$d, d$

FIG. 2. General  $2 \times 2$  Games.

the risk-dominant equilibrium captures all of the long-run probability in the invariant distribution, as  $\varepsilon \rightarrow 0$ .

### 3. GENERAL $2 \times 2$ COORDINATION GAMES

Consider the general class of  $2 \times 2$  games in Fig. 2. Attention is initially limited to games which are symmetric in the sense that the payoff to a particular agent depends only on that agent's action and on the action of her opponent but not on the names of the agents. (Section 5 relaxes this restriction.) It is assumed in this section that both  $(s_1, s_1)$  and  $(s_2, s_2)$  are strict Nash equilibria, so that  $a > c$  and  $d > b$ . In addition,  $(s_1, s_1)$  is taken to be strictly Pareto-superior to  $(s_2, s_2)$ , so that  $a > d$ .<sup>6</sup>

The population is composed of  $N$  individuals. In each period  $t = 1, 2, \dots$ , they are randomly and independently matched in pairs for a total of  $v$  rounds to play the above game. (The integer  $N \geq 2$  is taken to be even so that there are no "odd men out.") In each round of play, there are  $(N-1) \cdot (N-3) \cdot \dots \cdot 3 \cdot 1$  possible ways of forming these  $N$  individuals into pairs and each of these is assumed to be equally likely. As before, let  $z_t$  denote the number of individuals playing  $s_1$  (individuals of "type 1"),  $N - z_t$  being the individuals playing  $s_2$  (that is, individuals of "type 2"). It is assumed that players do not change their actions across the rounds in each period. The state of the system at time  $t$  is identified with  $z_t \in \Omega \equiv \{0, 1, \dots, N\}$ , as observed at the *start* of each period. (This then incorporates mutation occurring after the previous period's adjustment process.)

Given any  $z_t \in \Omega$ , let  $\tilde{p}_t^1$  be the random variable representing the number of cross-pairings between type 1's and type 2's in any *given round* at  $t$ . The support of  $\tilde{p}_t^1$  is denoted by  $P^1(z_t)$ , where<sup>7</sup>

$$P^1(z_t) = \begin{cases} \{0, 2, \dots, \min[z_t, N - z_t]\}, & \text{if } z_t \text{ is even,} \\ \{1, 3, \dots, \min[z_t, N - z_t]\}, & \text{if } z_t \text{ is odd.} \end{cases}$$

<sup>6</sup> The case where  $a = d$  is discussed in Section 4, Example 2.

<sup>7</sup> In particular, of course, if  $z = 0$  or  $z = N$ , then  $P^1(z) = \{0\}$ , whereas, if  $z = 1$  or  $z = N - 1$ , then  $P^1(z) = \{1\}$ .

Define the random variable  $\tilde{p}_t$  as the *total* number of cross-pairings taking place across all  $v$  rounds of period  $t$  and let  $P(z_t)$  denote its support. The random average payoffs for type 1 and 2,  $\pi_1(z_t, \tilde{p}_t)$  and  $\pi_2(z_t, \tilde{p}_t)$ , respectively, are then

$$\begin{aligned} \pi_1(z_t, \tilde{p}_t) &= [a(vz_t - \tilde{p}_t) + b\tilde{p}_t]/vz_t \quad \text{and} \\ \pi_2(z_t, \tilde{p}_t) &= [c\tilde{p}_t + d(v(N - z_t) - \tilde{p}_t)]/[v(N - z_t)], \end{aligned} \quad (3)$$

provided  $z_t > 0$ , or  $N - z_t > 0$ , respectively.

Since<sup>8</sup>  $E(\tilde{p}_t) = v \cdot E(\tilde{p}_t^1) = v \cdot z_t(N - z_t)/(N - 1)$ , it follows (again, provided  $z_t > 0$  or  $N - z_t > 0$ , respectively) that the expected payoffs obtained in period  $t$  by each action are given by

$$\begin{aligned} E(\pi_1(z_t, \tilde{p}_t)) &\equiv \pi_1(z_t) = [a(z_t - 1) + b(N - z_t)]/[N - 1], \\ E(\pi_2(z_t, \tilde{p}_t)) &\equiv \pi_2(z_t) = [cz_t + d(N - z_t - 1)]/[N - 1]. \end{aligned}$$

In the KMR model, the revision of player strategies is directly linked to these expressions  $\pi_1(z_t)$  and  $\pi_2(z_t)$ , which could derive there from play of a round-robin tournament in each period.

In the present model, payoffs are *random* variables. Given any realization  $p_t \in P(z_t)$  and associated realized average payoffs, the distribution of revised strategies is described by an *additional* i.i.d. *random* variable  $\tilde{B}(z_t, p_t) \in \Omega \equiv \{0, \dots, N\}$ . This is the number of individuals playing strategy  $s_1$  in period  $t$ , right after strategy revision. It is required to satisfy the following restrictions, for  $z_t \in \{1, \dots, N - 1\}$ ,

$$\tilde{B}(z_t, p_t) \begin{cases} \geq z_t & \text{if } \pi_1(z_t, p_t) > \pi_2(z_t, p_t), \\ \leq z_t & \text{if } \pi_1(z_t, p_t) < \pi_2(z_t, p_t), \end{cases} \quad (4)$$

where each of these weak inequalities is *strict* with positive probability.<sup>9</sup> It is unnecessary to restrict the dynamics in the case that both strategies yield identical average payoffs. It is further required that,

$$\tilde{B}(0, 0) = 0 \quad \text{and} \quad \tilde{B}(N, 0) = N, \quad (5)$$

with probability one.<sup>10</sup>

<sup>8</sup> Lemma 5 in the Appendix proves this.

<sup>9</sup> The restriction that  $z_t \in \{1, \dots, N - 1\}$  ensures that the payoffs are well defined and that strict inequalities are feasible.

<sup>10</sup> That is, mutation is needed to introduce a new strategy. This assumption is maintained throughout the present paper, in contrast to KMR.

As in the KMR model, our players are naive about the dynamics in that they evaluate future payoffs in terms of present realizations. Furthermore, they are unaware of the role that luck might have played in the matching outcome realized. Given this assumption, however, the above specification is quite general. It would be satisfied, for example, if no individual who plays a strategy yielding a higher average payoff ever abandons it, and there is a positive probability that at least one new individual adopts such a strategy.<sup>11</sup>

After the completion of the process of strategy revision, each individual independently changes her strategy with a small probability  $\varepsilon > 0$ . This ensures that there exists a unique invariant distribution  $\mu: \Omega \rightarrow [0, 1]$ , summarizing the long-run behavior of the system, regardless of initial conditions. We are interested in exploring such long-run behavior in the limit as  $\varepsilon \rightarrow 0$ . This is captured by the *stochastically stable distribution*  $\mu^* \equiv \lim_{\varepsilon \rightarrow 0} \mu$ , which is shown to be well defined.<sup>12</sup>

KMR analyze an analogous evolutionary process where, as mentioned above, the deterministic payoffs  $\pi_1(z_i)$  and  $\pi_2(z_i)$  play the role of the random variables  $\pi_1(z_i, \tilde{p}_i)$  and  $\pi_2(z_i, \tilde{p}_i)$ . In their model, the risk-dominant equilibrium is selected by the stochastically stable distribution, whether or not this is also the Pareto-efficient equilibrium. That is, if  $(a - c) < (d - b)$ , then the state in which all individuals play strategy  $s_2$  is assigned probability 1 by this distribution. Furthermore, the speed of convergence, as measured by the maximum expected first passage time, is approximately of order  $\varepsilon^{-\gamma N}$  where  $\gamma \in (0, 1)$  is determined by the game's payoff structure. Of course,  $\gamma N$  grows without bound as  $N$  becomes large. The conclusions should be contrasted with the following.

**THEOREM 1.** *Consider any  $2 \times 2$  coordination game and evolutionary process as described above. There exists some  $\bar{N} > 0$  such that if  $N > \bar{N}$  then  $\mu^*(N) = 1$ . Moreover, there is some  $\alpha > 0$  such that the expected first passage time to  $z = N$  from any state  $z \in \Omega \sim \{N\}$  is no greater than  $\alpha \varepsilon^{-k}$ , for all small enough  $\varepsilon$ , where  $k \in \{1, 2, \dots\}$  is independent of both  $v$  and  $N$ .*

*Proof.* See the Appendix.

The intuition underlying the above result extends that for the example in Section 2. Again, the stochastically stable distribution puts all its weight on the *stochastically stable states*  $z = N$  and  $z = 0$ .<sup>13</sup> The *basin of attraction* of each of these two states is the set of states from which a transition to the

<sup>11</sup> Note that only the individuals who adopt new strategies need to be aware of the random average payoffs.

<sup>12</sup> This term is inspired by Foster and Young [6] and is used, for example, by Samuelson [14].

<sup>13</sup> The term *stochastically stable state* is due to Foster and Young [6].

given state occurs with positive probability in a finite number of periods without mutation. These two basins of attraction may overlap. To evaluate the probabilities of the two stochastically stable states, it is sufficient to consider the number of mutations required to induce a transition into these two basins of attraction. Keep in mind that  $(s_1, s_1)$  yields the highest possible payoff of all strategy combinations. The transition into the basin of attraction of  $z = N$  from  $z = 0$  can occur with  $k$  mutations of type  $s_1$ , for some  $k$  which is independent of  $N$ . That is, if sufficiently many type 1's are matched among themselves to the maximum extent possible,<sup>14</sup> they must obtain a payoff higher than that for the type 2 players, regardless of the number of type 2's. If  $k$  mutations are sufficient with a single round of matching, this number remains sufficient with any fixed number of rounds, since the appropriate matching occurs with positive probability in *every* round. Consider, however, the reverse transition. For any  $m \geq k$ , the process requires more than  $m$  type 2 mutations to go from  $z = N$  into the basin of attraction of  $z = 0$ , if  $N$  is chosen large enough. That is, if there are  $m$  type 2's and  $N$  is large enough, most type 1's must be matched with other type 1's and the average type 1 payoff always exceeds the average payoff for the type 2's. If this situation holds for a single round of matching, it also holds for any number of rounds. If  $N$  is large enough, then, the transition into the basin of attraction of  $z = N$  becomes arbitrarily more likely than the transition into the basin of attraction of  $z = 0$ , as  $\varepsilon \rightarrow 0$ . It follows that the state  $z = N$  is given probability one by the stochastically stable distribution.

Theorem 1 allows a clear-cut comparison between our approach and that of KMR. Consider the consequences of increasing  $v$ , the number of rounds per period. On the one hand, since the conclusion of Theorem 1 is valid for all  $v$ , the Pareto-efficient equilibrium is selected even if the limit as  $v \rightarrow \infty$  is taken *after* the limit on  $\varepsilon$ . Alternatively, we could invert the order of these limit operations and take the limit on  $v$  first. By the strong law of large numbers,  $\tilde{p}_i/v \rightarrow E\{\tilde{p}_i^1\} = z_i(N - z_i)/(N - 1)$ , almost surely, as  $v \rightarrow \infty$ , and we would then obtain the KMR model, for each given  $\varepsilon > 0$ . If the limit as  $\varepsilon \rightarrow 0$  were *then* taken, of course, the risk-dominant equilibrium of the game, whether or not this is Pareto-efficient, would be selected.<sup>15</sup>

<sup>14</sup> That is, except only for one mutant when the total number of mutants is odd.

<sup>15</sup> Given a fixed  $\varepsilon$ , increasing  $v$  reduces dramatically the probability of the string of luck needed to induce a transition to  $z = N$ . A similar effect occurs as  $N$  increases. Ultimately, the procedure of taking the limit as  $\varepsilon \rightarrow 0$  is empirically meaningful only to the extent that mutation rates can be safely considered small *relative* to the various other probabilities involved here. If the mutation rate and the other probabilities are small in absolute value, but the relative magnitudes are not specified, the present paragraph illustrates why there is no uniquely defined limit for the model.

4. OTHER SYMMETRIC  $2 \times 2$  GAMES

To complete our analysis, we now turn to the study of other symmetric  $2 \times 2$  games. Besides coordination games, there are two other *generic* types of such games, those with a dominant strategy and those for which the only symmetric equilibrium is in mixed strategies. KMR show that these equilibria are selected under suitable conditions. The motivation in the present section is to verify that the present evolutionary approach is also able to provide sensible answers even when there is a unique symmetric equilibrium. If this were not so, it might cast doubt on the results already obtained for coordination games.

For games having a strictly dominant strategy, where  $a > c$  and  $b > d$ , for example, it is straightforward to show that the dominant strategy equilibrium is selected by the stochastically stable distribution, if  $N$  is large enough. Furthermore, the maximum expected first passage time is of order  $\varepsilon^{-1}$ .

The analysis of games where  $c > a$  and  $b > d$ , for which the unique symmetric equilibrium is in mixed strategies, is more involved. As in KMR, adjustment must be sufficiently slow to rule out overshooting of the equilibrium. To illustrate this, suppose that the process of strategy revision is the fast one of Section 2, satisfying (1) in particular. Now, whenever the random matching produces average payoffs which are not exactly identical for the two strategies, the adjustment process jumps to one of the two *monomorphic* or pure states. In general, that is, the system will oscillate from one extreme state to the other, following the arrival of mutations. Since the random matching mechanism is an additional source of noise here, it is perhaps less clear than in KMR that slowness of the adjustment process is sufficient to approximate the mixed strategy equilibrium. However, such approximation is still obtained if the size of the population is large.

Consider then the limit as  $N \rightarrow \infty$ , while dampening the tendency of the system to oscillate by fixing the size of the support of the revised strategy distribution. More precisely, suppose the adjustment mechanism still satisfies (4) and (5), but the maximum change in the number of individuals choosing either strategy is 1.<sup>16</sup> As a minor technicality, it is also assumed

<sup>16</sup> It is always necessary to pay close attention to the transitions involving the monomorphic states. To prevent these states from acquiring positive probability under the stochastically stable distribution, in particular, it should not be possible to enter them without mutation. It can be shown that the maximum change induced by the adjustment process must then, in general, be no greater than 1 near the two extremes. This is less restrictive than it might seem at first in the sense that a change of 1 near the endpoints is still large in percentage terms. Moreover, there seems no reason to expect the present result to be substantially affected if this maximum change is permitted to be greater than 1 away from the two extremes, as long as it remains bounded.

that the probability of a revision toward a favored strategy is uniformly bounded away from zero in  $N$ . Thus there is some  $\Delta > 0$  such that, for all  $z_t \in \{1, \dots, N-1\}$  and all  $p_t \in P(z_t)$ ,

$$|\tilde{B}(z_t, p_t) - z_t| \leq 1,$$

$$\text{but } \Pr\{\tilde{B}(z_t, p_t) = z_t + 1\} > \Delta, \quad \text{if } \pi_1(z_t, p_t) > \pi_2(z_t, p_t), \quad (6)$$

$$\text{and } \Pr\{\tilde{B}(z_t, p_t) = z_t - 1\} > \Delta, \quad \text{if } \pi_2(z_t, p_t) > \pi_1(z_t, p_t),$$

for all  $N$ .

Given the model otherwise as in Section 3, we then establish that the stochastically stable distribution,  $\mu^*$ , converges in probability to the symmetric mixed strategy equilibrium, as  $N \rightarrow \infty$ .

**THEOREM 2.** *Consider any  $2 \times 2$  game, as in Fig. 2, where  $c > a$  and  $b > d$ , so that there is a symmetric mixed-strategy equilibrium  $(\alpha, 1 - \alpha)$ , where  $\alpha = (b - d) / ((b - d) + (c - a)) \in (0, 1)$  is the probability of playing  $s_1$ . Suppose the evolutionary process now satisfies (6) but is otherwise as in Section 3. Then,  $\forall \eta, \delta > 0, \exists \bar{N} > 0$ , such that, if  $N > \bar{N}$ , then  $\mu^*\{z \in \Omega: |(z/N) - \alpha| < \eta\} > 1 - \delta$ .*

*Proof.* See the Appendix.

The intuition here is as follows. The crucial new feature of the present model is the random nature of the processes of matching and adjustment in the absence of mutation. If the number of individuals playing both strategies is large, however, the present random matching process still turns out to imply that the type in short supply relative to the symmetric mixed strategy equilibrium is likely to do better than the other.<sup>17</sup> If the magnitude of the change in the population under the adjustment process is also restricted, such change is again likely to lead closer to this equilibrium.

This completes the examination of the class of *generic* symmetric  $2 \times 2$  games. Overall, it has been shown that there is a reasonable sense in which the efficient symmetric Nash equilibrium is selected for these games, even when mixed strategies are allowed.<sup>18</sup> The next section considers more general two-person games.

<sup>17</sup> This remains true even if the number of one of the two types is small, so that it is enough that the *total* number of individuals be large.

<sup>18</sup> Note that, in the case of a symmetric  $2 \times 2$  coordination game, the (symmetric) mixed strategy equilibrium is Pareto-dominated by the efficient pure strategy equilibrium, so the selection of this efficient equilibrium is consistent with this claim.

	2	$s_1$	$s_2$
1			
$s_1$	$a, a$	$b, c$	
$s_2$	$c, b$	$a, a$	

FIG. 3. Example 2.

Before addressing these, we illustrate some issues arising in non-generic games. Suppose, in particular, that efficiency does not discriminate between the two coordination equilibria, as in Fig. 3. Assume that  $a > b > c$ , so that both  $(s_1, s_1)$  and  $(s_2, s_2)$  are strict Nash equilibria, with identical payoff vectors, but  $(s_1, s_1)$  risk-dominates  $(s_2, s_2)$ . A natural question then is: Is the tie in terms of Pareto-efficiency between the two equilibria now broken in favor of the risk-dominant equilibrium?

Take the model as in Section 3, but take the number of rounds,  $v$ , as one, for simplicity. In the previous notation,

$$\pi_1(z_t, p_t) - \pi_2(z_t, p_t) = \frac{[(2a - b - c)z_t - (a - b)N]p_t}{z_t(N - z_t)}.$$

Define  $z^* = (a - b)N / (2a - b - c) < N/2$ . It follows that  $\pi_1(z_t, p_t) > \pi_2(z_t, p_t)$ , whenever  $z_t > z^*$ , for all  $p_t > 0$ , and  $\pi_1(z_t, p_t) < \pi_2(z_t, p_t)$ , whenever  $z_t < z^*$ , for all  $p_t > 0$ . However, if  $p_t = 0$ , as is possible for any even  $z_t \in \{2, \dots, N - 2\}$ , but impossible for any odd  $z_t \in \{1, \dots, N - 1\}$ , then  $\pi_1(z_t, p_t) = \pi_2(z_t, p_t) = a$ .

The stochastically stable distribution now depends on the details of the adjustment process. Suppose, for example, that this adjustment process, as in (4) and (5), also changes  $z_t$  by at most 1. Suppose it generates a movement in *either* direction with positive probability in the case of ties when  $z_t \in \{2, \dots, N - 2\}$ . Without mutation, it is impossible to avoid the odd states, and these push the process towards 0 if  $z_t < z^*$  but towards  $N$  if  $z_t > z^*$ . It follows that the situation is quite similar to that in KMR and the risk-dominant equilibrium is selected, if  $N$  is large enough.<sup>19</sup> On the other hand, suppose that, in the case of ties when  $z_t \in \{2, \dots, N - 2\}$ , the adjustment process generates a change of 2 in  $z_t$  in *either* direction with positive probability. Now it is possible to use the even states, which generate ties with positive probability, as stepping stones over the odd states, which cannot generate such ties. Only two mutations are then

<sup>19</sup> The proof of Theorem 1 can be adapted to show that  $\mu^*(N) = 1$ , if  $N$  is large enough. Note that there are more odd states in the basin of attraction of  $z = N$  than in that of  $z = 0$ , when  $N$  is large enough.

	2	$s_1$	$s_2$	$s_3$
1				
$s_1$	$\sqrt{3}, \sqrt{3}$	0, 1	0, -1	
$s_2$	1, 0	1, 1	2, -1	
$s_3$	-1, 0	-1, 2	-1, -1	

FIG. 4. Example 3.

sufficient to induce a transition from either equilibrium to the other, so that the stochastically stable distribution puts positive probability on both equilibria, if  $N$  is large enough.<sup>20</sup>

## 5. GAMES OF COMMON INTEREST

The approach proposed in the previous sections can be extended beyond symmetric  $2 \times 2$  games to games which may be asymmetric and involve more than two strategies. For some such games, our previous conclusions may be significantly altered. For example, a Pareto-efficient Nash equilibrium in a symmetric  $3 \times 3$  game may be given a limiting probability less than one. Consider Example 3 in Fig. 4.

This example is derived from Example 1 by adding an additional strategy,  $s_3$ , which is strictly dominated by both  $s_1$  and  $s_2$ . Hence  $(s_1, s_1)$  remains the unique Pareto-efficient Nash equilibrium. Consider, for simplicity, a natural generalization of the rapid process of strategy revision proposed in Section 2 and take the state of the system as that observed in each period immediately after the adjustment process. The strategy  $s_3$  can never survive this adjustment process, when any other types are present. Ties between strategies  $s_1$  and  $s_2$  are still impossible, and so the evolutionary process of strategy revision and experimentation remains essentially a two-state Markov chain.<sup>21</sup>

Suppose, however, that an  $s_2$  mutant and an  $s_3$  mutant arise in a population of size  $N \geq 4$  originally all playing  $s_1$ .<sup>22</sup> If these two mutants are

<sup>20</sup> If  $N > (2a - b - c)/(a - b)$ , then a single individual of either type must do worse than the remaining  $(N - 1)$  individuals of the other type, so that neither monomorphic state can be left with a single mutation.

<sup>21</sup> To be more precise, it is possible to observe the monomorphic state "all  $s_3$ " after adjustment. However, although it cannot be left without mutation, a *single* mutation is sufficient to accomplish this. On the other hand, to enter this state  $N$  mutations are required, given that the two other monomorphic states are the only other states possible after adjustment. The monomorphic state "all  $s_3$ " is then given zero weight by the stochastically stable distribution.

<sup>22</sup> The restriction that  $N \geq 4$  ensures that a single  $s_2$  mutant cannot invade a population of type 1's.

paired, the  $s_2$  mutant obtains a higher payoff than that obtained by the remaining type 1's, thus inducing a switch to the state in which all play  $s_2$ .<sup>23</sup> The reverse transition to  $(s_1, s_1)$  still requires two  $s_1$  mutants. Hence the state where all play  $s_1$  and the state where all play  $s_2$  are both given positive probability by the stochastically stable distribution.<sup>24</sup>

The following restricted class of two-person games might suggest itself. Suppose there is a strict Nash equilibrium which strictly Pareto-dominates all other strategy combinations, but there may be other strict equilibria. Does the result of Theorem 1 then extend?

A two-person game in which there is a feasible payoff vector which strictly Pareto-dominates every other payoff vector is a game of *common interest*, as this concept is defined by Aumann and Sorin [1]. Their definition permits the Pareto-efficient payoff vector to be attained by more than one strategy vector. However, in order to avoid the complications introduced by Example 2 above, it is assumed here that there is a *single* pure strategy vector which generates the Pareto-efficient payoff vector.

As in Aumann and Sorin, asymmetry between the players is permitted, each player having available an arbitrary number of pure strategies. This potential asymmetry is treated by considering two different populations.<sup>25</sup> Consider then two separate populations, population 1 and population 2, each composed of  $N$  individuals. In each period  $t$ , there is a given number of independent matchings,  $v$ , in which individuals of both populations are randomly paired up, one player from each population, to play a bilateral game of common interest. Let the strategy spaces of each population be, respectively,  $S^1 = \{1, \dots, n_1\}$  and  $S^2 = \{1, \dots, n_2\}$ . Correspondingly, denote the payoff functions by  $u_i: S^1 \times S^2 \rightarrow \mathfrak{R}$ ,  $i = 1, 2$ . There is assumed to be some  $\hat{s} \in S^1 \times S^2$  such that  $\forall s \in S^1 \times S^2 \sim \{\hat{s}\}$ ,  $u_i(\hat{s}) > u_i(s)$ ,  $i = 1, 2$ . Without loss of generality, take  $\hat{s} = (1, 1)$ .

Suppose that the profile of types in population  $i = 1, 2$  at the start of period  $t$  is given by  $z_t^i = (z_{1,t}^i, \dots, z_{n_i,t}^i)$ , where  $z_{k,t}^i \in \{0, \dots, N\}$  denotes the number of individuals in population  $i$  who then play strategy  $k$ . Given that each population has  $N$  members, the (finite) state space here is, say,

$$\Omega = \left\{ (z^1, z^2) \in \{0, 1, \dots, N\}^{n_1 + n_2} \mid \sum_{k=1}^{n_i} z_k^i = N, \quad i = 1, 2 \right\}.$$

<sup>23</sup> The role of the dominated strategy  $s_3$  is then to be a sacrificial lamb for  $s_2$ .

<sup>24</sup> However, if the probability of introducing  $s_3$  were proportional to  $\epsilon^2$ , as suggested by the notion of properness due to Myerson [12], this would restore  $(s_1, s_1)$  as the unique stochastically stable state.

<sup>25</sup> This two-population structure permits additional asymmetric outcomes even in symmetric games. The results of Section 3 can also be generalized to a situation where individuals drawn from a *single* population are randomly matched to play a symmetric game. This game should have a symmetric pure strategy equilibrium strictly Pareto-dominating all other outcomes.

For each  $z_t \equiv (z_t^1, z_t^2) \in \Omega$ , define an associated random variable  $\tilde{r}_t$  to describe the matching process at time  $t$  between populations 1 and 2 consistent with  $z_t$ . Every possible matching outcome in each independent round of play is assumed equally likely. A realization  $r_t$  of  $\tilde{r}_t$  specifies the set of numbers  $r_{k,\ell,t} \geq 0$ , for  $k = 1, 2, \dots, n_1$ ,  $\ell = 1, 2, \dots, n_2$ , representing the total matchings, in all  $v$  rounds at time  $t$ , between individuals from population 1 playing strategy  $k$  and those from population 2 playing  $\ell$ . The support of  $\tilde{r}_t$  is denoted  $P(z_t)$ . Of course,

$$\sum_{\ell=1}^{n_2} r_{k,\ell,t} = v z_{k,t}^1, \quad \sum_{k=1}^{n_1} r_{k,\ell,t} = v z_{\ell,t}^2, \\ k = 1, \dots, n_1, \quad \ell = 1, \dots, n_2, \quad t = 1, 2, \dots$$

For any  $r_t$ , denote the associated realizations of the random average payoff functions by  $\pi_{ik}(z_t, r_t)$ ,  $i = 1, 2$ ,  $k = 1, 2, \dots, n_i$ , defined whenever  $z_{k,t}^i > 0$ . These are analogous to the payoffs in (3) and are given by

$$\pi_{1k}(z_t, r_t) = \sum_{\ell=1}^{n_2} r_{k,\ell,t} u_1(k, \ell) / (v z_{k,t}^1), \quad k = 1, \dots, n_1, \quad \text{for } z_{k,t}^1 > 0, \\ \pi_{2\ell}(z_t, r_t) = \sum_{k=1}^{n_1} r_{k,\ell,t} u_2(k, \ell) / (v z_{\ell,t}^2), \quad \ell = 1, \dots, n_2, \quad \text{for } z_{\ell,t}^2 > 0.$$

Consider now a suitable extension of the adjustment process used in Section 3. Define  $\tilde{B}_k^i(z_t, r_t)$  as the *random* variable representing the number of players in population  $i$  who play strategy  $k$ ,  $k = 1, \dots, n_i$ ,  $i = 1, 2$ , after the process of strategy revision at  $t$  has taken place, given the population vector  $z_t$  and the realization  $r_t \in P(z_t)$  of the matching mechanism. These random variables are independent across the two populations, i.i.d. for each population over time and satisfy

$$\text{IF } i \in \{1, 2\} \text{ and } k \in \{1, \dots, n_i\} \text{ are s.t. (i) } z_{k,t}^i \in \{1, \dots, N-1\} \text{ and} \\ \text{(ii) } \pi_{ik}(z_t, r_t) > \pi_{i\ell}(z_t, r_t), \forall \ell \in \{1, \dots, n_i\} \sim \{k\} \text{ s.t. } z_{\ell,t}^i > 0, \\ \text{THEN } \Pr\{\tilde{B}_k^i(z_t, r_t) \geq z_{k,t}^i\} = 1 \text{ and } \Pr\{\tilde{B}_k^i(z_t, r_t) > z_{k,t}^i\} > 0. \quad (7)$$

Furthermore, for all  $z_t \in \Omega$ , all  $r_t \in P(z_t)$ , and  $i = 1, 2$ , it is assumed that

$$\Pr\{\tilde{B}_k^i(z_t, r_t) = z_{k,t}^i\} > 0. \quad (8)$$

As before, the process of strategy revision does not introduce new strategies so that  $\tilde{B}_k^i(z_t, r_t) = 0$  with probability 1 whenever  $z_{k,t}^i = 0$  for all  $k = 1, \dots, n_i$  and  $i = 1, 2$ . Of course,  $\sum_{k=1}^{n_i} \tilde{B}_k^i(z_t, r_t) = N$ , for  $i = 1, 2$ .<sup>26</sup>

<sup>26</sup> Thus the  $\tilde{B}_k^i(z_t, r_t)$  cannot be independent across strategies for a given population at a given point in time.



In other words, consider the set of *active* strategies, those for which  $z_{k,i}^i > 0$ , in a given population  $i = 1$  or  $2$ . If a particular strategy  $k$ , say, uniquely realizes the *highest* average payoff of these active strategies, then the number of individuals playing  $k$  has a positive probability of strictly increasing and cannot decrease. This seems the weakest possible natural generalization of (4). The further assumption that, in all cases, there is a positive probability that the revised number of players of any type remains constant is a technicality which simplifies the proof. It requires that the adjustment process has a minimal level of inertia.<sup>27</sup> Again, the adjustment process need not be restricted in situations not covered above.<sup>28</sup>

Finally, at the end of every period  $t$ , each individual independently changes her strategy, adopting each other strategy with probability  $\varepsilon > 0$ .<sup>29</sup> Denoting the state

$$\underbrace{(N, 0, \dots, 0)}_{n_1 \text{ entries}}, \underbrace{(N, 0, \dots, 0)}_{n_2 \text{ entries}}$$

by  $\hat{N}$  and the stochastically stable distribution by  $\hat{\mu}^*$ , the following result shows that Pareto-dominance is sufficient here to ensure that  $\hat{N}$  is selected.

**THEOREM 3.** *Consider any game of common interest and evolutionary process as defined above. There exists some  $\bar{N} > 0$  such that, if  $N > \bar{N}$  then  $\hat{\mu}^*(\hat{N}) = 1$ . Moreover, there exists  $\alpha > 0$  such that the expected first passage time to  $\hat{N}$  from any  $z \in \hat{\Omega} \sim \{\hat{N}\}$  is no greater than  $\alpha \varepsilon^{-2}$ , for all small enough  $\varepsilon$ .*

*Proof.* See the Appendix.

Note the sharper result here concerning the expected first passage times than held in Theorem 1. Indeed, the two-population structure facilitates coordination, implying that the basin of attraction of  $\hat{N}$  includes all profiles with at least one type 1 from each population. The intuition for this claim is as follows. Suppose, for example, that there is a positive number of type 1's in population 1 and this is no greater than the number of type 1's in population 2. It is then possible to match all of the type 1's in

<sup>27</sup> The present assumption helps, in particular, to tighten the bound here on the expected first passage times. It is not, by any means, the only assumption which yields the desired result. In fact, a natural extension of the fast adjustment process of Section 2 produces similar conclusions.

<sup>28</sup> It is irrelevant, for example, if the strategy with the second highest realized average payoff fares worse than the strategy with the lowest such payoff. Furthermore, it is unnecessary to restrict the adjustment process in the case of ties, even if there is such a tie for the *highest* realized average payoff.

<sup>29</sup> This could be generalized so that each transition was just first-order in  $\varepsilon$ .

population 1 with type 1's from population 2. Given an adjustment process satisfying (7) and (8), there is then a strictly positive probability of the total number of type 1's increasing, in the absence of mutation, as is sufficient to establish this claim.

## 6. OTHER RELATED LITERATURE

A key additional paper related to the present work is Young [16]. Young considers an  $n$ -person game, which may be asymmetric, and which is played once in each period by a randomly drawn set of players. Each of these players knows only a sample of the history of the game and even then may choose the wrong strategy with a small probability. Each player may die after playing the game and be replaced by a naive newcomer. The sampling procedure allows relatively fast convergence when the sample size is small. Young's model also casts doubt on the generality of the KMR result for symmetric  $2 \times 2$  coordination games, presenting an example of a symmetric  $3 \times 3$  game where an equilibrium which is both Pareto-dominant and risk-dominant is not selected.<sup>30</sup> This example also serves to highlight how Young's results differ from those of the present paper. That is, the example is a common interest game in which the approach of Section 5 would select the common interest equilibrium.

Also related to the present work are papers by Binmore *et al.* [4] and Binmore and Samuelson [3]. Both of these papers consider a model in which the adjustment process is noisy, for reasons differing from the random matching considered here. In the first of these papers, the interpretation is literally biological. Selection is noisy in that a particular pair of strategies determines survival probabilities rather than survival itself. In the second paper, learning is similarly noisy. For example, an individual may obtain a new strategy by copying that used by a randomly chosen other individual. In either case, the term mutation is reserved for changes in strategy beyond those arising from the adjustment process. These models may generate rapid convergence to equilibrium. Furthermore, there are circumstances under which the Pareto-efficient equilibrium, *per se*, is selected.

Bergin and Lipman [2] provide a different cautionary message from that of the present paper, also concerning the evolutionary approach to equilibrium selection. In general, that is, it seems reasonable that mutation rates might differ according to the particular transition involved. However, Bergin and Lipman prove the stark result that *any* invariant distribution of the mutationless process can be approximated by an invariant distribution of the process with suitably chosen small mutation rates.

<sup>30</sup> See [16, Example 3, p. 73].

## APPENDIX

*Proof of Theorem 1.* Let  $Q$  be the transition matrix of the complete process of adjustment and mutation of Section 3, so  $Q(z, z')$  is the probability of transition from  $z$  to  $z'$ , for all  $z, z' \in \Omega$ . Let  $T$  be the Markov transition matrix representing this process *excluding* mutation. Note that  $N$  is an *absorbing state* of  $T$  in that,  $\forall z \in \Omega \sim \{N\}$ ,  $T(N, z) = 0$ . Define its *basin of attraction under  $T$* ,  $A_T(N)$ , says, as follows:<sup>31</sup>

$$A_T(N) \equiv \{z \in \Omega \mid T^n(z, N) > 0 \quad \text{for some } n = 1, 2, \dots\} \quad (9)$$

A technique due to Friedlin and Wentzell [7] and applied by KMR [9] can be used to prove Theorem 1. To this end, define for each  $z \in \Omega$ , a *z-tree*  $H$ , as a collection of ordered pairs  $z' \rightarrow s(z')$  such that (i) every  $z' \in \Omega \sim \{z\}$  is the first element of exactly one such ordered pair and (ii) from every  $z' \in \Omega \sim \{z\}$  there exists a sequence of such ordered pairs leading to  $z$ . (That is,  $s(z')$  denotes the end-point of the unique arrow exiting  $z'$ .) The set of all such *z-trees* is denoted by  $\mathcal{H}_z$ . Now consider, for each  $z \in \Omega$ ,

$$q(z) = \sum_{H \in \mathcal{H}_z} \prod_{(z' \rightarrow s(z')) \in H} Q(z', s(z')) > 0.$$

The key result

$$\mu(z) = \frac{q(z)}{\sum_{z \in \Omega} q(z)} > 0$$

then holds, where  $\mu$  is the unique invariant distribution associated with the matrix  $Q$ .

Each  $q(z)$  is a polynomial in  $\varepsilon$  so that  $\mu^* = \lim_{\varepsilon \rightarrow 0} \mu$  is well defined. In addition, the state or states assigned positive probability by  $\mu^*$  are precisely those whose polynomials involve the lowest powers of  $\varepsilon$ .

To amplify this last observation, consider the following definitions. The set of states accessible in one period from a given state  $z \in \Omega$  with no mutations is

$$T(z) = \{z' \in \Omega \mid T(z, z') > 0\}.$$

Now the minimum number of mutations needed to induce a given transition from any  $z \in \Omega$  to any  $z'' \in \Omega$  in one period is

$$c(z, z'') = \min\{|z' - z''| \mid z' \in T(z)\}.$$

<sup>31</sup> Recall that the probability that a Markov process represented by  $T$  is in state  $z'$  after  $n$  periods, given that starts in state  $z$ , is  $T^n(z, z')$ , where  $T^n$  is the  $n$ th power of the matrix  $T$ .

The total number of such mutations, taken as the "mutation cost" of a given tree,  $H$ , is then

$$c(H) = \sum_{(z' \rightarrow s(z')) \in H} c(z', s(z')),$$

which is the lowest power of  $\varepsilon$  in the polynomial expression

$$\prod_{(z' \rightarrow s(z')) \in H} Q(z', s(z')).$$

(Note that a similar argument applies, in particular, to an arbitrary path,  $\bar{P}$ , yielding its cost,  $c(\bar{P})$ , say.) It follows that the lowest power of  $\varepsilon$  in  $q(z)$  is  $\min_{H \in \mathcal{H}_z} c(H)$ . Hence the state or states  $z \in \Omega$  given positive probability under  $\mu^*$  are precisely those for which  $\min_{H \in \mathcal{H}_z} c(H)$  attains its minimum.

The proof of Theorem 1 then relies on four Lemmas.

LEMMA 1.  $\exists k \in \{1, 2, \dots\}$  such that  $\{z' \in \Omega : z' \geq k\} \subseteq A_T(N)$ ,  $\forall N \in \{2, 4, \dots\}$ . It follows that,  $\forall z \in \Omega \sim \{N\}$ ,  $\exists z' \in A_T(N)$  such that  $c(z, z') \leq k$ .

*Proof.*<sup>32</sup> Choose  $k$  as the smallest integer such that

$$\frac{a(k-1) + b}{k} \geq \max\{d, c\} \equiv f. \quad (10)$$

Note that  $k$  depends only on the payoffs of the game and not on  $N$ . The assertion of the Lemma is trivial if  $N \leq k$ . Thus, take  $N > k$ . If  $z_t \geq k$  at any time  $t$ , it follows that there is positive probability that  $\pi_1(z_t, \tilde{p}_t) > \pi_2(z_t, \tilde{p}_t)$ . If  $z_t$  is even, this is immediate since the agents playing  $s_1$  are with positive probability matched only among themselves in every round at  $t$ , thereby obtaining  $a > d$ . (That is, with positive probability,  $\tilde{p}_t = 0 \in P(z_t)$ .) On the other hand, if  $z_t$  is odd, (10) ensures that the average payoff of  $s_1$  exceeds that of  $s_2$  in the positive probability event that *only one* of the players of type 1 is not matched to a player also of type 1 in every round. (That is, in the event that  $\tilde{p}_t = v \in P(z_t)$ .)

<sup>32</sup> Note that two  $s_1$  mutants might not suffice to induce a transition to  $z = N$ . There is a positive probability that two such  $s_1$  mutants are paired and receive the unmatchable payoff of  $a$ , and this does generate a positive probability of increasing the number of  $s_1$  players. It is the next step which might be problematic. For example, there might now be three  $s_1$  players, at most two of which can be paired up. The cross-matching of one of the three  $s_1$  players could then lower the average payoff of the  $s_1$  players below that of the  $s_2$  players, preventing any further increase in the number of  $s_1$  players.

By (4), if  $\pi_1(z_i, p) > \pi_2(z_i, p)$ , there is a positive probability that  $z_{i+1} > z_i$ . It follows that

$$\{z' \in \Omega : z' \geq k\} \subseteq A_T(N),$$

so at most  $k$  mutations always suffice to enter the basin of attraction of  $N$ , from any state  $z \in \Omega \sim \{N\}$ . ■

LEMMA 2.  $\forall m \in \{1, 2, \dots\}$ ,  $\exists \bar{N} > 0$  such that,  $\forall N > \bar{N}$  and  $\forall z \in \{N - m, \dots, N - 1\}$ ,  $\pi_1(z, p) > \pi_2(z, p)$ ,  $\forall p \in P(z)$ . It follows that  $c(\bar{P}) > m$ , where  $\bar{P}$  is any path from  $N$  to any  $z \notin A_T(N)$ .

*Proof.* Define

$$\bar{N} = \left\lceil \frac{(2a - b - f) \cdot m}{(a - f)} \right\rceil,$$

where  $f = \max\{c, d\}$  and  $\lceil x \rceil$  denotes the smallest integer greater than or equal to  $x$ . It follows that, if  $N > \bar{N}$ , then  $N/2 > m$  and

$$\frac{a(N - 2m) + b \cdot m}{N - m} > f. \quad (11)$$

Hence, if  $z \geq N - m$ , then  $\pi_1(z, p) > \pi_2(z, p)$ ,  $\forall p \in P(z)$ . It follows, in the light of the adjustment dynamics (4), that more than  $m$  mutations are needed altogether on any path  $\bar{P}$  from  $N$  to any  $z \notin A_T(N)$ . ■

Given the observations preceding Lemma 1, the following completes the proof that the stochastically stable distribution selects the Pareto-efficient equilibrium.

LEMMA 3.  $\exists \bar{N}$  such that

$$\min_{H \in \mathcal{H}_N} c(H) < \min_{H' \in \mathcal{H}_z} c(H'),$$

$\forall N > \bar{N}$  and  $z \in \Omega \sim \{N\}$ .

*Proof.* Choose  $m \geq k$  and  $N > \bar{N}$  as in Lemmas 1 and 2. If  $z \in \Omega \sim \{N\}$  and  $H' \in \mathcal{H}_z$  are arbitrary, the result follows if  $H \in \mathcal{H}_N$  can be found such that  $c(H) < c(H')$ . Note, for future reference, that there exists an  $N$ -tree  $Y$ , say, defined only on  $A_T(N)$ , such that  $c(Y) = 0$ .<sup>33</sup>

<sup>33</sup> Samuelson [14, Lemma 3, p. 45] presents a result based on Young [16, Theorem 4, p. 78] that basins of attractions can effectively be suppressed for the present purpose. The costless tree  $Y$  associated with the basin of attraction of  $N$  is considered here for completeness.

Suppose first that  $z \notin A_T(N)$ . Now, in any given  $H' \in \mathcal{H}_z$ , there must exist, in particular, a path leading from  $N$  to  $z$ . Label the “subpath” which leads from  $N$  to the first point *not* in  $A_T(N)$  as  $\bar{P}$ , taken to be composed of points  $P \in \Omega$ . The desired  $N$ -tree  $H$  can be obtained from  $H'$  by the following two alterations. First, switch arrows from each  $z' \in A_T(N) \cap P \sim \{N\}$  to its immediate successor on the corresponding costless path to  $N$  in  $Y$ .<sup>34</sup> Furthermore, as required, switch all subsequent arrows along each such path appropriately. Note that each such switch preserves the “tree” property that each point have a unique successor point. Delete entirely the arrow from  $N$  itself. This first step reduces overall cost by more than  $m$ , from Lemma 2. The resulting graph links every point via a unique path *either* to  $z$  *or* to  $N$ . Second, add the arrow  $z \rightarrow z'$ , where  $z' \in A_T(N)$  is as in Lemma 1 and also switch arrows, as necessary, at subsequent points on the costless path in  $Y$  from  $z'$  to  $N$ . This second step increases cost by at most  $k$ . It also produces a graph which is the required  $N$ -tree  $H$ , recalling that  $k \leq m$ .

Suppose now that  $z \in A_T(N) \sim \{N\}$ . Construct  $H$  in a similar fashion to that above. However, in the first step, take the entire path from  $N$  to  $z$  in the role of  $\bar{P}$ . This first step reduces cost by at least 1, since  $c(N, z) \geq 1$ , for all  $z \neq N$ . As the second step, introduce the costless path in  $Y$  from  $z$  to  $N$ . Since this cannot increase cost, the result that  $c(H) < c(H')$  again follows. ■

The assertion concerning the expected first passage times to  $z = N$  can now be derived from Lemma 1 to complete the proof of Theorem 1:

LEMMA 4. Let  $\tau(z)$  denote the expected first passage time from any  $z \in \Omega \sim \{N\}$  to  $N$ . Then  $\exists \alpha > 0$  such that  $\tau(z) \leq \alpha \varepsilon^{-k}$ , for all small enough  $\varepsilon$ , where  $k$  is as in Lemma 1.

*Proof.* Lemma 1 implies that the transition probability,  $Q(z, z')$ , from any  $z \in \Omega \sim \{N\}$  to some  $z' \in A_T(N)$ , is no less than  $\beta \varepsilon^k$ , for all sufficiently small  $\varepsilon$ , for some  $\beta > 0$ . By the definition of this basin of attraction, there is some  $\bar{n} \in \{2, 3, \dots\}$  and some  $\zeta \in (0, 1/2]$  such that  $T^{\bar{n}-1}(z', N) \geq 2\zeta$ , for all  $z' \in A_T(N)$ . It follows that  $Q^{\bar{n}-1}(z', N) \geq \zeta > 0$ , for all  $z' \in A_T(N)$  and for sufficiently small  $\varepsilon$ . Hence  $Q^{\bar{n}}(z, N) \geq \beta \varepsilon^k \zeta = \xi$ , say, for all  $z \in \Omega \sim \{N\}$  and for sufficiently small  $\varepsilon$ . Define  $\hat{Q}$  as the associated Markov process which samples every  $\bar{n}$  periods and makes  $z = N$  absorbing. That is, define  $\hat{Q}(N, N) = 1$  and  $\hat{Q}(N, z) = 0$ , for all  $z \in \Omega \sim \{N\}$ , but  $\hat{Q}(z, z') = Q^{\bar{n}}(z, z')$ , for all  $z \in \Omega \sim \{N\}$  and all  $z' \in \Omega$ . The expected time to absorption at  $N$  for  $\hat{Q}$

<sup>34</sup> The set  $A_T(N) \cap P \sim \{N\}$  may be empty, in which case this sentence and the one following are vacuously satisfied.

is the expected first passage time to  $N$  for  $Q^{\bar{n}}$ , which is no less than the expected first passage time to  $N$  for  $Q$ . Furthermore, the expected time to absorption for  $\tilde{Q}$  is finite, for all initial  $z \in \Omega$ .<sup>35</sup> Setting  $Q^0(z, N) \equiv 0$ , note that  $\tilde{Q}^i(z, N) \rightarrow 1$ , as  $i \rightarrow \infty$ . Define  $\rho^i(z)$  as the probability of absorption under  $\tilde{Q}$  at time  $t = i\bar{n}$ , given state  $z$  at  $t = 0$ , and given that absorption did not occur at or before  $t = (i-1)\bar{n}$ . It follows that

$$\rho^i(z) \equiv \frac{\tilde{Q}^i(z, N) - \tilde{Q}^{i-1}(z, N)}{1 - \tilde{Q}^{i-1}(z, N)} \geq \xi, \quad i = 1, 2, \dots,$$

for all  $z \in \Omega \sim \{N\}$  and all sufficiently small  $\varepsilon$ . It also follows that there exists  $\bar{\tau} \in [\bar{n}, \infty)$  such that, for all  $z \in \Omega \sim \{N\}$ ,

$$\tau(z) \leq \bar{n} \sum_{i=1}^{\infty} i [\tilde{Q}^i(z, N) - \tilde{Q}^{i-1}(z, N)] \leq \bar{\tau},$$

where the middle expression above represents the expected time to absorption under  $\tilde{Q}$ , given state  $z \in \Omega \sim \{N\}$  at  $t = 0$ .

We now show, by induction, for  $i = 1, 2, \dots$ , that

$$\tau(z) \leq \bar{n}\xi + \bar{n}(1-\xi)\xi + \dots + \bar{n}i(1-\xi)^{i-1}\xi + (1-\xi)^i \tau^i(z),$$

where, for  $i = 0, 1, \dots$ ,

$$\tau^i(z) \equiv \bar{n} \sum_{j=i}^{\infty} (j+1) [\tilde{Q}^{j+1}(z, N) - \tilde{Q}^j(z, N)] / [1 - \tilde{Q}^i(z, N)]$$

represents the expected time of absorption under  $\tilde{Q}$ , given state  $z$  at  $t = 0$  and given that such absorption has not occurred at or before  $t = i\bar{n}$ . Recall first that  $\tau(z) \leq \tau^0(z)$ .<sup>36</sup> For  $i = 0, 1, \dots$ , it can be shown that  $\tau^i(z) \in [\bar{n}(i+1), \bar{n}i + \bar{\tau}]$  and that

$$\tau^i(z) = \bar{n}(i+1) \rho^{i+1}(z) + (1 - \rho^{i+1}(z)) \tau^{i+1}(z),$$

which is maximized over  $\rho^{i+1}(z)$  at  $\rho^{i+1}(z) = \xi$ , given that  $\tau^{i+1}(z) > \bar{n}(i+1)$ . Hence, as suffices to complete the proof by induction,

$$\tau^i(z) \leq \bar{n}(i+1)\xi + (1-\xi)\tau^{i+1}(z), \quad i = 0, 1, \dots$$

Since

$$0 \leq (1-\xi)^i \tau^i(z) \leq (1-\xi)^i (\bar{n}i + \bar{\tau}) \rightarrow 0, \quad \text{as } i \rightarrow \infty,$$

<sup>35</sup> This follows since  $Q^{\bar{n}}$  is regular. See Kemeny and Snell [10, Chap. IV], for example.

<sup>36</sup> Note that  $\tau^0(z) = \bar{n} \sum_{i=1}^{\infty} i [\tilde{Q}^i(z, N) - \tilde{Q}^{i-1}(z, N)]$ .

it follows that

$$\tau(z) \leq \bar{n} \sum_{i=1}^{\infty} i(1-\xi)^{i-1} \xi = \bar{n}/\xi = \bar{n}/\beta e^k \xi = \alpha e^{-k},$$

say, for all  $z \in \Omega \sim \{N\}$  and all small enough  $\varepsilon$ , as required. ■

The following preliminary result is useful in the proof of Theorem 2.

LEMMA 5. Suppose an even number,  $N \geq 4$ , of individuals are paired at random for  $v$  independent rounds, where every pairing in each round is equally likely, and where  $z > 0$  of these individuals are of type 1 and  $N - z > 0$  are of type 2. Then the random total number of cross-pairings,  $\tilde{p}$ , has mean and variance as follows:

$$E\{\tilde{p}\} = v \frac{z(N-z)}{(N-1)} \quad \text{and} \quad \text{var}\{\tilde{p}\} = 2v \frac{z(N-z)[z(N-z) - (N-1)]}{(N-1)^2(N-3)}.$$

*Proof.* Without loss of generality suppose that the individuals of type 1 are labelled  $i = 1, \dots, z$ . Consider first the case  $v = 1$  and define the following random variables, for  $i = 1, \dots, z$ ,

$$\tilde{e}_i = \begin{cases} 1 & \text{if individual } i \text{ is cross-paired,} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, then,

$$\tilde{p} = \sum_{i=1}^z \tilde{e}_i \quad \text{and} \quad E\{\tilde{p}\} = zE\{\tilde{e}_1\} = \frac{z(N-z)}{(N-1)}.$$

Furthermore,

$$\tilde{p}^2 = \sum_{i,j=1}^z \tilde{e}_i \tilde{e}_j = \sum_{i=1}^z \tilde{e}_i^2 + \sum_{i \neq j, i,j=1}^z \tilde{e}_i \tilde{e}_j,$$

so that<sup>37</sup>

$$\begin{aligned} E\{\tilde{p}^2\} &= zE\{\tilde{e}_1^2\} + z(z-1)E(\tilde{e}_1 \tilde{e}_2) \\ &= z \Pr\{\tilde{e}_1 = 1\} + z(z-1) \cdot \Pr\{\tilde{e}_1 = 1\} \cdot \Pr\{\tilde{e}_2 = 1 \mid \tilde{e}_1 = 1\} \\ &= \frac{z(N-z)}{(N-1)} + z(z-1) \cdot \frac{(N-z)}{(N-1)} \cdot \frac{(N-z-1)}{(N-3)}. \end{aligned}$$

<sup>37</sup> The formula to follow is formally valid for  $z = 1$  and for  $z = N - 1$ , when  $\text{var}\{\tilde{p}\} = 0$ .

The result for the variance then follows, after some algebra, from  $\text{var}\{\tilde{p}\} = E\{\tilde{p}^2\} - (E\{\tilde{p}\})^2$ . For arbitrary  $v$ , the desired expressions follow immediately from the independence of the matching across rounds. ■

*Proof of Theorem 2.* The stochastically stable distribution  $\mu^*$  can be shown to exist as in the proof of Theorem 1. Let  $\eta$  and  $\delta$  be as in the statement of Theorem 2, and denote the integers by  $Z$ . It will first be shown that

$$\sum_{z \in [0, N(\alpha - \eta)] \cap Z} \mu^*(z) < \frac{\delta}{2}.$$

In particular, the following considerations apply at  $z = 0$ . A single mutation induces a transition from  $z = 0$  to  $z = 1$ , but a strictly larger number of mutations is required for a transition to any other  $z \in \Omega \sim \{0\}$ . At least one mutation is required to induce any transition to  $z = 0$  from any  $z \in \Omega \sim \{0\}$ , if

$$N > \bar{N}_1 = 1 + \max\{(c - d)/(b - d), 0\}, \quad (12)$$

which ensures that a lone  $s_1$  player must do better than the other  $s_2$  players. A single mutation might then induce the transition from  $z = 1$  to  $z = 0$ , given the adjustment process as in (4), (5), and (6).<sup>38</sup> Indeed, a single mutation might also induce the transition from  $z = 2$  to  $z = 0$ .<sup>39</sup> On the other hand, a single mutation cannot suffice for the transition from any other  $z \in \Omega \sim \{0\}$  to  $z = 0$ . The definition of the invariant distribution  $\mu$  in terms of the transition matrix  $Q$  implies that

$$\mu(0) \sum_{z=1}^N Q(0, z) = \sum_{z=1}^N \mu(z) Q(z, 0).$$

Taking the limit as  $\varepsilon \rightarrow 0$ , assuming  $N > \bar{N}_1$  and considering further details of the previous claims, it follows that

$$\mu^*(0) \leq \mu^*(1) + \mu^*(2). \quad (13)$$

If it can be shown that  $\mu^*(1)$  and  $\mu^*(2)$  are small, then so is  $\mu^*(0)$ .

<sup>38</sup> This would be if the adjustment process left  $z = 1$  unaltered, so that this solitary  $s_1$  player could then mutate.

<sup>39</sup> This would be if  $d > a$ , so that a pairing of the only two  $s_1$  players would yield a positive probability of a transition to only one. A single mutation of that  $s_1$  player would then complete the transition to  $z = 0$ .

With  $\eta$  still as in the statement of Theorem 2, take

$$N > \bar{N}_2 = 1 + \max\left(\frac{4(a - d)}{\eta[(c - a) + (b - d)]}, 0\right)$$

and consider all integers  $z \geq 1$  such that  $z/N < \alpha - \eta/2$ . It follows that

$$E(\pi_1(z, \tilde{p}) - \pi_2(z, \tilde{p})) > [(c - a) + (b - d)] \eta/4 > 0.$$

In addition, since

$$\text{var}\{\pi_1(z, \tilde{p}) - \pi_2(z, \tilde{p})\} = \left(\frac{(b - a)}{z} - \frac{(c - d)}{N - z}\right)^2 \cdot \frac{\text{var}\{\tilde{p}\}}{v^2},$$

it follows readily from Lemma 5 that, if  $N \geq 4$ , then

$$\frac{\text{var}\{\tilde{p}\}}{z^2}, \frac{\text{var}\{\tilde{p}\}}{(N - z)^2}, \quad \text{and} \quad \frac{\text{var}\{\tilde{p}\}}{(N - z)z} \leq \frac{2v}{(N - 3)}.$$

Chebyshev's inequality now implies that the probability that  $\pi_1(z, \tilde{p}) - \pi_2(z, \tilde{p}) > 0$  can be made arbitrarily high, uniformly in  $z \in [1, N(\alpha - \eta/2)]$ , by choosing  $N$  large enough. If  $N > 2/\eta$ , then  $(z + 1)/N < \alpha - \eta/2$ , for all  $z \in [1, N(\alpha - \eta)]$ . Hence, in the light also of (4) and (6), there exists  $\bar{N}_3 \geq \max\{4, 2/\eta, \bar{N}_2\}$  such that, if  $N > \bar{N}_3$ , then  $0 \leq T(z + 1, z) < \delta/(2(4 + \delta))$ , whereas  $T(z, z + 1) > \delta/2 > 0$ , for all integers  $z \in [1, N(\alpha - \eta)]$ .

From the definition of the invariant distribution  $\mu$ , it follows that, for all  $z \in \Omega$ ,

$$\sum_{z' \leq z} \mu(z') \sum_{z'' \geq z+1} Q(z', z'') = \sum_{z'' \geq z+1} \mu(z'') \sum_{z' \leq z} Q(z'', z'). \quad (14)$$

Now, for all integers  $z \in [1, N(\alpha - \eta)]$ , in particular,

$$T(z, z + 1) \mu^*(z) = T(z + 1, z) \mu^*(z + 1),$$

as follows by taking the limit of (14), since the mutationless adjustment process as in (4) and (6) changes  $z$  by at most 1 in each period. Hence, for  $N > \bar{N}_3$  and for all integers  $z \in [1, N(\alpha - \eta)]$ ,

$$0 \leq \mu^*(z) \leq \frac{\delta}{4 + \delta} \mu^*(z + 1).$$

In particular, this implies  $\mu^*(1)$  and  $\mu^*(2) \rightarrow 0$ , as  $N \rightarrow \infty$ , so that  $N(\alpha - \eta) \rightarrow \infty$ . Hence, from (13), there exists  $\bar{N}_4 \geq \bar{N}_1$ , where  $\bar{N}_1$  is as in

(12), such that, if  $N > \bar{N}_4$ , then  $\mu^*(0) < \delta/4$ . It follows that, if  $N > \max\{\bar{N}_3, \bar{N}_4\}$ , then, as required,

$$\sum_{z \in [0, N(\alpha - \eta)] \cap Z} \mu^*(z) < \sum_{i=1}^{\infty} \left( \frac{\delta}{4 + \delta} \right)^i + \frac{\delta}{4} = \frac{\delta}{2}.$$

Entirely analogous considerations apply for the integers  $z \in [N(\alpha + \eta), N]$ . Altogether, it follows that there exists  $\bar{N}$  such that  $N > \bar{N}$  implies

$$\sum_{z \in [0, N(\alpha - \eta)] \cap Z} \mu^*(z) + \sum_{z \in [N(\alpha + \eta), N] \cap Z} \mu^*(z) < \delta,$$

completing the proof of Theorem 2.

*Proof of Theorem 3.* Define  $\hat{Q}$  as the Markov matrix on  $\hat{\Omega}$  for the two-population model of Section 5, with  $\hat{T}$  as the associated Markov matrix for the process *excluding* mutation. Retain the notation for the mutation cost of a transition from  $z$  to  $z'$  as  $c(z, z')$ . Recall the notation for the common interest equilibrium,

$$\hat{N} \equiv (\overbrace{N, 0, \dots, 0}^{n_1 \text{ entries}}, \overbrace{N, 0, \dots, 0}^{n_2 \text{ entries}}),$$

which is an absorbing state with a basin of attraction,  $A_{\hat{T}}(\hat{N})$ , say, defined in the analogous fashion to (9) under  $\hat{T}$ . The key to the present proof is then:

LEMMA 6. *In the two-population model of Section 5,*

$$A_{\hat{T}}(\hat{N}) = \{z = (z^1, z^2) \in \hat{\Omega} \mid z_1^1 > 0 \quad \text{and} \quad z_1^2 > 0\}.$$

Hence,  $\forall z \in \hat{\Omega} \sim \{\hat{N}\}$ ,  $\exists z' \in A_{\hat{T}}(\hat{N})$  such that  $c(z, z') \leq 2$ .

*Proof.* Consider any  $z$  such that  $z_1^1 > 0$  and  $z_1^2 > 0$  and suppose, without loss of generality, that  $z_1^1 \geq z_1^2$  and  $z_1^2 < N$ . Under the random matching mechanism, there is a positive probability that all these  $z_1^2$  “type 1” players in population 2 will be matched only within the set of  $z_1^1$  “type 1” players of population 1 in every round within the period. It follows that, in this event, the type 1 players in population 2 must attain the highest average payoff in population 2 and there is a positive probability, given (7), that the number of type 1 players in population 2 increases. By (8), there is an independent positive probability that the number of type 1 players in population 1 remains at  $z_1^1$ . Altogether, there is then a positive probability that the total number of type 1 players in both populations strictly

increases.<sup>40</sup> It follows that  $\hat{T}^{2N-2}(z, \hat{N}) > 0$ , so that  $z \in A_{\hat{T}}(\hat{N})$ , as required. Hence two mutations always suffice to reach  $A_{\hat{T}}(\hat{N})$ , from any  $z \in \hat{\Omega} \sim \{\hat{N}\}$ . ■

Analogous to Lemma 2 is that, if the total number of individuals who are not of type 1 is bounded above while the total number of players is large enough, then the average payoff for type 1 in each population is strictly higher than for any other active strategy, regardless of the outcome of the matching mechanism.

LEMMA 7. *For  $m = 1, 2, \dots$ , define  $Z(m) = \{z \in \hat{\Omega} \sim \{\hat{N}\} \mid \sum_{k=2}^{n_1} z_k^1 + \sum_{\ell=2}^{n_2} z_{\ell}^2 \leq m\}$ . Then  $\forall m \in \{1, 2, \dots\}$ ,  $\exists \bar{N}$  such that,  $\forall N > \bar{N}$ , and  $\forall z_t \in Z(m)$ , then  $z_{1,t}^i > 0$  and  $\pi_{i1}(z_t, r_t) > \pi_{ik}(z_t, r_t)$ , for all  $k \neq 1$  for which  $z_{k,t}^i > 0$ , for all  $r_t \in P(z_t)$ , and for  $i = 1, 2$ .*

*Proof.* Recall that the payoff functions for each of the two populations are given by

$$\pi_{1k}(z_t, r_t) = \sum_{\ell=1}^{n_2} r_{k,\ell,t} u_1(k, \ell) / (v z_{k,t}^1), \quad k = 1, \dots, n_1, \quad \text{for } z_{k,t}^1 > 0,$$

$$\pi_{2\ell}(z_t, r_t) = \sum_{k=1}^{n_1} r_{k,\ell,t} u_2(k, \ell) / (v z_{\ell,t}^2), \quad \ell = 1, \dots, n_2, \quad \text{for } z_{\ell,t}^2 > 0,$$

where  $r_{k,\ell,t}$  stands for the number of total matchings between type  $k$  players of population 1 and type  $\ell$  players of population 2 at  $t$ , so that

$$\sum_{\ell=1}^{n_2} r_{k,\ell,t} = v z_{k,t}^1, \quad \sum_{k=1}^{n_1} r_{k,\ell,t} = v z_{\ell,t}^2, \quad k = 1, \dots, n_1, \quad \ell = 1, \dots, n_2.$$

In addition,  $u_i(1, 1) > u_i(k, \ell)$ , for all  $(k, \ell) \neq (1, 1)$ ,  $i = 1, 2$ . Note that  $z_{1,t}^i > 0$ ,  $i = 1, 2$ , whenever  $N > \bar{N} > m$ . The stated result is then immediate. ■

“Tree surgery” as in Lemma 3 again establishes that the tree or trees with the lowest mutational cost are  $\hat{N}$ -trees, for all  $N > \bar{N}$ , taking  $m \geq 2$  in Lemma 6 and  $\bar{N}$  as in Lemma 7. This completes the proof that the common interest equilibrium,  $\hat{N}$ , is selected by the stochastically stable distribution, given that  $N > \bar{N}$ .

<sup>40</sup> There are clearly alternative assumptions to (8) which would serve to ensure that there is a positive probability of the total number of type 1 players increasing for all such initial population vectors.

The assertion concerning the expected first passage times is then a consequence of Lemma 6. This follows from Lemma 4, which applies equally as well to the Markov processes given by  $\hat{Q}$  and  $\hat{T}$  as to those given by  $Q$  and  $T$ . ■

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