Repeated Games
Long Run versus Short Run Player

a fixed simultaneous move stage game

Player 1 is long-run with discount factor $\delta$
actions $a^1 \in A^1$ a finite set
utility $u^1(a^1,a^2)$

Player 2 is short-run with discount factor 0
actions $a^2 \in A^2$ a finite set
utility $u^2(a^1,a^2)$
What it is about

the “short-run” player may be viewed as a kind of “representative” of many “small” long-run players

♦ the “usual” case in macroeconomic/political economy models
♦ the “long run” player is the government
♦ the “short-run” player is a representative individual
Example 1: Peasant-Dictator
**Example 2: Backus-Driffil**

<table>
<thead>
<tr>
<th></th>
<th>Low</th>
<th>High</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low</td>
<td>0,0</td>
<td>-2,-1</td>
</tr>
<tr>
<td>High</td>
<td>1,-1</td>
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**Inflation Game: LR=government, SR=consumers**

consumer preferences are whether or not they guess right

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with a hard-nosed government
Repeated Game

history $h_t = (a_1, a_2, \ldots, a_t)$
null history $h_0$
behavior strategies $\alpha^i_t = \sigma^i(h_{t-1})$
long run player preferences
average discounted utility

$$
(1 - \delta) \sum_{t=1}^{T} \delta^{t-1} u^i(a_t)
$$

note that average present value of 1 unit of utility per period is 1
Equilibrium

Nash equilibrium: usual definition – cannot gain by deviating

Subgame perfect equilibrium: usual definition, Nash after each history

 Observation: the repeated static equilibrium of the stage game is a subgame perfect equilibrium of the finitely or infinitely repeated game

♦ strategies: play the static equilibrium strategy no matter what
“perfect equilibrium with public randomization”

may use a public randomization device at the beginning of each period to pick an equilibrium

key implication: set of equilibrium payoffs is convex
Example: Peasant-Dictator

normal form: unique Nash equilibrium **high, eat**

<table>
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<tr>
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<th>grow</th>
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<tr>
<td>low</td>
<td>0*,1</td>
<td>1,2*</td>
</tr>
<tr>
<td>high</td>
<td>0*,1*</td>
<td>3*,0</td>
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**Static Benchmarks**

payoff at static Nash equilibrium to LR player: 0

precommitment or Stackelberg equilibrium
precommit to low get 1
mixed precommitment to 50-50 get 2

minmax payoff to LR player: 0
Payoff Space

utility to long-run player

- mixed precommitment/Stackelberg = 2
- best dynamic equilibrium = ?
- pure precommitment/Stackelberg = 1
- Set of dynamic equilibria
- static Nash = 0
- worst dynamic equilibrium = ?
- minmax = 0
Repeated Peasant-Dictator

finitely repeated game
final period: high, eat, so same in every period
Do you believe this??

♦ Infinitely repeated game

begin by low, grow
if low, grow has been played in every previous period then play low, grow
otherwise play high, eat (reversion to static Nash)
claim: this is subgame perfect
When is this an equilibrium?

clearly a Nash equilibrium following a history with high or eat
SR play is clearly optimal

for LR player
may high and get \((1 - \delta)3 + \delta 0\)
or low and get 1

so condition for subgame perfection
\((1 - \delta)3 \leq 1, \delta \geq 2/3\)
Equilibrium Utility

equilibrium utility for LR
General Deterministic Case
Fudenberg, Kreps and Maskin

max $u^1(a)$
mixed precommitment/Stackelberg
$v^1$ best dynamic equilibrium
pure precommitment/Stackelberg
Set of dynamic equilibria

static Nash
$v^1$ worst dynamic equilibrium
minmax
min $u^1(a)$
Characterization of Equilibrium Payoff

\[ \alpha = (\alpha^1, \alpha^2) \] where \( \alpha^2 \) is a b.r. to \( \alpha^1 \)

\( \alpha \) represent play in the first period of the equilibrium

\( w^1(a^1) \) represents the equilibrium payoff beginning in the next period

\[ \nu^1 \geq (1 - \delta)u^1(a^1, \alpha^2) + \delta w^1(a^1) \]

\[ \nu^1 = (1 - \delta)u^1(a^1, \alpha^2) + \delta w^1(a^1), \alpha^1(a^1) > 0 \]

\[ \underline{\nu}^1 \leq w^1(a^1) \leq \overline{\nu}^1 \]
Simplified Approach

impose stronger constraint using $n$ static Nash payoff

for best equilibrium $n \leq w^1(a^1) \leq \bar{v}^1$

for worst equilibrium $v^1 \leq w^1(a^1) \leq n$

avoids problem of best depending on worst

remark: if we have static Nash = minmax then no computation is needed for the worst, and the best calculation is exact.
max problem

fix \( \alpha = (\alpha^1, \alpha^2) \) where \( \alpha^2 \) is a b.r. to \( \alpha^1 \)

\[
\bar{v}^1 \geq (1 - \delta)u^1(a^1, \alpha^2) + \delta w^1(a^1)
\]

\[
\bar{v}^1 = (1 - \delta)u^1(a^1, \alpha^2) + \delta w^1(a^1), \alpha^1(a^1) > 0
\]

\( n^1 \leq w^1(a^1) \leq \bar{v}^1 \)

how big can \( w^1(a^1) \) be in = case?

Biggest when \( u^1(a^1, \alpha^2) \) is smallest, in which case

\[
w^1(a^1) = \bar{v}^1
\]

\[
\bar{v}^1 = (1 - \delta)u^1(a^1, \alpha^2) + \delta \bar{v}^1
\]
Summary

conclusion for fixed $\alpha$

$$\min_{a^1|\alpha(a^1)>0} u^1(a^1, \alpha^2)$$

i.e. worst in support

$$\bar{v}^1 = \max_{\alpha^2 \in BR^2(\alpha^1)} \min_{a^1|\alpha(a^1)>0} u^1(a^1, \alpha^2)$$

observe:

mixed precommitment $\geq \bar{v}^1 \geq$ pure precommitment
**Peasant-Dictator Example**

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<table>
<thead>
<tr>
<th>$p$(low)</th>
<th>BR</th>
<th>worst in support</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>grow</td>
<td>1</td>
</tr>
<tr>
<td>$\frac{1}{2}&lt;p&lt;1$</td>
<td>grow</td>
<td>1</td>
</tr>
<tr>
<td>$p=1/2$</td>
<td>any mixture</td>
<td>$\leq 1$(low)</td>
</tr>
<tr>
<td>$0&lt;p&lt;\frac{1}{2}$</td>
<td>eat</td>
<td>0</td>
</tr>
<tr>
<td>$p=0$</td>
<td>eat</td>
<td>0</td>
</tr>
</tbody>
</table>
Check the constraints

\[ w^1(a^1) = \frac{\bar{v}^1 - (1-\delta)u^1(a^1, \alpha^2)}{\delta} \geq n^1 \]

as \( \delta \to 1 \) then \( w^1(a^1) \to \bar{v}^1 \geq n^1 \)
**min problem**

fix $\alpha = (\alpha^1, \alpha^2)$ where $\alpha^2$ is a b.r. to $\alpha^1$

$$v^1 \geq (1 - \delta)u^1(a^1, \alpha^2) + \delta w^1(a^1)$$

$$v^1 \leq w^1(a^1) \leq n^1$$

Biggest $u^1(\alpha^1, \alpha^2)$ must have smallest $w^1(a^1) = v^1$

$$v^1 = (1 - \delta)u^1(a^1, \alpha^2) + \delta v^1$$

conclusion

$$v^1 = \max u^1(a^1, \alpha^2)$$

or

$$v^1 = \min_{\alpha^2 \in BR^2(a^1)} \max u^1(a^1, \alpha^2), \text{ that is, constrained minmax}$$
**Worst Equilibrium Example**

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>M</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>0, -3</td>
<td>1,2</td>
<td>0,3</td>
</tr>
<tr>
<td>D</td>
<td>0,3*</td>
<td>2,2</td>
<td>0,0</td>
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- static Nash gives 0
- minmax gives 0
- worst payoff in fact is 0
- pure precommitment also 0
mixed precommitment

$p$ is probability of up

to get more than 0 must get SR to play M

\[-3p + (1 - p)3 \leq 2 \quad \text{and} \quad 3p \leq 2\]

first one
\[-3p + (1 - p)3 \leq 2\]
\[-3p - 3p \leq -1\]
$p \geq 1/6$

second one
\[3p \leq 2\]
\[p \leq 2/3\]

want to play D so take $p = 1/6$

get $1/6 + 10/6 = 11/6$
Utility to long-run player

\[
\max u^1(a) = 2
\]

mixed precommitment/Stackelberg = \( \frac{11}{16} \)

\[
\bar{v}^1 \ \text{best dynamic equilibrium} = 1
\]

pure precommitment/Stackelberg = 0

Set of dynamic equilibria

static Nash = 0

\[
\underline{v}^1 \ \text{worst dynamic equilibrium} = 0
\]

\[
\min max = 0
\]

\[
\min u^1(a) = 0
\]
calculation of best dynamic equilibrium payoff

$p$ is probability of up

<table>
<thead>
<tr>
<th>$p$</th>
<th>$BR^2$</th>
<th>worst in support</th>
</tr>
</thead>
<tbody>
<tr>
<td>$&lt;1/6$</td>
<td>L</td>
<td>0</td>
</tr>
<tr>
<td>$1/6&lt;p&lt;5/6$</td>
<td>M</td>
<td>1</td>
</tr>
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so best dynamic payoff is 1