Repeated Games
Long Run versus Short Run Player

a fixed simultaneous move stage game

Player 1 is long-run with discount factor $\delta$
actions $a^1 \in A^1$ a finite set
utility $u^1(a^1, a^2)$

Player 2 is short-run with discount factor 0
actions $a^2 \in A^2$ a finite set
utility $u^2(a^1, a^2)$
What it is about

the “short-run” player may be viewed as a kind of “representative” of many “small” long-run players

♦ the “usual” case in macroeconomic/political economy models
♦ the “long run” player is the government
♦ the “short-run” player is a representative individual
Example 1: Peasant-Dictator

Diagram showing the interaction between peasants and dictators with states and transitions labeled with actions Eat and Grow, and rewards Low and High.
### Example 2: Backus-Driffl

<table>
<thead>
<tr>
<th></th>
<th>Low</th>
<th>High</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low</td>
<td>0,0</td>
<td>-2,-1</td>
</tr>
<tr>
<td>High</td>
<td>1,-1</td>
<td>-1,0</td>
</tr>
</tbody>
</table>

**Inflation Game:** LR=government, SR=consumers

Consumer preferences are whether or not they guess right

<table>
<thead>
<tr>
<th></th>
<th>Low</th>
<th>High</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low</td>
<td>0,0</td>
<td>0,-1</td>
</tr>
<tr>
<td>High</td>
<td>-1,-1</td>
<td>-1,0</td>
</tr>
</tbody>
</table>

With a hard-nosed government
Repeated Game

history \( h_t = (a_1, a_2, \ldots, a_t) \)

null history \( h_0 \)

behavior strategies \( \alpha_t^i = \sigma^i(h_{t-1}) \)

long run player preferences

average discounted utility

\[
(1 - \delta) \sum_{t=1}^{T} \delta^{t-1} u^i(a_t)
\]

note that average present value of 1 unit of utility per period is 1
Equilibrium

Nash equilibrium: usual definition – cannot gain by deviating
Subgame perfect equilibrium: usual definition, Nash after each history
Observation: the repeated static equilibrium of the stage game is a subgame perfect equilibrium of the finitely or infinitely repeated game
♦ strategies: play the static equilibrium strategy no matter what
“perfect equilibrium with public randomization”

may use a public randomization device at the beginning of each period to pick an equilibrium

key implication: set of equilibrium payoffs is convex
Example: Peasant-Dictator

normal form: unique Nash equilibrium **high, eat**

<table>
<thead>
<tr>
<th></th>
<th>eat</th>
<th>grow</th>
</tr>
</thead>
<tbody>
<tr>
<td>low</td>
<td>0*,1</td>
<td>1,2*</td>
</tr>
<tr>
<td>high</td>
<td>0*,1*</td>
<td>3*,0</td>
</tr>
</tbody>
</table>
Static Benchmarks

payoff at static Nash equilibrium to LR player: 0

precommitment or Stackelberg equilibrium
precommit to low get 1
mixed precommitment to 50-50 get 2

minmax payoff to LR player: 0
Payoff Space

utility to long-run player

mixed precommitment/Stackelberg = 2

best dynamic equilibrium = ?

pure precommitment/Stackelberg = 1

Set of dynamic equilibria

static Nash = 0

worst dynamic equilibrium = ?

minmax = 0
Repeated Peasant-Dictator

finitely repeated game
final period: high, eat, so same in every period
Do you believe this??

♦ Infinitely repeated game

begin by low, grow
if low, grow has been played in every previous period then play low, grow
otherwise play high, eat (reversion to static Nash)
claim: this is subgame perfect
When is this an equilibrium?

clearly a Nash equilibrium following a history with high or eat
SR play is clearly optimal

for LR player
may high and get \((1 - \delta)3 + \delta 0\)
or low and get 1

so condition for subgame perfection
\((1 - \delta)3 \leq 1, \delta \geq 2/3\)
equilibrium utility for LR
General Deterministic Case
Fudenberg, Kreps and Maskin

$\max u^1(a)$

mixed precommitment/Stackelberg

$\bar{v}^1$ best dynamic equilibrium

pure precommitment/Stackelberg

Set of dynamic equilibria

static Nash

$v^1$ worst dynamic equilibrium

minmax
\[
\min u^1(a)
\]

**Characterization of Equilibrium Payoff**

\[\alpha = (\alpha^1, \alpha^2) \text{ where } \alpha^2 \text{ is a b.r. to } \alpha^1\]

\(\alpha\) represent play in the first period of the equilibrium

\(w^1(a^1)\) represents the equilibrium payoff beginning in the next period

\[v^1 \geq (1 - \delta)u^1(a^1, \alpha^2) + \delta w^1(a^1)\]

\[v^1 = (1 - \delta)u^1(a^1, \alpha^2) + \delta w^1(a^1), \alpha^1(a^1) > 0\]

\[v^1 \leq w^1(a^1) \leq \bar{v}^1\]
Simplified Approach

impose stronger constraint using $n$ static Nash payoff

for best equilibrium $n\leq w^1(a^1)\leq \bar{v}^1$

for worst equilibrium $\underline{v}^1 \leq w^1(a^1) \leq n$

avoids problem of best depending on worst

remark: if we have static Nash = minmax then no computation is needed for the worst, and the best calculation is exact.
**max problem**

fix $\alpha = (\alpha^1, \alpha^2)$ where $\alpha^2$ is a b.r. to $\alpha^1$

$$\bar{v}^1 \geq (1 - \delta)u^1(a^1, \alpha^2) + \delta w^1(a^1)$$

$$\bar{v}^1 = (1 - \delta)u^1(a^1, \alpha^2) + \delta w^1(a^1), \alpha^1(a^1) > 0$$

$$n^1 \leq w^1(a^1) \leq \bar{v}^1$$

how big can $w^1(a^1)$ be in = case?

Biggest when $u^1(a^1, \alpha^1)$ is smallest, in which case

$$w^1(a^1) = \bar{v}^1$$

$$\bar{v}^1 = (1 - \delta)u^1(a^1, \alpha^2) + \delta \bar{v}^1$$
Summary

conclusion for fixed $\alpha$

$$\min_{a^1_{\alpha(a^1)>0}} u^1(a^1, \alpha^2)$$

i.e. worst in support

$$\bar{v}^1 = \max_{a^2 \in BR^2(\alpha^1)} \min_{a^1_{\alpha(a^1)>0}} u^1(a^1, \alpha^2)$$

observe:

mixed precommitment $\geq \bar{v}^1 \geq$ pure precommitment
Peasant-Dictator Example

<table>
<thead>
<tr>
<th></th>
<th>eat</th>
<th>grow</th>
</tr>
</thead>
<tbody>
<tr>
<td>low</td>
<td>0*,1</td>
<td>1,2*</td>
</tr>
<tr>
<td>high</td>
<td>0*,1*</td>
<td>3*,0</td>
</tr>
</tbody>
</table>

$p(\text{low})$  | BR        | worst in support |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>grow</td>
<td>1</td>
</tr>
<tr>
<td>$\frac{1}{2}&lt;p&lt;1$</td>
<td>grow</td>
<td>1</td>
</tr>
<tr>
<td>p=1/2</td>
<td>any mixture</td>
<td>$\leq 1(\text{low})$</td>
</tr>
<tr>
<td>$0&lt;p&lt;\frac{1}{2}$</td>
<td>eat</td>
<td>0</td>
</tr>
<tr>
<td>p=0</td>
<td>eat</td>
<td>0</td>
</tr>
</tbody>
</table>
Check the constraints

\[ w^1(a^1) = \frac{\bar{v}^1 - (1-\delta)u^1(a^1, \alpha^2)}{\delta} \geq n^1 \]

as \( \delta \to 1 \) then \( w^1(a^1) \to \bar{v}^1 \geq n^1 \)
**min problem**

fix $\alpha = (\alpha^1, \alpha^2)$ where $\alpha^2$ is a b.r. to $\alpha^1$

$v^1 \geq (1-\delta)u^1(a^1, \alpha^2) + \delta w^1(a^1)$

$v^1 \leq w^1(a^1) \leq n^1$

Biggest $u^1(a^1, \alpha^1)$ must have smallest $w^1(a^1) = v^1$

$v^1 = (1-\delta)u^1(a^1, \alpha^2) + \delta v^1$

conclusion

$v^1 = \max u^1(a^1, \alpha^2)$

or

$v^1 = \min_{\alpha^2 \in BR^2(a^1)} \max u^1(a^1, \alpha^2)$, that is, constrained minmax
Worst Equilibrium Example

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>M</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>0,-3</td>
<td>1,2</td>
<td>0,3</td>
</tr>
<tr>
<td>D</td>
<td>0,3*</td>
<td>2,2</td>
<td>0,0</td>
</tr>
</tbody>
</table>

static Nash gives 0
minmax gives 0
worst payoff in fact is 0
pure precommitment also 0
mixed precommitment

$p$ is probability of up

to get more than 0 must get SR to play $M$

$-3p + (1 - p)3 \leq 2$ and $3p \leq 2$

first one

$-3p + (1 - p)3 \leq 2$

$-3p - 3p \leq -1$

$p \geq 1/6$

want to play $D$ so take $p = 1/6$

get $1/6 + 10/6 = 11/6$
Utility to long-run player

\[ \max u^1(a) = 2 \]

mixed precommitment/Stackelberg = 11/16

\[ \bar{v}^1 \text{ best dynamic equilibrium} = 1 \]

pure precommitment/Stackelberg = 0

Set of dynamic equilibria

static Nash = 0

\[ \underline{v}^1 \text{ worst dynamic equilibrium} = 0 \]

minmax = 0

\[ \min u^1(a) = 0 \]
calculation of best dynamic equilibrium payoff

$p$ is probability of up

<table>
<thead>
<tr>
<th>$p$</th>
<th>$BR^2$</th>
<th>worst in support</th>
</tr>
</thead>
<tbody>
<tr>
<td>$&lt;1/6$</td>
<td>L</td>
<td>0</td>
</tr>
<tr>
<td>$1/6 &lt; p &lt; 5/6$</td>
<td>M</td>
<td>1</td>
</tr>
<tr>
<td>$p &gt; 5/6$</td>
<td>R</td>
<td>0</td>
</tr>
</tbody>
</table>

so best dynamic payoff is 1