Equity Premium Puzzle
Present Value vs. Average Present Value

infinite discounted utility
\[ \sum_{t=1}^{\infty} \delta^{t-1} u_t \]

average discounted utility
\[ (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_t \]

note that average present value of 1 unit of utility per period is 1

- macro and finance use present value
- game theory uses average present value
- why? Common units across different discount factors
Risk Aversion for Wealth and Consumption

relative risk aversion for wealth versus consumption at steady state

wealth the present value of consumption

\[ W = \frac{c}{1 - \delta} \]

utility of wealth the present value of the utility of consumption

\[ U(W) = u(c)/(1 - \delta) = U((1 - \delta)W)/(1 - \delta) \]

calculate coefficient of relative risk aversion

\[ \rho_W = -U''(W)W/U'(W) = -(1 - \delta)^2 u''(c)W/(1 - \delta)u'(c) \]
\[ = u''(c)c/u'(c) = \rho_c \]
Simple Portfolio Choice Model

have initial wealth $W$

invest a fraction $1 - \alpha$ in safe bonds with certain return $r_b$

$\alpha$ in risky stock with risky return

$r_s = \bar{r}_s + \sigma y$ where $Ey = 0, Ey^2 = 1$

equity premium is defined as $\lambda = \bar{r}_s - r_b$

final wealth is

$$W + W(\alpha r_s + (1 - \alpha)r_b) = W + W(r_b + \alpha(\lambda + \sigma y))$$
Fundamental Risk Equation

\[ U(W + W(r_b + \alpha \lambda + \alpha \sigma y)) \]

derivative with respect to \( \alpha \)

\[ U'(W + Wr_b + W\alpha \lambda + W\alpha \sigma y)(W\lambda + W\sigma y) \]

set \( W' = W + Wr_b + W\alpha \lambda \)

linear approximation to the derivative

\[ U'(W')(W\lambda + W\sigma y) + U''(W')W\alpha \sigma y(W\lambda + W\sigma y) \]

take the expectation and equate to zero

\[ U'(W')W\lambda + U''(W')W^2\alpha \sigma^2 = 0 \] gives

\[ \rho = -U''W/U' = \lambda/(\alpha \sigma^2) \]
Equity Premium and Relative Risk Aversion

Mean real return on bonds $r_b = 1.9\%$;
Mean real return on S&P $\bar{r}_s = 7.5\%$
Equity premium $\lambda = 0.056$
Standard error of real stock return $\sigma = 0.181$
$\rho = \lambda/(\alpha\sigma^2) = 1.81\alpha^{-1}$
that is, at least 1.81
What is the portfolio?

Assume consumption proportional to wealth \( c = \phi W \)

recall final wealth

\[
W_1 = W + W(r_b + \alpha(\lambda + \sigma y))
\]

define

\[
s^2 = \text{var}c/Ec = \text{var}W_1/EW_1 = \alpha^2 \sigma^2 W/W' \approx \alpha^2 \sigma^2
\]

(wealth does not change much in a single period)

in the data \( s = .035 \)

hence \( \alpha^{-1} = \sigma/s = 5.17 \) so \( \rho = 8.84 \)
The real equity premium puzzle

suppose CRRA $u(c) = c^{1-\rho}/(1 - \rho)$

$u'(c) = c^\rho$

cconsumption grows at a constant rate $c_t = \gamma^t$

interest rate determined by indifference condition

$$\frac{1}{1 + r} = \frac{\delta u'(x_{t+1})}{u'(x_t)} = \frac{\delta \gamma^{-\rho(t+1)}}{\gamma^{-\rho t}} = \delta \gamma^{-\rho}$$

average real US per capita consumption growth rate 1.8%

with $\delta = 1$ and $\rho = 8.84$ this gives $r = 17\%$

rather hard to reconcile with mean real return on bonds 1.9%; Mean real return on S&P 7.5%
How does the market react to good news?

Value of claims to future consumption relative to current consumption

\[ c_1 = 1 \]

\[ \sum_{t=2}^{\infty} \frac{\delta^{t-1} u'(c_t)c_t}{u'(1)} \]

\[ \sum_{t=2}^{\infty} \delta^{t-1} \gamma^{-(t-1)\rho} \gamma^{t-1} = \sum_{t=1}^{\infty} \left[ \delta \gamma^{1-\rho} \right]^t = \frac{\delta \gamma^{1-\rho}}{1 - \delta \gamma^{1-\rho}} \]

to be finite we need \( \delta \gamma^{-\rho} < 1 \)

\[ \frac{\partial}{\partial \gamma} \frac{\delta \gamma^{1-\rho}}{1 - \delta \gamma^{1-\rho}} = \frac{\delta (1 - \rho) \gamma^{-\rho}}{(1 - [\delta \gamma^{-\rho}])^2} \]

\( \rho > 1 \) this is negative
Separability

we can’t have both separability between states (expected utility) and separability between periods

we have a strong reason for expected utility and none at all for intertemporal separability

various theories of non-separable time preferences
Risky Drinking

Suppose that all consumption takes place in a nightclub at the beginning of the year before you see your stock return you choose the quality of nightclub you will attend $c^q$.

If you are anticipating low income you choose the cheap beer place. If you are anticipating high income you choose the expensive champagne place.

Utility $u(c|c^q)$

We are going to assume $u(c|c) = \log(c)$

$$u(c|c^q) = \log c^q - \frac{(c/c^q)^{1-\rho}}{\rho-1}$$

Conditional on $c^q$ you have relative risk aversion $\rho$.

But with growth you have intertemporal separability $1$. 


Types of Models

- habit formation
- indivisibility (houses)