Notes on Game Theory

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6 From MWG 9B
7 From (Fudenberg & Levine, )
8 From FT 5.1, 5.2
9 From OR 6.2
10 From MWG 6B
11 From MWG 6C
12 Directly taken from (Angeletos, Laibson, Repetto, Tobacman, & Weinberg, ), with very minor adjustments for readability.
13 From MWG 9C
1. Static games

1 Notation and basic concepts

We are interested in describing the underlying structure of the decision making process. We will assume that there is no coordination between the players (no ‘mind-reading’).

1.1 Strategies

A strategy is a complete contingent plan or decision rule that specifies how the player will act in every possible distinguishable circumstance in which she might be called upon to move. A player’s strategy may include plans for actions that her own strategy makes irrelevant.

More formally:
Let \( \mathcal{H}_i \) denote the collection of player \( i \)'s information sets, \( \mathcal{A} \) the set of possible actions in the game, and \( C(H) \subset \mathcal{A} \) the set of actions possible at information set \( H \). A strategy for player \( i \) is a function \( s_i : \mathcal{H}_i \rightarrow \mathcal{A} \) such that \( s_i(H) \in C(H) \ \forall H \in \mathcal{H}_i \).

Example: Matching Pennies
There are two players with a penny: Player 1 and player 2. Player 1 puts her penny down first with either heads up (H) or tails up (T). After seeing player 1’s choice, player 2 puts her penny down. Strategies for player 1 are H or T, call them respectively \( s^1_1 \) and \( s^2_1 \) while strategies for player 2 specify how she will play after seeing player 1’s move. Player 2 has four strategies:

- \( s^3_2 \): play T if player 1 plays H; play H if player 1 plays T.
- \( s^4_2 \): play T if player 1 plays H; play T if player 1 plays T.

Strategy profile
In an \( I \)-players game a strategy profile is a vector \( s = (s_1, \ldots , s_I) \) where \( s_i \) is the strategy chosen by player \( i \). Call \( S_i \) the set of strategies for player \( i \), then the set of different strategy-profiles is \( S_1 \times \cdots \times S_I \).

In the previous example \( S_1 = \{s^1_1, s^2_1\} \), \( S_2 = \{s^3_2, s^4_2, s^2_2, s^4_2\} \). An example of a strategy profile is \( s = (s^1_1, s^2_2) \). To refer to a strategy profile the we will also use the notation \( (s_i, s_{-i}) \), where \( s_{-i} \) is

\(^1\)From MWG 7D
the vector \( s \) in which the element \( s_i \) has been removed:

\[
s = (s_1, \ldots, s_i, \ldots, s_I) = (s_i, (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_I))
\]

**Normal form**

It is a way of specifying the game directly in terms of strategies and their associated payoffs. A game in strategic form has three elements:

1. The set of players \( i = 1, \ldots, I \);
2. The pure strategy space \( S_i \), which we take to be finite. The strategy profile is \( s \in S \equiv \times_{i=1}^{I} S_i \), which is the Cartesian product of the individual strategies. For player \( i \)'s opponents, we can state \( s_{-i} \in S_{-i} \equiv \times_{j \neq i} S_j \). Note that \( S \) is an \( I \) vector, while \( S_{-i} \) is an \( I - 1 \) vector. Hence, we can write \( S = (S_i, S_{-i}) \).

A **strategy** is a complete contingent plan, or decision rule, that specifies how the player will act in every possible distinguishable circumstance in which she might be called upon to move.

3. The payoff function \( u_i \), which is of the vNM utility function form \( u_i(s) \) for each profile \( s = (s_1, \ldots, s_I) \).

We represent normal form games in a payoff matrix, with player 1 in the rows and player 2 in the columns. Formally, we write:

\[
\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]
\]

For example, the normal form of the Matching Pennies example above:

|       | P2
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>( s_1 )</td>
<td>( s_2 )</td>
<td>( s_3 )</td>
<td>( s_4 )</td>
</tr>
<tr>
<td>( s_1 )</td>
<td>( H )</td>
<td>-1,-1</td>
<td>-1,1</td>
<td>1,-1</td>
</tr>
<tr>
<td>( s_2 )</td>
<td>( T )</td>
<td>1,-1</td>
<td>-1,1</td>
<td>1,-1</td>
</tr>
</tbody>
</table>

The deterministic strategies considered above are referred to as **pure strategies**. We also consider the possibility that a player could randomize when faced with a choice.

### 1.2 Mixed strategy

Given player \( i \)'s (finite) pure strategy set \( S_i \), a mixed strategy for player \( i \), \( \sigma_i : S_i \to [0,1] \), assigns to each pure strategy \( s_i \in S_i \) a probability \( \sigma_i(s_i) \geq 0 \) that it will be played, where \( \sum_{s_i \in S_i} \sigma_i(s_i) = 1 \).

Take an individual \( i \) with \( M \) pure strategies. Then his set of pure strategies is \( S_i = (s_1, \ldots, s_m) \).
The set of possible mixed strategies of individual $i$ is:

$$
\Delta(S_i) = \left\{ (\sigma_1^i, \ldots, \sigma_M^i) \in \mathbb{R}^M : \sigma_m^i \geq 0 \quad \forall m = 1, \ldots, M \text{ and } \sum_{m=1}^{M} \sigma_m^i = 1 \right\}
$$

This simplex $\Delta(S_i)$ is called the mixed extension of $S_i$. Notice that a pure strategy is a special case of a mixed strategy where the probability distribution over the elements of $S_i$ degenerates (to one).

Player $i$’s payoff given a profile of mixed strategies $\sigma = (\sigma_1, \ldots, \sigma_I)$ is:

$$
E_{\sigma}[u_i(s)] = \sum_{s \in S} [\sigma_1(s_1)\sigma_2(s_2)\ldots\sigma_I(s_I)]u_i(s) \equiv u_i(\sigma)
$$

Normal form games where mixed and pure strategies are allowed are defined as: $\Gamma_N = [I, \{\Delta S_i\}, \{u_i(\cdot)\}]$.

Example:

<table>
<thead>
<tr>
<th></th>
<th>P1</th>
<th>P2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>C</td>
<td>D</td>
</tr>
<tr>
<td>A</td>
<td>a,c</td>
<td>a,d</td>
</tr>
<tr>
<td>B</td>
<td>b,c</td>
<td>d,b</td>
</tr>
</tbody>
</table>

Take a mixed strategy profile $\sigma = (\sigma_1, \sigma_2)$, where $\sigma_1 = (\sigma_1(A), \sigma_1(B))$ and $\sigma_2 = (\sigma_2(C), \sigma_2(D))$. The expected utility for player 1 of playing $\sigma_1$ is:

$$
u_1(\sigma_1, \sigma_2) = \sigma_1(A)\sigma_2(C)u_1(A, C)+\sigma_1(B)\sigma_2(C)u_1(B, C)+\sigma_1(A)\sigma_2(D)u_1(A, D)+\sigma_1(B)\sigma_2(D)u_1(B, D)$$

The expected utility is the sum of the payoff of the pure strategies times the probability that each strategy occurs.
2 Dominant and dominated strategies

In this section we will compare player’s possible strategies, distinguishing between situations when randomization is not and is allowed.\(^3\)

2.1 Pure strategies

Let’s first restrict our focus to situations when we do not allow for randomization. Consider the prisoner’s dilemma game:

<table>
<thead>
<tr>
<th></th>
<th>P1</th>
<th>P2</th>
</tr>
</thead>
<tbody>
<tr>
<td>NC</td>
<td>-2, -2</td>
<td>-10, -1</td>
</tr>
<tr>
<td>NC</td>
<td>-2, -2</td>
<td>-10, -1</td>
</tr>
<tr>
<td>C</td>
<td>-1, -10</td>
<td>-5, -5</td>
</tr>
</tbody>
</table>

**Strictly dominant strategy**

A strategy is a strictly dominant strategy for player \(i\) if it maximizes uniquely player \(i\)’s payoff for any strategy that player \(i\)’s rivals might play.

More formally:

A strategy \(s_i \in S_i\) is a strictly dominant strategy for player \(i\) in game \(\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]\) if for all \(s_i' \neq s_i\), we have:

\[ u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}) \quad \forall s_{-i} \in S_{-i} \]

In the prisoner’s dilemma, confess (\(C\)) is a strictly dominant strategy for both players.

**Strictly dominated strategy**

A strategy \(s_i \in S_i\) is strictly dominated for player \(i\) in game \(\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]\) if there exist another strategy \(s_i' \in S_i\) such that for all \(s_{-i} \in S_{-i}\) we have:

\[ u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}) \]

In this case, we say that strategy \(s_i'\) strictly dominates strategy \(s_i\). Note how, using this definition, a strictly dominant strategy for player \(i\) strictly dominates every other strategy in \(S_i\).

In the prisoner’s dilemma game, not confess (\(NC\)) is a strictly dominated strategy.

**Weakly dominated strategy**

A strategy is weakly dominated if another strategy does at least as well for all \(s_{-i}\) and strictly better for some \(s_{-i}\).

\(^3\)From MWG 8B
Formally:
A strategy \( s_i \in S_i \) is weakly dominated for player \( i \) in game \( \Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}] \) if there exist another strategy \( s_i' \in S_i \) such that for all \( s_{-i} \in S_{-i} \) we have:
\[
u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i})
\]
with strictly inequality for some \( s_{-i} \). In this case, we say that strategy \( s_i' \) weakly dominates strategy \( s_i \). A strategy is a weakly dominant strategy for player \( i \) in game \( \Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}] \) if it weakly dominates every other strategy in \( S_i \).

2.2 Mixed strategies

Let’s now allow for mixed strategies.

**Strictly dominated mixed strategy**
A strategy \( \sigma_i \in \Delta(S_i) \) is strictly dominated for player \( i \) in game \( \Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}] \) if there exists another strategy \( \sigma_i' \in \Delta(S_i) \) such that for all \( \sigma_{-i} \in \prod_{j \neq i} \Delta S_j \) we have:
\[
u_i(\sigma_i', \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i})
\]

**Small trick:** when you test whether \( \sigma_i \) is strictly dominated by \( \sigma_i' \) for player \( i \), you only need to consider these two strategies’ payoff against the pure strategies of \( i \)’s opponents. That is:
\[
u_i(\sigma_i', \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i}) \quad \forall \sigma_{-i}
\]
if and only if
\[
u_i(\sigma_i', s_{-i}) > u_i(\sigma_i, s_{-i}) \quad \forall s_{-i}
\]

**Proposition:** Player \( i \)’s pure strategy \( s_i \in S_i \) is strictly dominated in game \( \Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}] \) if and only if there exists another strategy \( \sigma_i' \in \Delta(S_i) \) such that:
\[
u_i(\sigma_i', s_{-i}) > u_i(s_i, s_{-i}) \quad \forall s_{-i} \in S_{-i}
\]

**Example:**

<table>
<thead>
<tr>
<th></th>
<th>P1</th>
<th>P2</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>10,1</td>
<td>0.4</td>
</tr>
<tr>
<td>M</td>
<td>4.2</td>
<td>4.3</td>
</tr>
<tr>
<td>D</td>
<td>0.5</td>
<td>10.2</td>
</tr>
</tbody>
</table>

Strategy \( M \) is strictly dominated by the randomized strategy \( \frac{1}{2}U + \frac{1}{2}D \).
3 Equilibrium concepts

3.1 Rationalizable strategies

Best response
In game $\Gamma_N = [I, \Delta(S_i), u_i(.)]$, strategy $\sigma_i$ is a best response for player $i$ to his rivals’ strategies $\sigma_{-i}$ if:

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}) \quad \forall \sigma_i' \in \Delta(S_i)$$

Strategy $\sigma_i$ is never a best response if there is no $\sigma_{-i}$ for which $\sigma_i$ is a best response.

In the above example, if we only allow for pure strategies:

$$BR_1(L) = U \quad BR_2(U) = R$$

$$BR_1(R) = D \quad BR_2(M) = R$$

$$BR_2(D) = L$$

Iterated elimination of strictly dominated strategies
A player who has a strictly dominant strategy is expected to play it, whereas strictly dominated strategies are not expected to be played.

The first thing to do when faced with an exercise is to spot the pure strictly dominated strategies and remove them (no mixed strategy will put positive probability on a pure strategy that is strictly dominated). Assuming that players know the structure of the game, each others payoff, and believe that the other players are rational, you can remove not only strictly dominated strategy but also the strategies that become strictly dominated because of previous deletion of strictly dominated strategy. This is called iterated deletion of strictly dominated strategies.

We iteratively eliminate all strictly dominated strategies up to the point where there are no more strictly dominated strategies. The order of deletion does not matter. If you try performing iterated dominance deletion with weak dominated strategies, however, the order of deletion will matter.

Rationalizable strategies
Performing iterated removal of never best response strategies leads to rationalizable strategies.

In game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(.)\}]$, the strategies in $\Delta(S_i)$ that survive the iterated removal of strategies that are never a best response are known as player $i$’s rationalizable strategies.

If there is only one strategy profile surviving iterated strict dominance, the game is called dominance solvable. Any dominance solvable game has a unique NE.

---

4From MWG 8C
3. EQUILIBRIUM CONCEPTS

3.2 Nash equilibrium

3.2.1 Definition and concepts

Pure strategy Nash equilibrium
A strategy profile \( s = (s_1, \ldots, s_I) \) constitutes a Nash equilibrium of the game \( \Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}] \) if for every \( i = 1, \ldots, I \):

\[
u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}) \quad \forall s'_i \in S_i \]

In a Nash equilibrium, each player’s strategy choice is a best response to the strategies played by his rivals. Hence, no player will find a profitable deviation from a Nash equilibrium.

We can also state the NE in terms of best responses. Define player \( i \)'s best response correspondence \( BR_i : S_{-i} \rightarrow S_i \) in the game \( \Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}] \) as the correspondence that assigns to each \( s_{-i} \in S_{-i} \) the set \( BR_i(s_{-i}) = \{s_i \in S_i : u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}) \quad \forall s'_i \in S_i\} \).

The strategy profile \( (s_1, \ldots, s_I) \) is a Nash equilibrium of game \( \Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}] \) if and only if \( s_i \in BR_i(s_{-i}) \) for \( i = 1, \ldots, I \).

Nash equilibrium with mixed strategies
A mixed strategy profile \( \sigma = (\sigma_1, \ldots, \sigma_I) \) constitutes a Nash equilibrium of game \( \Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}] \) if for every \( i = 1, \ldots, I \):

\[
u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma'_i, \sigma_{-i}) \quad \forall \sigma'_i \in \Delta(S_i) \]

From this definition we can make the following two propositions, which are useful for solving exercises.

**Proposition.** Let \( S^+ \subset S_i \) denote the set of pure strategies that player \( i \) plays with positive probability in the mixed strategy profile \( \sigma = (\sigma_1, \ldots, \sigma_I) \). Strategy profile \( \sigma \) is a Nash equilibrium in game \( \Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}] \) if \( \forall i = 1, \ldots, I \):

\[
u_i(s_i, \sigma_{-i}) = u_i(s'_i, \sigma_{-i}) \quad \forall s_i, s'_i \in S^+_i \quad ***
\]

\[
u_i(s_i, \sigma_{-i}) \geq u_i(s'_i, \sigma_{-i}) \quad \forall s_i \in S^+_i \quad \text{and} \quad \forall s'_i \notin S^+_i
\]

Condition \( *** \) is very powerful for finding a mixed strategy NE.

The proposition states that a necessary and sufficient condition for the mixed strategy profile \( \sigma \) to be a NE is that each player, given the distribution of strategies played by his opponents, is indifferent among all the pure strategies that he plays with positive probability and that these pure strategies are at least as good as any pure strategy he plays with zero probability.

**Example:** Find the MSNE

---

\(^5\)From MWG 8D
CHAPTER 1. STATIC GAMES

\[
\begin{array}{c|cc}
  & P1 & P2 \\
\hline
U & 1,1 & -1,1 \\
D & -1,-1 & 0,0 \\
\end{array}
\]

Call \( \sigma_u \) the probability that P1 playS \( U \) and \( 1 - \sigma_u \) the probability that he plays D. Then equate the pure strategy payoff of P2:

- Payoff of playing L given P1’s strategy: \( 1 \sigma_u + (1 - \sigma_u)(-1) \)
- Payoff of playing R given P1’s strategy: \( -1(\sigma_u) \)
- Equate them to find \( \sigma_u \): \( 1 \sigma_u + (1 - \sigma_u)(-1) = -1(\sigma_u) \rightarrow \sigma_u = \frac{1}{3} \).

To find the MSNE we’d do the same for the other player.

**Proposition** (Corollary). The pure strategy profile \( s = (s_1, \ldots, s_I) \) is a NE of game \( \Gamma_N = [I, \{(S_i)\}, \{u_i(\cdot)\}] \) iff it is a degenerate mixed strategy NE of the game \( \Gamma'_N = [I, \{\Delta S_i\}, \{u_i(\cdot)\}] \).

This corollary tells you that to identify the pure strategy equilibria of the game \( \Gamma'_N = [I, \{\Delta S_i\}, \{u_i(\cdot)\}] \), it suffices to restrict attention to the game \( \Gamma_N = [I, \{(S_i)\}, \{u_i(\cdot)\}] \) in which randomization is not permitted. In other words, the pure strategy NE of the mixed game are the same than those in the pure strategy game.

### 3.2.2 NE, strict and weak dominance

**Proposition.**

A profile is a NE of a game \( \Gamma \) if and only if it is the NE of the game in which strategies have been removed by \( N \) iterated strict dominance \( \Gamma_N \).

**Proposition.** A NE of a game in which strategies have been removed by iterated weak dominance \( \Gamma^W_N \) is a NE of the original game. The converse is not true: a NE of the original game can be removed by weak iteration.

**Proofs.** PS1 Excercise 2.

### 3.3 Existence of a NE

**Existence Theorem**

Every finite strategic form game, in which the sets \( S_1, \ldots, S_N \) have a finite number of elements, has a mixed strategy equilibrium.

---

\(^6\)From FT 1.3
3. EQUILIBRIUM CONCEPTS

The theorem fails in pure strategies. For example, matching pennies.

**Proof.** To prove this theorem we apply Kakutani’s fixed point theorem for correspondences. \(^7\)

Suppose that \(A \subset \mathbb{R}^N\) is a nonempty compact, convex set, and that \(f : A \rightrightarrows A\) is an upper hemicontinuous correspondence from \(A\) into itself, with the property that the set \(f(x) \subset A\) is nonempty and convex for every \(x \in A\). Then \(f(\cdot)\) has a fixed point; that is, there is an \(x \in A\) such that \(x \in f(x)\).

Applied to our context: an upper hemicontinuous (UHC) convex-valued correspondence \(BR\) from a compact, nonempty convex subset \(BR : \Sigma \rightrightarrows \Sigma\) to itself has a fixed point \(\sigma \in BR(\sigma)\).

It means that when everyone plays a BR, we are back in a BR. A sequence of BR converges to a BR. Mathematically: when \(b\) is a BR to \(\sigma\), then \(b \in BR(\sigma)\).

From Kakutani’s fixed point theorem, the following are sufficient conditions for \(BR : \Sigma \rightrightarrows \Sigma\) to have a fixed point.

1. \(\Sigma\) is compact, convex, non-empty subset of a finite-dimensional Euclidean space.
2. \(BR(\sigma)\) is nonempty for all \(\sigma\).
3. \(BR(\sigma)\) is convex for all \(\sigma\).
4. UHC: \(BR(\sigma)\) has a closed graph. If \((\sigma^n, \hat{\sigma}^n) \to (\sigma, \hat{\sigma})\) with \(\hat{\sigma}^n \in BR(\sigma^n)\), then \(\hat{\sigma} \in BR(\sigma)\).

Going through each item separately:

1. Consider the \(\Sigma\) simplex of mixed strategies, with dimension \((\#S_i - 1)\). Its main characteristic is that the elements (probability vectors) are non-negative and they sum up to 1.

   It is compact and convex: the convex combination of two different mixed strategies remains in the set. The product of two convex sets is convex.

2. Trially non-empty.

3. The \(BR\) correspondence must be continuous: the correspondence (straight line) must be convex. Let \(BR_i(\sigma)\) be the set of best responses of \(i\) to \(\sigma_{-i}\). This set is convex valued: the convex combination (randomization) of a BR is itself a BR. Specifically, since we must be indifferent between all pure strategies played with positive probability, the BR set is the set of all convex combinations of the pure strategies that are BR.

4. That a correspondence \(BR : \Sigma \to \Sigma\) is UHC means that if \(\sigma^n \to \sigma\) such that \(b^n \in BR(\sigma^n)\), when \(b^n \to b\) then \(b \in BR(\sigma)\).

---

\(^7\)In game theory correspondences are much more frequent than functions. To grasp the general idea, consider the same theorem but applied to functions.

Brouwer’s Fixed Point Theorem:

Suppose that \(A \subset \mathbb{R}^N\) is a nonempty, compact, convex set, and that \(f : A \to A\) is a continuous function from \(A\) into itself. Then \(f(\cdot)\) has a fixed point; that is, there is an \(x \in A\) s.t. \(x = f(x)\).
Figure 1.1: Illustration of BR correspondence

\[ b^n \in BR(\sigma^n), \text{ with } b^n \to b, \text{ means that } u_i(b^n_i, \sigma^n_{-i}) \geq u_i(\sigma^n_i, \sigma^n_{-i} - 1). \] Suppose by contradiction that \( b \notin BR(\sigma), \) but for every other point the strategies do belong to \( BR. \) This means that \( u_i(\hat{\sigma}, \sigma_{-i}) > u_i(b_i, \sigma_{-i}) \) for some \( \hat{\sigma}. \) Since \( \sigma^n_{-i} \to \sigma_{-i} \) for \( n \) sufficiently large, and \( u_i \) is continuous in \( \sigma_{-i} \), we have \( u_i(\hat{\sigma}, \sigma^n_{-i}) > u_i(b_i, \sigma^n_{-i}) \to u_i(\hat{\sigma}, \sigma_{-i}) > u_i(b_i, \sigma_{-i}) \), as the limit preserves the strict inequality.

Since \( b^n_i \to b_i \) and \( u_i \) is continuous (linear) in \( \sigma_i \), also for \( n \) sufficiently large \( u_i(\hat{\sigma}, \sigma^n_{-i}) > u_i(b^n_i, \sigma^n_{-i}). \) This contradicts \( b^n_i \in BR(\sigma^n). \)

The condition of a finite number of elements for the strategy set of an individual can be a very strict requirement for economic applications. We frequently encounter games in which players have strategies naturally modeled as continuous variables: a production game or an auction. Then, we can apply the following alternative theorem:

**Proposition.**
A NE exists in an infinite game \( \Gamma = [I, \{\Delta S_i\}, \{u_i(\cdot)\}] \) if, \( \forall i: \)

- \( S_i \) is a non-empty, convex and compact subset of some Euclidean subspace \( \mathbb{R}^S \)

- \( u_i(s_1, \ldots, s_N) \) is continuous in \( (s_1, \ldots, s_N) \) and quasi-concave in \( s_i. \)

\(^8\text{Recall that } u_i(\sigma) \text{ is continuous (multilinear): small changes in probabilities have little effect in utilities. Following the EU theorem, utilities are in fact continuous in probabilities.}\)
3. EQUILIBRIUM CONCEPTS

3.4 Correlated equilibrium

3.4.1 Introduction and motivation

Correlated equilibria are closely related to the notion of mixed strategies. To see this, let us first give somewhat of an alternative interpretation to what a mixed strategy is. Before selecting his action, a player may receive random private information, inconsequential from the point of view of the other players, on which his action may depend. The player may consciously or unconsciously use this private information (or signal) to make the decision on this action. What is important is that, under a MSNE, we assume that the other players do not observe this signal or consider it irrelevant for the equilibrium outcome, and hence do not take it into account when playing their strategy. 9 10

In many cases, the outcome predicted by MSNE can be unsatisfactory. Take as example the Meeting in New York game: Miguel and Ismael have to meet in New York but they cannot communicate. There are only two well-known places to meet in New York: the Empire State building and Grand Central station. They both arrive with the train at Grand Central, so it is better to meet there than to meet at the Empire State.

<table>
<thead>
<tr>
<th>Miguel</th>
<th>Ismael</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ES</td>
</tr>
<tr>
<td>ES</td>
<td>100,100</td>
</tr>
<tr>
<td>GC</td>
<td>0,0</td>
</tr>
</tbody>
</table>

\[ MSNE = \left\{ (\sigma_1(ES) = 1, \sigma_2(ES) = 1); (\sigma_1(ES) = 0, \sigma_2(ES) = 0); \left( \sigma_1(ES) = \frac{10}{11}, \sigma_2(ES) = \frac{10}{11} \right) \right\} \]

Note first that in the model there are players, strategies and utility functions, but there is no public device to randomize choices. Second, note that if the guys try to meet 121 times, 100 hundred of times they meet in the worst place (Empire State), 20 times they do not meet at all (and that is pretty frustrating), and only 1 time they meet in Grand Central, which is the outcome yielding the highest utility for both players.

9As an example, consider a coin toss. Everyone agrees that the probability of getting heads or tails is one half each. But this is not quite right. Once the coin is flipped, if I know the strength I applied to the coin when I flipped it, the height from which I flipped it and the point of arrival, and I have a friend who is very good in physics, we can exactly say if it would land as heads or tails. But we don’t usually do this calculation and so once I flip the coin the probability is one half, one half.

This line of reasoning can be applied to any phenomenon. If we were suffering from gambling addiction, we could bet everyday on whether the price of the same schiacciata at Piatti e Fagotti goes up or down. If you have gone to P&F enough, you will probably agree that the price is random, so the probability that it goes up is one half, and that it goes down is one half. If corona has precluded you from going to P&F to verify this, check with Damiano.

10A note on coordination:
An alternative interpretation comes from the theory of focal points. It suggest that in some real-life situations, players may be able to coordinate on a particular equilibrium by using information that is abstracted away by the strategic form. For example, when guessing a time, noon versus 1:43; or when deciding on plays based on the names of the strategies, say “top-left”.

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\[ 9 \] As an example, consider a coin toss. Everyone agrees that the probability of getting heads or tails is one half each. But this is not quite right. Once the coin is flipped, if I know the strength I applied to the coin when I flipped it, the height from which I flipped it and the point of arrival, and I have a friend who is very good in physics, we can exactly say if it would land as heads or tails. But we don’t usually do this calculation and so once I flip the coin the probability is one half, one half.

\[ 10 \] A note on coordination:
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Correlated equilibria will allow us to model players’ randomization processes and produce outcomes that are more satisfying. In a correlated equilibria we assume that the signals are not private and independent, but public and less-than-perfectly correlated. Before defining correlated equilibria, let us define some useful concepts using the toss of a coin as an event.

1. The set of all possible outcomes is called $\Omega = \{\omega_1, \ldots, \omega_l\}$. So, in our example heads $H$ and tails $T$ are the two states of the world.

2. The probability measure over the states of the world is $\pi$. In our case $\pi(H) = \pi(T) = \frac{1}{2}$.

3. A partition $P_i$ of a set $\Omega$ is a collection of subsets $P_i \in \Omega$ so that every element $\omega \in \Omega$ occurs in exactly one of the partitions $P_i$. Clearly, two different $\omega$ can belong to the same partition. We will assume that if two elements belong to the same partition the player cannot disentangle between them but only knows that an element of the partition has occurred.

**Example: Dice as a randomization device**

Two players coordinate on an outcome using a dice as a randomization device. The first player is almost blind and cannot recognize between 1, 2 and 3, but can tell if one of those numbers is drawn that is smaller or equal than 3, and the same for the number greater than 3. The second player is tipsy and can only tell apart 1 and 6, and confounds all numbers in-between.

1. $\Omega = \{1, 2, 3, 4, 5, 6\}$

2. Probability measure $\pi(\omega) = \frac{1}{6} \forall \omega$.

3. The partition of $P_1$ is $P_1 = \{P_1^1 = \{1, 2, 3\}; P_1^2 = \{4, 5, 6\}\}$. So, if for example the outcome of the dice is 2, $P_1$ does not know if it is 1, 2, or 3 but from Bayes rules he knows that the probability that it is one among 1, 2, or 3 is one third.

$$
\Pr(1|P_1^1 = \{1, 2, 3\}) = \frac{\Pr(P_1^1 | 1) \cdot \Pr(\omega = 1)}{\Pr(P_1^1)} = \frac{1 \cdot \frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}
$$

4. The partition for $P_2$ is $P_2 = \{\{1\}, \{2, 3, 4, 5\}, \{6\}\}$. Every time a number gets drawn he will do the same kind of computation (or at least he will try!).

One might also encounter the special case where there are as many partitions as states of the world. In this case, whichever state of the world is realized, every player can recognize it, so there is no uncertainty.

Finally, it is important to understand in this setting what a strategy for a player is. A strategy becomes a decision rule that associates an action to any state of the world.  

---

11Going back to our schiacciata in San Domenico example: the possible action of the players are buy or don’t buy the schiacciata. The possible strategy of a player are: $\sigma_1 = (\sigma(UP) = \text{don’t buy, } \sigma(DOWN) = \text{buy})$, or $\sigma_2 = (\sigma(DOWN) = \text{don’t buy, } \sigma(UP) = \text{buy})$. Note that now a strategy is a function that associates an action to each state of the world.
3.4.2 Definition of correlated equilibrium \(^{12}\)

**Correlated equilibrium**

A correlated equilibrium of a strategic game \(\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]\) consist of:

1. A finite probability space \((\Omega, \pi)\), where \(\Omega\) is a set of states and \(\pi\) is a probability measure on \(\Omega\).
2. For each player \(i \in I\) a partition \(P_i\) of \(\Omega\), that is, player \(i\)'s information partition.
3. For each player \(i \in I\) a function \(\sigma : \Omega \rightarrow S_i\) with \(\sigma_i(\omega) = \sigma_i(\omega')\) whenever \(\omega \in P_i\) and \(\omega' \in P_i\) for some \(P_i \in P_i\), where \(\sigma_i\) is player \(i\)'s strategy.

such that for every \(i \in I\) and every function \(\tau_i : \Omega \rightarrow S_i\) for which \(\tau(\omega) = \tau(\omega')\) whenever \(\omega \in P_i\) and \(\omega' \in P_i\) for some \(P_i \in P_i\) (i.e. for every strategy of player \(i\)) we have:

\[
\sum_{\omega \in \Omega} \pi(\omega)u_i(\sigma_i(\omega), \sigma_{-i}(\omega)) \geq \sum_{\omega \in \Omega} \pi(\omega)u_i(\tau_i(\omega), \sigma_{-i}(\omega)) \tag{1.1}
\]

A couple of things to note:

- The probability space \((\Omega, \pi)\) and the information partition are part of the equilibrium. You have to define your probability space and the information partitions when you talk about correlated equilibrium. A different probability space or a different partition define a different equilibrium.

- The strategy of a player has to attribute the same action to all the states of the world belonging to the same partition (players cannot disentangle between states of the world holding to the same \(P_i\)).

- A strategy is a function that goes from the states of the world to the actions. A strategy is a decision rule that attaches, to any state of the world, an action.

- The rest is the usual Nash equilibrium: for every state of the world \(\omega\) that occurs with positive probability the action \(\sigma_i(\omega)\) is optimal given the other players’ strategies and player \(i\)'s knowledge about \(\sigma\).

- Players know the other players’ partitions.

We have started this section by explaining that we will explore correlated equilibria because mixed strategy equilibria can be unsatisfactory. To show that the correlated equilibrium concept improves on the mixed strategy equilibrium concept, we will first show that the set of correlated equilibria contains the set of mixed strategy Nash equilibria.

\(^{12}\)From OR 3.3
Proposition. For every mixed strategy Nash equilibrium $\alpha$ of a finite strategic game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$, there is a correlated equilibrium $[(\Omega, \pi), (P_i), \sigma_i]$ in which for each player $i \in I$ the distribution on $S_i$ induced by $\sigma_i$ is $\alpha_i$.

That is, we can construct a correlated equilibrium in which the probability of a player playing a strategy is the same as in the mixed strategy equilibrium of the game.

Let’s see this using an example:

$$\begin{array}{c|cc}
P1 & P2 \\
\hline
L & 2,1 & 0,0 \\
U & 0,0 & 1,2 \\
D & 0,0 & 1,2 \\
\end{array}$$

There are three mixed strategy equilibria:

$$MSNE = \left\{ \{\sigma_1(u) = 1, \sigma_2(L) = 1\}; \{\sigma_1(L) = 0, \sigma_2(U) = 0\}; \{\sigma_1(U) = \frac{2}{3}, \sigma_2(L) = \frac{1}{3}\} \right\}$$

We will take as example the pure mixed strategy NE $\{\sigma_1(U) = \frac{2}{3}, \sigma_2(L) = \frac{1}{3}\}$

1. Let $\Omega = \{UL, UR, DL, DR\}$ and define $\pi(\omega) = \prod_{j \in I} \alpha_j(\omega_j)$ In our example, for the fully mixed strategy equilibrium, $\pi(UL) = \frac{2}{9}, \pi(UR) = \frac{4}{9}, \pi(DL) = \frac{1}{9}, \pi(DR) = \frac{2}{9}$.

2. For each $i \in I$, and $s_i \in \Omega_i$, for $P1$ $U$ and $D$, let $P_i(s_i) = \{\omega \in \Omega : \omega_i = s_i\}$ and let $P_i$ consists of the $\Omega_i$ sets $P_i(s_i)$. So, in our case for $s_1 = U$ $P_1(U) = \{UL, UR\}$ which are the elements in $\Omega$ for which the strategy played by player 1 is $U$.

Hence, the partition for each player is:

$$P_1 = \{\{UL, UR\}, \{DL, DR\}\}$$
$$P_2 = \{\{UL, DL\}, \{UR, DR\}\}$$

3. Define $\sigma_i(\omega) = \omega_i$ for each $\omega \in \Omega$. That is, in our example:

$$\sigma_1(UL) = \sigma_1(UR) = U; \sigma_1(DL) = \sigma_1(DR) = D$$
$$\sigma_2(UL) = \sigma_2(DL) = L; \sigma_2(UR) = \sigma_2(DR) = R$$

This $[(\Omega, \pi), (P_i), \sigma_i]$ is a correlated equilibrium, since there is no profitable deviation for any player. Equation (1.1) is satisfied for every strategy $\tau_i$: the LHS of (1.1) is player $i$’s payoff in the mixed strategy NE $\alpha$ and the RHS is his payoff using a different mixed strategy $\tau_i(s)$ when every other player follows the mixed strategy $\alpha_{-i}$. Further, the distribution on $S_i$ induced by $\sigma_i$ is $\alpha_i$.

In the game above there are three mixed strategy Nash equilibrium payoff profiles: $(2, 1)$, $(1, 2)$, $(\frac{2}{3}, \frac{2}{3})$. 
Here, thanks to the correlated equilibrium, we are able to generate a new equilibrium payoff profile. Let me construct a correlated equilibrium that will yield \((\frac{3}{2}, \frac{3}{2})\) as a payoff profile. This time, I will use a slightly different notion of correlated equilibria. Suppose that we can have a device sending signals \(x\) and \(y\), so that the players associated these signals with actions:

1. \(\Omega = \{x, y\}\), \(\pi(x) = \pi(y) = \frac{1}{2}\)
2. \(\mathcal{P}_1 = \mathcal{P}_2 = \{\{x\}, \{y\}\}\)
3. Construct the strategies such that the desired payoff profile is achieved: one half of the time \(x\) is drawn and leads payoff \((2,1)\), and one half \(y\) is drawn and leads payoff \((1,2)\).

\[
\begin{align*}
\sigma_1(x) &= U \quad \sigma_1(y) = D, \\
\sigma_2(x) &= L \quad \sigma_2(y) = R
\end{align*}
\]

So, the payoff profile is \((\frac{3}{2}, \frac{3}{2})\).

**Proposition.** Let \(\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]\) be a strategic game. Any convex combination of correlated equilibrium payoff profiles of \(\Gamma_N\) is a correlated equilibrium payoff profile of \(\Gamma_N\).

Intuitively, this means that we can construct multiple correlated equilibria and then consider a public randomization device that determines which of the constructed correlated equilibria is to be played.

If we combine the last two propositions, we get a very powerful result: any convex combination of mixed strategy equilibrium payoffs can be reached by a correlated equilibrium. More importantly, one can reach payoff profiles that are outside the convex hull of the mixed strategy NE payoff profiles.

**Example:**

<table>
<thead>
<tr>
<th>Game</th>
<th>Messages</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>P2</td>
</tr>
<tr>
<td>L</td>
<td>R</td>
</tr>
<tr>
<td>6,6</td>
<td>2,7</td>
</tr>
<tr>
<td>D</td>
<td></td>
</tr>
<tr>
<td>7,2</td>
<td></td>
</tr>
<tr>
<td>0,0</td>
<td></td>
</tr>
</tbody>
</table>

The MSNE payoffs of the game are \((2, 7), (7, 2), (\frac{14}{3}, \frac{14}{3})\).

1. Let \(\Omega = \{x, y, z\}\) and \(\pi(x) = \pi(y) = \pi(z) = \frac{1}{3}\).
2. \(\mathcal{P}_1 = \{\{x\}, \{y, z\}\}\) and \(\mathcal{P}_2 = \{\{x, y\}, \{z\}\}\).

\(^{13}\) Use the probability of the realization of each outcome, computed earlier in the “proof”, and multiply it by each players’ payoff.
3. Define the following strategies:

$$
\begin{align*}
\sigma_1(x) &= D & \sigma_1(y) &= \sigma_1(z) &= U \\
\sigma_2(z) &= R & \sigma_2(y) &= \sigma_2(x) &= L
\end{align*}
$$

See how player 1’s behavior is optimal given player 2’s strategy: in state $x$, player 1 recognizes with certainty it is $x$. As player 1 knows player 2’s strategy, he knows that upon $x$ being drawn, player 2 will associate it to the partition requiring him to play $L$, and so $D$ is his best response; in state $y$ P1 does not know if it is $y$ or $z$. From Bayes rule he assumes it is with probability one half $y$ and one half $z$. Player 1 expects P2 to play with one half probability $R$ and with one half $L$, and so $U$ is his best response. By symmetry we would find the same results for player 2.

Hence this is a correlated equilibrium. When $y$ is drawn the outcome is $(U, L)$, when $x$ is drawn $(D, L)$, and when $z$ is drawn $(U, R)$. Given the probabilities assigned to $x, y, z$ the payoff profile is $(5, 5)$.

### 3.4.3 Constructing payoff profiles

**Proposition.** Let $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ be a finite strategic game. Every probability distribution over outcomes that can be obtained in a correlated equilibrium of $\Gamma_N$ can be obtained in a correlated equilibrium in which the set of states is $\Omega$, and for each $i \in I$ player $i$’s information partition consists of all sets of the form $\{\omega \in \Omega : \omega_i = s_i\}$ for some action $s_i \in \Omega_i$.

You would be fine then with strategy $\sigma_i(\omega) = \omega_i$.

Now I will construct a correlated equilibrium like the one above. In particular I want to make them play the outcomes $(UL, DL, UR)$ with one third of probability:

<table>
<thead>
<tr>
<th></th>
<th>P2</th>
<th>P1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>L</td>
<td>R</td>
</tr>
<tr>
<td>U</td>
<td>6.6</td>
<td>2.7</td>
</tr>
<tr>
<td>D</td>
<td>7.2</td>
<td>0.0</td>
</tr>
</tbody>
</table>

1. Let $\Omega = \{(LU), (UR), (DL), (DR)\}$ and $\pi(LU) = \pi(UR) = \pi(DL) = \frac{1}{3}$ and $\pi(DR) = 0$.
2. $\mathcal{P}_1 = \{UL, UR\}, \{DL, DR\}$, $\mathcal{P}_2 = \{UL, DL\}, \{UR, DR\}$.
3. Consider the strategy $\sigma_1(UL) = \sigma_1(UR) = U$, $\sigma_1(DL) = \sigma_1(DR) = D$ and $\sigma_2(UL) = \sigma_2(DL) = L$, $\sigma_2(DR) = \sigma_2(UR) = R$.

Now, when $UL$ is drawn P1 does not distinguish between $UL$ or $UR$ and so for him P2 plays with equal probability $R$ or $L$, so that $U$ is optimal. When $UR$ is drawn the same reasoning applies. When $DL$ is drawn P1 is sure that is DL, as DR has probability 0 of being drawn. Hence, he is sure that P2 plays $L$ and $D$ is his optimal choice.
For player 2 when $UL$ is drawn he does not distinguish between $UL$ or $DL$, and so if expecting $U$ and $D$ with equal probability, he plays $L$. Similarly, when $DL$ is drawn he plays $L$. When $UR$ is drawn, however, he is sure about it and so he knows that P1 plays $U$ and so he plays $R$.

So, the above strategy profile, $\Omega, \pi$ and its partitions are a correlated equilibirum. In addition, the probability induced over the outcomes is exactly the probability assigned to the outcome as a state of the world: when $UL$ is drawn, the outcome is $(U, L)$; when $UD$ is drawn the outcome is $(U, D)$; and when $UR$ is drawn the outcome is $(U, R)$. The states of the world are drawn one third of the time each except for $(D, R)$, which is drawn with zero probability, and so is the frequency of the occurrence of the outcome.

**Summing up: procedure to calculate correlated equilibria**

1. Define the possible outcomes $\Omega$ and assign to each outcome the probability you want in the correlated equilibrium $\Pi$.
2. Define the partition over outcomes for each player $\mathcal{P}_1, \mathcal{P}_2$. This partition insures that the players will play a BR.
3. Define the strategies for each player given the set $\Omega$ and the partition over outcomes.
4. Check if this is a correlated equilibrium by insuring each player is playing a BR to each value of $\Omega$. 

4 Equilibrium refinements

4.1 Trembling hand perfection

Trembling hand perfection (THP) is an equilibrium refinement. It identifies NE that are robust to the possibility that, with some very small probability $\varepsilon$, the opponent may make a mistake. The idea of trembling hand perfection is tied to the problem of weakly dominated strategy. The idea can be better understood with an example:

\[
\begin{array}{c|cc}
   & P2 & \\
 P1 & L & R \\
 U & 1,1 & 0,-3 \\
 D & -3,-3 & 0,0 \\
\end{array}
\]

The NE of the game are: $\{(U, L), (D, R)\}$. Note how, for player 1, D is a weakly dominated strategy. In the $(D, R)$ equilibrium, a small mistake by P2 could lead P1 to have a loss of -3. That strategy is a strategy that the concept of trembling-hand perfection rules out. Now let’s have a look at this formally.

Perturbed games

For any normal form game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$, we can define a perturbed game $\Gamma_\varepsilon = [I, \{\Delta_\varepsilon(S_i)\}, \{u_i(\cdot)\}]$ by choosing, for each player $i$ and strategy $s_i \in S_i$, a number $\varepsilon(s_i) \in (0, 1)$ with $\sum_{s_i \in S_i} \varepsilon_i(s_i) < 1$ and then defining player $i$’s perturbed strategy set to be:

\[\Delta_\varepsilon(S_i) = \{ \sigma_i : \sigma_i(s_i) \geq \varepsilon(s_i) \quad \forall s_i \in S_i \quad \text{and} \quad \sum_{s_i \in S_i} \sigma_i(s_i) = 1 \}\]

The perturbed game $\Gamma_\varepsilon$ is derived from the original game $\Gamma_N$ by requiring that each player $i$ play every one of his strategies $s_i$ with at least some minimal positive probability $\varepsilon_i(s_i)$. This $\varepsilon_i(s_i)$ can be thought of as the unavoidable probability that the other player’s hand trembles and makes a mistake.

The THP refinement focuses on the NE $\sigma$ that are robust to the possibility that players make mistakes. Let’s look at this notion formally.

THP NE

A NE $\sigma$ of game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ is (normal-form) THP if there is some sequence of perturbed games $\{\Gamma_{\varepsilon_k}\}_{k=1}^\infty$ that converges to $\Gamma_N$ for which there is some associated sequence of NE $\{\sigma^k\}_{k=1}^\infty$ that converges to $\sigma$. \(^{14}\)

\(^{14}\)From MWG 8F

\(^{15}\)Convergence in the sense that $\lim_{k \to \infty} \varepsilon^k_i(s_i) = 0 \forall i$ and $s_i \in S_i$ for the game, and $\lim_{k \to \infty} \sigma^k = \sigma$ for the strategies.
Proposition. A NE $\sigma$ of game $\Gamma = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ is (normal-form) trembling-hand perfect if and only if there is some sequence of totally mixed strategies $\{\sigma^k\}_{k=1}^\infty$ such that $\lim_{k \to \infty} \sigma^k = \sigma$ and $\sigma_i$ is a best response to every element of the sequence $\{\sigma^k\}_{k=1}^\infty \forall i = 1, \ldots, I$.\(^{16}\)

Proof. Exercise 5 in PS1. \qed

Now, a well deserved trick!

Proposition. If $\sigma = (\sigma_1, \ldots, \sigma_I)$ is a (normal-form) THP NE, then $\sigma_i$ is not a weakly dominated strategy for any $i = 1, \ldots, I$. Hence, in any (normal-form) THP NE, no weakly dominated pure strategy can be played with positive probability.

The converse is not generally true, but there is an exception! Any NE not involving the play of a weakly dominated strategy is necessarily THP for two-player games. BUT, it is not generally true for games with more than two players.

Here are some examples of strategy profiles that converge, for $0 < \varepsilon < 1$:

\[
\sigma^n = \left\{ \sigma_1 = \left(p_{s_i} - \frac{1}{n}\right); \sigma_j = \left(p_{s_j} - \frac{1}{n}\right) \right\} \xrightarrow{n \to \infty} \sigma = \left\{ \sigma_i = (p_{s_i}); \sigma_j = (p_{s_j}) \right\}
\]

\[
\sigma^n = \left\{ \sigma_1 = (p_{s_i} - \varepsilon^n); \sigma_j = (p_{s_j} - \varepsilon^n) \right\} \xrightarrow{n \to \infty} \sigma = \left\{ \sigma_i = (p_{s_i}); \sigma_j = (p_{s_j}) \right\}
\]

Example of an equilibrium that is THP: Show that $\sigma = (\sigma_1(U) = 1, \sigma_2(L) = 1)$ is THP.

<table>
<thead>
<tr>
<th></th>
<th>P1</th>
<th>P2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>L</td>
<td>R</td>
</tr>
<tr>
<td>U</td>
<td>2,1</td>
<td>0,0</td>
</tr>
<tr>
<td>D</td>
<td>0,0</td>
<td>1,2</td>
</tr>
</tbody>
</table>

1. Define your sequence of totally mixed strategies that converges to your NE profile. In this case, you can pick

\[
\sigma^n = \left\{ \sigma_1^n = \sigma_1(U) = \left(1 - \frac{1}{3n}\right); \sigma_2^n = \sigma_2(L) = \left(1 - \frac{1}{3n}\right) \right\}
\]

Show it converges at $\lim_{n \to \infty} = \sigma$.

2. Then show that $\sigma_1 = (\sigma_1(U) = 1)$ is best response to $\sigma_2^n$ and $\sigma_2 = (\sigma_2(L) = 1)$ is a best response to $\sigma_1^n$. For P1:

\[
u_1(U, \sigma_2^n) = 2 - \frac{2}{3n} > \frac{1}{3n} = \sigma(D, \sigma_2^n) \forall n > 1
\]

Do the same for P2. The proposition is satisfied.

Remember that you only have to show that there exist one sequence, whichever this may be, with $n$ or $k$ from 1 to $+\infty$.

\(^{16}\)Totally mixed strategies are mixed strategies in which every pure strategy receives positive probability of being played.
2. Dynamic games

1 Extensive forms and their normal form representation

1.1 Extensive form \(^1\)

The extensive form of a game captures who moves when, what actions each player can take, what players know when they move, what the outcome is as a function of the actions taken by the players, and the players’ payoffs from each possible outcome. The extensive form relies on the conceptual apparatus known as game tree, with a unique connected path of branches from the initial decision node to each point of the tree. The structure of the game is common knowledge.

The extensive form accommodates games of perfect and imperfect information.

Perfect information games
A game is one of perfect information if each information set contains a single decision node. When it is a player’s time to move, she is able to observe all her rival’s previous moves. Otherwise, it is a game of imperfect information.

Information set
Call \(X\) a finite set of nodes \(x\) and a collection of information sets \(J\), and a function \(H : X \rightarrow J\) assigning each decision node \(x \in X\) to an information set \(H(x) \in J\). Thus the information sets in \(J\) form a partition of \(X\), and the elements of an information set are a subset of a particular player’s decision nodes.

A natural restriction on information sets is that at every node within a given information set, a player must have the same set of possible actions. \(^2\) Every decision node has to belong to an information set, which can be also formed by itself alone (a singleton), and no decision node can belong to two different information sets. We require that all decision nodes assigned to the same information set have the same choices available. In addition, for two nodes belonging to the same information set, when a player has reached one of the decision nodes in the information set and it is her time to move, she does not know which of these nodes she is actually in.

The use of information sets also allows us to capture play that is simultaneous rather than sequential.

---

\(^1\)From MWG 7C
\(^2\)Another restriction is perfect recall, which we will assume and not touch upon during the course.
1.2 Normal form

The normal form is a condensed version of the extensive form: think of a strategy as describing the actions a player will play at each information set. Remember that a strategy has to describe also the actions taken at information sets that will never be reached in equilibrium, because it is a complete contingent plan.

Any extensive form representation of a game has a unique normal form representation (the converse is not true).

Recall the very first example, the Matching Pennies game. The normal form represents not only

\[
\begin{array}{c|cc|cc}
\text{P1} & s_2^1 & s_2^2 & s_2^3 & s_2^4 \\
\hline
s_1^1 = H & -1,1 & -1,1 & 1,-1 & 1,-1 \\
\frac{s_1^2 = T}{1,-1} & 1,-1 & -1,1 & 1,-1 & -1,1 \\
\end{array}
\]

the game in Figure 2.1 but also the following extensive form (taking into account the relabelling of the strategies):
1.3 Normal form with Nature as a player

In many games, the outcome is an element of chance. In these cases, we introduce nature as an additional player who must play its actions with fixed probabilities.

Take as an example exercise 1 in problem set 2. Let us focus on the game starting from the node where Nature (N) plays: with one half of probability one of these two normal form games is played: The outcome arising from strategies played in this subgame, starting from nature and going through

<table>
<thead>
<tr>
<th>P1</th>
<th>P2</th>
<th>P1</th>
<th>P2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>L₃</td>
<td>R₃</td>
<td></td>
</tr>
<tr>
<td>U₃</td>
<td>1,4</td>
<td>2,2</td>
<td>U₃</td>
</tr>
<tr>
<td>D₃</td>
<td>0,1</td>
<td>2,2</td>
<td>D₃</td>
</tr>
</tbody>
</table>

the terminal nodes, is the expected outcome of those strategies: one half the payoff of the game in the left and one half the on in the right:

<table>
<thead>
<tr>
<th>P1</th>
<th>P2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>L₃</td>
</tr>
<tr>
<td>U₃</td>
<td>1,5,4</td>
</tr>
<tr>
<td>D₃</td>
<td>0,5,1</td>
</tr>
</tbody>
</table>
2 Sequential rationality, backward induction and subgame perfection

2.1 Sequential rationality and backward induction

Consider the following entry game.

\[
\begin{array}{c|cc}
\text{Firm E} & s_{I1} & s_{I2} \\
\hline
\text{O} & 0,2 & 0,2 \\
\text{I} & 2,1 & -3,-1 \\
\end{array}
\]

PSNE = \{(I, s_{I1}), (O, s_{I2})\}

In the pure strategy NE of this game, the entrant enters and the incumbent accommodates, or the entrant stays out and the incumbent fights if the entrant enters. This second NE is based on an empty threat: fight if enter. However, once the entrant has entered, accommodate is a dominant strategy for the incumbent. The concept of subgame perfect NE will rule out equilibria consisting of empty threats.

In particular, we want that players’ equilibrium strategies satisfy the principle of sequential rationality: a player’s strategies need to specify optimal actions at every point in the game tree. That is, given that a player finds herself at some point in the tree, her strategy should prescribe a play that is optimal from that point on given her opponents’ strategy. Then, firm I’s strategy “fight if firm E plays in” cannot be part of an equilibrium strategy satisfying sequential rationality.

In order to identify equilibria that are sequential rationally we apply backward induction. This procedure ensures that the Nash equilibria are sequentially rational. It works as follows: start by

\[\text{From MWG 9B}\]
determining the optimal actions for moves at the final decision nodes in the tree (those for which the only successor nodes are terminal nodes). Then given that these will be the actions taken at the the final decision nodes, we can proceed to the next-to-last decision nodes and determine the optimal actions to be taken there by players who correctly anticipate the actions that will follow at the final decision nodes, and so on backward through the game tree.

This procedure is readily implemented using reduced games. At each stage, after solving for the optimal actions at the current final decision nodes, we can derive a new reduced game by deleting the part of the game following these nodes and assigning to these nodes the payoffs that result from the already determined continuation play.

Let’s do backwards induction with an example.

Start by looking at the nodes preceding the terminal nodes. Form left to right: in the first one P3 prefers to play $r$ as $6 > 1$; in the second one P3 still prefers $r$ as $4 > 2$; in the furthest right he prefers $l$. We can rewrite the game as that depicted in (b) where P1 and P2 foresee P3’s choices (everyone knows the payoffs and everyone knows that the other players are rational). Then, reapply backward induction to the new game: P2 prefers $a$ to $b$. The final stage is at the reduced game
(c), where P1 prefers $R$.

The sequential rational NE found through the backward induction procedure is:

$$\sigma^{SR} = (\sigma_1, \sigma_2, \sigma_3) = (R; a \text{ if } P1 \text{ plays } R;$$

$$\text{r if } P1 \text{ plays } L, r \text{ if } P1 \text{ plays } R \text{ and } P2 \text{ plays } a, l \text{ if } P1 \text{ plays } R \text{ and } P2 \text{ plays } b)$$

Note that this strategy profile is a NE of this three-player game but that the game also has other pure strategy NE (but they do not satisfy the principle of sequential rationality).

**Zermelo’s theorem:**

Every finite game of perfect information $\Gamma_E$ has a pure strategy NE that can be derived through backward induction. Moreover, if no players has the same payoffs at any two terminal nodes, then there is a unique Nash equilibrium that can be derived in this manner.

### 2.2 Subgame perfect NE

SPNE allows us to identify NE that satisfy the principle of sequential rationality in more general games, particularly those involving imperfect information.

**Subgames**

A subgame of an extensive form game $\Gamma_E$ is a subset of the game having the following properties:

- It begins with an information set containing a single decision node, contains all the decision nodes that are successors (both immediate and later) of this node, and contains only these nodes.
- If decision node $x$ is in the subgame, then every $x' \in H(x)$ is also, where $H(x)$ is the information set that contains decision node $x$. (That is, there are no broken information sets.)

Note that the game as a whole is a subgame, and in a finite game of perfect information, every decision node initiates a subgame. In the entry game there are two subgames: the game as a whole, and the part of the tree that begins with the node that follows E’s entry decision node.

Let’s apply the NE concept to each subgame. In the discussion that follows, we say that a strategy profile $\sigma$ in the extensive form game $\Gamma_E$ induces a Nash equilibrium in a particular subgame of $\Gamma_E$ if the moves specified in $\sigma$ for information sets within the subgame constitute a NE when this subgame is considered in isolation.

**Subgame perfect Nash equilibrium (SPNE)**

A profile of strategies $\sigma = (\sigma_1, \ldots, \sigma_I)$ in an I-player extensive form game $\Gamma_E$ is a SPNE if it induces a Nash equilibrium in every subgame of $\Gamma_E$. 
Note that the set of SPNE is a subset of Nash equilibria, but not every NE is subgame perfect.

In finite games of perfect information, the set of SPNEs coincides with the set of Nash equilibria that can be derived through the backward induction procedure. Recall that in finite games of perfect information every decision node initiates a subgame. Therefore, by backward induction you will find a profile of strategies $\sigma$ that induce a NE in every subgame.

**Proposition.** Every finite game of perfect information $\Gamma_E$ has a pure strategy subgame perfect Nash equilibrium. Moreover, if no player has the same payoffs at any two terminal nodes, then there is a unique subgame perfect Nash equilibrium.

To find SPNE equilibria in a general finite dynamic game $\Gamma_E$ (and so also in games of imperfect information) you can use the *generalized backward induction procedure*:

1. Start at the end of the game tree, and identify the Nash equilibria for each of the final subgames (i.e. those that have no other subgames nested within them).
2. Select one Nash equilibrium in each of these final subgames, and derive the reduced extensive form game in which these final subgames are replaced by the payoffs that result in these subgames when players use these equilibrium strategies.
3. Repeat steps 1 and 2 for the reduced game. Continue the procedure until every move in $\Gamma_E$ is determined. This collection of moves at the various information sets of $\Gamma_E$ constitutes a strategy profile that is SPNE.
4. If multiple equilibria are never encountered in any step of this process, this profile of strategies is the unique SPNE. If multiple equilibria are encountered, the full set of SPNEs is identified by repeating the procedure for each possible equilibrium that could occur for the subgames in question.

The formal justification for using this generalized backward induction procedure to identify the set of SPNE comes from the following proposition.

**Proposition.** Consider an extensive form game $\Gamma_E$ and some subgame $S$ of $\Gamma_E$. Suppose that strategy profile $\sigma^s$ is a SPNE in subgame $s$, and let $\hat{\Gamma}_E$ be the reduced game formed by replacing subgame $s$ by a terminal node with payoffs equal to those arising from play $\sigma^s$. Then:

- In any SPNE $\sigma$ of $\Gamma_E$ in which $\sigma^s$ is the play in subgame $s$, players’ move at information sets outside subgame $s$ must constitute a SPNE of reduced game $\hat{\Gamma}_E$.
- If $\hat{\sigma}$ is a SPNE of $\hat{\Gamma}_E$, then the strategy profile $\sigma$ that specifies the moves in $\sigma^s$ at information sets in subgame $s$ and that specifies the moves in $\hat{\sigma}$ at information sets not in $s$ is a SPNE of $\Gamma_E$.

Consider an extended version of the *entry game*, where firms now play a simultaneous-move game after entry. We want to find the SPNE of this game. Let’s start by the final subgames. Here, we
have only a final subgame that is the game starting after the decision of firm E to enter. From the normal form for this subgame we can find its NE.

\[
\begin{array}{c|cc}
E & I \\
\hline
A & 3,1 & -2,-1 \\
F & 1,-2 & -3,-1 \\
\end{array}
\]

In this subgame there is only one pure NE that is \((A,A)\). We can replace the subgame with the payoff of that NE you have just found. In the new reduced game, firm E has to choose between Out with a zero payoff or playing In and getting the payoff of \((A,A)\), which is 3. The entrant chooses In as it leads to a higher payoff. The unique SPNE is:

\[
\sigma = (\sigma_E, \sigma_I) = ((\text{In}, \text{Accommodate if In}), (\text{Accommodate if firm E plays In}))
\]

This equilibrium strategy is the only one that induces a Nash equilibrium in every subgame of the extensive form game. There are two subgames in this game. One is the final node shown above, and the second is the game as a whole: Here there are three pure NE:

<table>
<thead>
<tr>
<th>Firm I</th>
<th>Accommodate if E Plays In</th>
<th>Fight if E plays IN</th>
</tr>
</thead>
<tbody>
<tr>
<td>Out, Acc If In</td>
<td>0,2</td>
<td>0,2</td>
</tr>
<tr>
<td>Out, fight if In</td>
<td>0,2</td>
<td>0,2</td>
</tr>
<tr>
<td>In, Acc if In</td>
<td>3,1</td>
<td>-2,-1</td>
</tr>
<tr>
<td>In, Fight if In</td>
<td>1,-2</td>
<td>-3,-1</td>
</tr>
</tbody>
</table>
\[ PSNE = \{ ((\text{Out, Accommodate if In}), \text{Fight if E plays In});
\]
\[ ((\text{Out, Fight if In}), \text{Fight if E plays In}); ((\text{In, Accommodate if In}), \text{Accommodate if E plays In})\] 

To be SPNE, a strategy profile to be has to induce one of the three NE in the whole game, and the NE (accommodate, accommodate) in the smaller subgame. The only strategy profile that satisfies this condition is the one we found previously using backward induction.
3 Self-confirming equilibrium

Nash equilibrium and its refinements describe a situation in which (i) each player’s strategy is a best response to his beliefs about the play of his opponents, and (ii) each player’s beliefs about the opponents’ play are exactly correct. (Fudenberg & Levine, ) propose a new equilibrium concept, self-confirming equilibrium (SCE), which weakens condition (ii) by requiring only that players’ beliefs are correct along the equilibrium path of play. Thus, each player may have incorrect beliefs about how his opponents would play in contingencies that do not arise when play follows the equilibrium, and moreover the beliefs of different players may be wrong in different ways.

We will study self-confirming equilibria in an I-players extensive form game of perfect recall (players do not forget what they have played). Let’s begin with some notation:

- Call the finite game tree X, and the nodes \( x \in X \). Call the set of terminal nodes \( Z \subset X \), with each terminal node \( z \in Z \).
- Call the set of information sets \( H \), and each information set \( h \in H \). They are disjoint subsets of \( X \) without the elements of \( Z \). The information sets where player \( i \) has the move are \( H_i \subset H \), and \( H_{-i} \) are the information sets of the other players.
- The feasible actions at \( h_i \in H \) are denoted by \( A(h_i) \); \( A_i = \bigcup_{h_i \in H_i} A(h_i) \) is the set of all feasible actions for player \( i \). A pure strategy for player \( i \), \( s_i \), is a map from information sets in \( H_i \) to actions satisfying \( s_i(h_i) \in A(h_i) \); \( S_i \) is the set of all such strategies. As usual, \( s \) is a strategy profile, and \( \sigma \) is a mixed strategy profile.
- Player \( i \)'s payoff function is denoted \( u_i : Z \to \mathbb{R} \), and each player knows is own payoff function.
- Call \( Z(s_i) \) the subset of terminal nodes that are reachable when \( s_i \) is played, and define \( H(s_i) \) as the set of all information set information sets that can be reached if \( s_i \) is played. Denote \( \bar{H}(\sigma) \) the information sets that are reached with positive probability under \( \sigma \). Note that if \( \sigma_{-i} \) is completely mixed then \( \bar{H}(s_i, \sigma_{-i}) = H(s_i) \).
- We call a behavior strategy for player \( i \) \( \pi_i \). \( \pi_i \) is a map from information sets in \( H_i \) to probability distribution over moves: \( \pi_i(h_i) \in \Delta(A(h_i)) \), and \( \Pi_i \) is the set of all such strategies, and \( \Pi = \prod_{i=1}^I \Pi_i \). Since the game has perfect recall, each mixed strategy induces an unique behavior strategy denoted \( \pi_i(\cdot|\sigma_i) \). So \( \pi_i(h_i|\sigma_i) \) is the probability distribution over actions at \( h_i \) induced by \( \sigma_i \).
- Let \( p(z|\pi) \) be the probability that \( z \) is reached under profile \( \Pi \), and define \( p(x|\pi) \) analogously.
- We will suppose that all players know the structure of the extensive form, and in particular they know the strategy spaces of their opponents. Players know their payoffs, and the probability distribution on nature’s moves. The only uncertainty each player faces concerns the strategies his opponents will play. To model this “strategic uncertainty”, we introduce

\(^7\text{From (Fudenberg & Levine, )}\)
3. SELF-CONFIRMING EQUILIBRIUM

\( \mu_i \): a probability measure over \( \Pi \). \( \mu_i \) are the beliefs of player \( i \) about the strategies of other players.

- Player \( i \)’s preferences are: \( u_i(s_i, \mu_i) = \sum_{z \in \mathcal{Z}(s_i)} p_i(z|s_i, \mu_i) u_i(z) \).

Now we can redefine NE using this notation.

**Nash Equilibrium:**
A mixed profile \( \sigma \) is NE if for each \( s_i \in \text{supp}(\sigma_i) \) \(^8\) there exist beliefs \( \mu_i \) s.t:

(i) Each player’s strategy is a best response to his beliefs about the play of his opponents: \( s_i \) maximizes \( u_i(\cdot, \mu_i) \)

(ii) Each player’s beliefs about the opponents’ play are exactly correct:
\[
\mu_i(\pi_{-i}|\pi_j(h_j) = \pi_j(h_j|\sigma_j)) = 1 \quad \forall h_j \in H_{-i}
\]

So each player optimizes given his beliefs, and his beliefs are a point mass on the true distribution, and are correct for each information set of the extensive form game.

Now, we have stated that SCE weakens the second requirement. In particular, it requires only that, for each \( s_i \) that is played with positive probability in the equilibrium strategy profile, beliefs are confirmed by the information revealed when \( s_i \) and \( \sigma_{-i} \) are played, which we take to be corresponding distribution on terminal nodes \( p(s_i, \sigma_{-i}) \). This corresponds to the idea that the terminal node reached is observed only at the end of each play of the game.

**Self-confirming equilibrium:**
Profile \( \sigma \) is self confirming if for each \( s_i \in \text{supp}(\sigma_i) \) there exist beliefs \( \mu_i \) such that:

1. \( s_i \) maximizes \( u_i(s_i, \mu_i) \) and
2. \( \mu_i(\pi_{-i}|\pi_j(h_j) = \pi_j(h_j|\sigma_j)) = 1 \) for all \( h_j \in H(s_i, \sigma_{-i}) \)

In words, condition 2 requires that player \( i \)’s beliefs be concentrated on the subset of \( \Pi \) that coincides with the true distribution at information sets that are reached with positive probability when player \( i \) plays \( s_i \). The flexibility on beliefs matters once beliefs are allowed to be wrong.

We can have two types of SCE: if agents have the same beliefs supporting the actions of other players, then it is a unitary SCE; with different beliefs to support the actions in the support of rivals’ strategies, it is an heterogeneous SCE.

**Example:**

In this game, there are four PSNE:

\[
\text{PSNE} = \{(D_1, A_2, L), (D_1, D_2, L), (D_1, D_2, R), (A_1, D_2, R)\}
\]

\(^8\) \( s_i \in \text{supp}(\sigma_i) \): each \( s_i \) played with positive probability
The equilibria where P1 plays $D_1$ embody what P2 would have played if P1 had played $A_1$. P1 and P3 are sure about what P2 would have done in the other case.

With self-confirming equilibrium we relax this constraint. Consider the following beliefs: P1 beliefs about P3 is that P3 plays $R$, while P2 beliefs are such that P3 plays $L$ for sure. Then, for P1 $A_1$ is a BR, and for P2 $A_2$ is a BR, and $\sigma = (A_1, A_2, \cdot)$ is a self-confirming equilibrium.

Notice that our proposed self-confirming equilibrium cannot be a Nash equilibrium, since for Nash equilibria beliefs need to be correct off-path. In this case, P1 and P2 have to have the same beliefs over P3’s strategy: both need to believe $L$ or both $R$. If both beliefs were P3 playing $L$, then for P1 $D_1$ is a profitable deviation, while if both beliefs are P3 playing $R$, $D_2$ is a profitable deviation.

To conclude, our self-confirming equilibrium is:

$$\sigma = (A_1, A_2, L)$$

$$\mu_1 = (\Pr(A_2) = 1, \Pr(R) = 1)$$

$$\mu_2 = (\Pr(A_1) = 1, \Pr(L) = 1)$$

**To sum up: elements needed in a self-confirming equilibrium**

To define a self-confirming equilibrium, you need a strategy profile and a set of beliefs for all the players. Beliefs must be correct on path, and agents maximize given the beliefs.
4 Repeated games

The best-understood class of dynamic games is that of repeated games, in which players face the same stage or constituent game in every period $t$ for a number of $T$ periods, and the player’s overall payoff is a weighted average of the payoffs in each stage. If the players’ actions can be observed at the end of each period, it becomes possible for players to condition their play on the past play of their opponents, which can lead to equilibrium outcomes that do not arise when the game is played only once. Note how repeated games do not allow for past play to influence the feasible actions or payoff functions in the current period (so games of investment in machinery cannot be studied using this type of games). Let us only focus on repeated games with observable actions at the end of each period.

The building block of a repeated game, the game which is repeated, is called the stage game. Formally, $I$ players play a strategic form game $\Gamma = [I, A, g]$ for $T$ periods. $A_i$ denotes the set of actions for player $i$ at each stage, and $\Delta A_i$ the space of probability distributions over $A_i$. We have $g_i : A \to \mathbb{R}$, where $A = \prod_{j \in I} A_j$, and $g_i(a_t^i, a_{-i}^t)$ is the stage payoff to player $i$ when action profile $a_t^i = (a_t^i, a_{-i}^t)$ is played. At the end of the game, players’ payoff is the sum of discounted payoffs at each stage. Future payoffs are discounted proportionally (exponentially) at some discount rate $\delta \in [0, 1)$. At each period, the outcomes of all past periods are observed by all players, what we call perfect monitoring.

Let’s suppose that the game begins at period 0 with the null history $h^0$. For each $t \geq 1$, let $h^t = (a^0, a^1, \ldots, a^{t-1})$ be the realized choices of actions at all periods before $t$, and let $H^t = (A)^t$ be the space of all possible period $t$ histories.

Since all players observe $h^t$, a pure strategy $s_i = \{s^t_i\}$ for player $i$ in this repeated game is a sequence of maps $s^t_i$ that maps possible period-$t$ histories $h^t \in H^t$ to actions $a_i \in A_i$, so that $s^t_i : H^t \to A_i$. A mixed strategy $\sigma_i$ in the repeated game is a sequence of maps $\sigma^t_i$ from $H^t$ to mixed actions $\alpha_i \in \Delta A_i$. Note that the strategy cannot depend on the past values of his opponents’ randomizing probabilities $\alpha_{-i}$, but only on the past values of $a_i$.

Finally, note that each period of play begins a proper subgame. Since moves are simultaneous in the stage game, these are the only proper subgames.

4.1 Finitely repeated games

Let’s focus on games where the terminal date is well and commonly foreseen, so that $T$ is finite.

---

10 From FT 5.1, 5.2
11 Again, a strategy must specify play in all contingencies, even those that are not expected to occur!
12 Recap on notation: $u_i, s_i, \sigma_i$ denote payoffs and strategies of the overall game. Payoffs and strategies of the stage game are denoted $g_i, a_i, \alpha_i$. 
The sequence of action profiles is denoted by \( a = \{a^t\}_{t=0}^T \), and \( \alpha = \{\alpha^t\}_{t=0}^T \) is the profile of mixed strategies. The payoff to player \( i \) in the repeated game is:

\[
u_i(a) = \sum_{t=0}^{T} \delta^t g_i(a^t_i, a^t_{-i})
\]

Where \( \delta \in [0, 1) \). Denote the \( T \)-period game with discount factor \( \delta \) by \( \Gamma^T(\delta) \). As each period begins a proper subgame, we will use as equilibrium notion subgame perfect Nash equilibrium. As previously discussed, to find the SPNE we apply backward-induction: start at the last period \( T \), find the NE, substitute the last game with its Nash payoff, move to \( T - 1 \)...

**Theorem:**
Consider a repeated game \( \Gamma^T(\delta) \) for \( T < \infty \). Suppose that the stage game \( \Gamma \) has a unique pure strategy equilibrium \( a^* \). Then \( \Gamma^T \) has a unique SPNE. In this unique SPNE, \( a^t = a^* \) for each \( t = 0, 1, \ldots, T \) regardless of history.

### 4.2 Infinitely repeated games

Now, consider the infinitely-repeated game \( \Gamma^\infty \) i.e. players play the game repeatedly at times \( t = 0, 1, \ldots \). Clearly, backward induction cannot be applied in this set up. \( a = \{a\}_{t=0}^\infty \) denotes the infinite sequence of action profiles.

Usually, the payoffs is scaled by the term \((1 - \delta)\). The normalization factor \( 1 - \delta \) serves to measure the stage-game and repeated game payoffs in the same units: the normalized value of 1 unit per period is 1.

To support equilibria that that are unreachable in static games, dynamic games use the notion of trigger strategies.

**Trigger strategies:**
A trigger strategy threatens other players with a worse punishment action if they deviate from an implicitly agreed action profile. A non-forgiving trigger strategy (or grim-trigger strategy) would involve this punishment forever after a single deviation.

A non-forgiving trigger strategy for a player \( i \) takes the following form:

\[
a^t_i = \begin{cases} 
\bar{a}_i & \text{if } a^\tau = \bar{a} \quad \forall \tau < t \\
a_i & \text{if } a^\tau \neq \bar{a} \quad \text{for some } \tau < t
\end{cases}
\]

Where \( \bar{a} \) is the implicitly agreed action and \( a_i \) is the punishment action. This strategy is non forgiving since a single deviation from \( \bar{a} \) induces player \( i \) to switch to \( a_i \) forever.

\(^{14}\) Not a lot of situations are repeated ad infinitum. Perhaps a better way to think about infinitely repeated games is to consider situations where the players always think that the game extends one more period with a high probability.

\(^{15}\) Remember that \( \sum_{t=0}^{\infty} \delta^t = \frac{1}{1-\delta} \) for \( \delta \in [0, 1) \).
Single-deviation principle

In an infinitely repeated game, we use the single-deviation principle in order to check whether a strategy profile is a subgame-perfect Nash equilibrium. The single deviation principle is applied through augmented stage games. Augmented refers to the fact that one simply augments the payoff in the stage game by adding the present value of future payoffs under the purported equilibrium. Call a payoff stream \( \pi = (\pi_0, \pi_1, \ldots, \pi_t, \ldots) \), then the present value at time \( t \) is simply:

\[
PV_{i,t}(\pi, \delta) = \sum_{s=t}^{\infty} \delta^{s-t}\pi_s = \pi_t + \delta \pi_{t+1} + \ldots
\]

Augmented Stage Game:
Consider a strategy profile \( s^* = (s_1, s_2, \ldots, s_I) \) in the repeated game. Consider any date \( t \) and any history \( h^t = (a_0, \ldots, a_{t-1}) \), where \( a_t \) is the outcome of the play at date \( t' \). The augmented stage game for \( s^* \) and \( h^t \) is the same game as the stage game in the repeated game except that the payoff of each player \( i \) from each terminal node \( z \) of the stage game is:

\[
u_i(z | s^*, h) = u_i(z) + \delta PV_{i,t+1}(\pi, \delta | s^*, (h, z))
\]

where \( u_i(z) \) is the stage game payoff of player \( i \) at \( z \) in the original stage game, and \( PV_{i,t+1}(\pi, \delta | s^*, (h, z)) \) is the present value of player \( i \) at \( t + 1 \) from the payoff stream that results when all players follow \( s^* \) starting with the history \( (h, z) = (a_0^*, \ldots, a_{t-1}^*, z) \), which is a history at the beginning of date \( t + 1 \).

Note that \( u_i(z | s^*, h) \) is the time \( t \) present value of the payoff stream that results when the outcome of the stage game is \( z \) in round \( t \) and everybody complies with the strategy profile \( s^* \) from the next period on. Note also that the only difference between the original stage game and the augmented stage game is that the payoff in the augmented game is \( u_i(z | s^*, h) \) while the payoff in the original game is \( u_i(z) \).

The single deviation principle stated that a strategy profile in the repeated game is subgame-perfect if it always yields a subgame-perfect Nash equilibrium in the augmented stage game. Let’s formalize this in a Theorem.

**Theorem:**
Strategy profile \( s^* \) is a subgame-perfect Nash equilibrium of the repeated game if and only if \( (s_1^* (h), \ldots, s_I^* (h)) \) is a subgame-perfect Nash equilibrium of the augmented stage game for \( s^* \), for every date \( t \) and every history \( h \) at the beginning of \( t \).

The key take-away from this theorem is that, to show that \( s^* \) is a subgame-perfect Nash equilibrium, one must check for all histories \( h \) and dates \( t \) that \( s^* \) yields a SPNE in the augmented stage game.
Example: Infinitely repeated entry deterrence\textsuperscript{17}

For what values of $\delta$ does the following grim-trigger strategy constitute a SPNE?

\[
ad^t_E = \begin{cases} 
    \text{Enter} & \text{if upon Entry, the Incumbent played Acc at least once in the past} \\
    X & \text{if Incumbent never played Acc after entry in the past} 
\end{cases}
\]

\[
ad^t_I = \begin{cases} 
    \text{Acc} & \text{if upon Entry, the Incumbent played Acc at least once in the past} \\
    \text{Fight} & \text{if Incumbent never played Acc after entry in the past} 
\end{cases}
\]

First note that there are two different histories: those that contain at least one $(\text{Entry}, \text{Acc})$; and the histories that do not contain any $(\text{Entry}, \text{Acc})$.

- Histories where there is at least one $(\text{Entry}, \text{Acc})$.

  Take first any date $t$ and any history $h$ where the incumbent has accommodated some entrants in the past. Now, independent of what happens at $t$, the histories at $t + 1$ and later will contain a past instance of $(\text{Entry}, \text{Acc})$ and according to the strategy profile, future entrants will always enter and incumbents will accommodate. The present value at $t + 1$ for any strategy is:

  \[
  V_A = 1 + \delta + \delta^2 + \ldots = \frac{1}{1 - \delta}
  \]

  Graphically, the augmented stage game for this $h$ and $s^*$ is:

\[\begin{array}{c|c|c}
1 & \text{Enter} & 2 \\
\hline
\text{X} & \text{Acc.} & (1+\delta V_A, 1+\delta V_A) \\
\hline
0+\delta V_A & \text{Fight} & -1+\delta V_A \\
2+\delta V_A & & -1+\delta V_A \\
\end{array}\]

\textsuperscript{17}This example differs from the exercises in the problem sets in that the stage game is an extensive form game. There is, however, one exercise like this in one of the past exams. When the stage game is a simultaneous action game, there is no distinction between subgame perfect Nash equilibrium and Nash equilibrium. Hence, in simultaneous moves games, to test the single deviation principle, it is enough to check whether $s^*(h)$ is a Nash equilibrium of the augmented stage game for every history $h$. 

The single-deviation principle requires that \((\text{Entry, Acc})\) be a subgame perfect equilibrium of the augmented stage game. This is in fact the case for any \(\delta\). Hence \(s^*\) passes the single deviation test for this particular history.

- Histories where there is no \((\text{Entry, Acc})\).

Now consider the second history where the incumbent has never played accommodated any entrant. The augmented stage game for this history is: According to their strategies, if the Entrant enters and the Incumbent accommodates the repeated game will yield \((\text{Entry, Acc})\) forever, with future payoffs \(V_A\). In any other case, we will remain in \((X, \text{Fight if Enter})\), that yields 0 for the Entrant and \(V_F = \frac{2}{1-\delta}\) for the Incumbent. At this history, the grim-trigger strategy profile prescribes \((X, \text{Fight if Enter})\). The single deviation principle requires then that \((X, \text{Fight if Enter})\) is a SPNE of the above augmented game.

Since \(X\) is a best response to \(\text{Fight}\), we only need to ensure that the incumbent weakly prefers \(\text{Fight}\) to \(\text{Accommodate}\) after an entry in the above game.

\[-1 + \delta V_F \geq 1 + \delta V_A \iff \delta \geq \frac{2}{3}\]

As we have considered all possible histories, the strategy profile passed the single deviation test for \(\delta \geq \frac{2}{3}\), and so for such parameter it is a SPNE of the infinitely repeated game.

### 4.3 Folk Theorem

The “folk theorems” for repeated games assert that if the players are sufficiently patient then any feasible, individually rational payoffs can be enforced by an equilibrium. Thus, in the limit of extreme patience, repeated play allows virtually any payoff to be an equilibrium outcome.

To make this assertion precise, we must define feasible and individually rational payoffs.

1. Individually rational

Define player \(i\)'s minmax value \(v_i\) to be the lowest payoff that player \(i\)'s opponent can hold him to by any choice of \(\alpha_{-i}\), provided that player \(i\) correctly foresees \(\alpha_{-i}\) and plays a best response to it.

\[v_i = \min_{\alpha_{-i}} \max_{\alpha_i} g_i(\alpha_i, \alpha_{-i})\]
A minmax strategy profile \((m^i_{-i})\) against player \(i\) by his opponents \(-i\) is the strategy that would attain this minimum. Let \(m^i_1\) be a strategy for player \(i\) such that \(g_i(m^i_1, m^i_{-i}) = v_i\).

Note that player \(i\)'s payoff is at least \(v_i\) in any static equilibrium and in any NE of the repeated game, regardless of the level of the discount factor. This is why we can also refer to \(v_i\) as the reservation utility. We thus know that no equilibrium of the repeated game can give any player a payoff lower than this amount.

**Theorem:**

- Let \(\alpha\) be a (possibly mixed) Nash equilibrium of \(\Gamma\) and \(g_i(\alpha)\) be the payoff to player \(i\) in equilibrium \(\alpha\). Then:
  \[g_i(\alpha) \geq v_i\]

- Let \(\sigma\) be a (possibly mixed) Nash equilibrium of \(\Gamma^\infty(\delta)\) and \(u_i(\sigma)\) be the payoff to player \(i\) in equilibrium \(\sigma\). Then:
  \[u_i(\sigma) \geq v_i\]

A payoff vector \(v \in \mathbb{R}^I\) is strictly individually rational if \(v_i > v_i\) \(\forall i\).

2. Socially feasible

Consider the stage game \(\Gamma = [I, A_i, g_i]\) and the infinitely repeated game \(\Gamma^\infty(\delta)\). The set of feasible payoff is:

\[V = \text{convex hull}\{v \in \mathbb{R}^I \mid \exists a \in A \text{ with } g(a) = v\}\]

That is, \(V\) is the convex hull of all \(I\)-dimensional vectors that can be obtained by some action profile. Convexity here is obtained by public randomization. \(^{19}\)

**Socially feasible individually rational (SFIR) payoffs** are the payoffs that are feasible and strictly individual rational. This set is denoted:

\[\text{SFIR} = \{v \in V : v_i > v_i \forall i\}\]

**Folk theorem**

For every feasible payoff vector \(v\) with \(v_i > v_i \forall i\), there exists a \(\delta < 1\) such that for all \(\delta \in (\bar{\delta}, 1)\), there is a NE of \(\Gamma^\infty(\delta)\) with payoffs \(v = (v_1, \ldots, v_I)\).

All payoff profiles inside the SFIR (yielding more than the min max) can be sustained as a NE in a repeated game with the correct grim-trigger strategy and discount factor \(\delta\).

Let’s look at an example: \(^{19}\)

\(^{19}\)For a short discussion on this issue, check the book. Here, it is enough to point out that the convex hull is based on pure strategies. The public randomization is linked to correlated equilibria, where the outcomes will be different pure-strategy profiles being played with different probability. Mathematically, note that \(V\) is not equal to:

\[V = \text{convex hull}\{v \in \mathbb{R}^I \mid \exists a \in \Delta A \text{ such that } g(a) = v\}\]
To compute player 1’s min max value, we first compute his payoffs as a function of the probability that player 2 plays $L$, denoted $\sigma_L$.

$$
\begin{align*}
\begin{array}{c|cc}
\sigma_L & 1 - \sigma_L \\
\hline
\sigma_U & U & -2, 2 \\
\sigma_M & M & 1, -2 \\
1 - \sigma_U - \sigma_M & D & 0, 1 \\
\end{array}
\end{align*}
$$

$$
\begin{align*}
&u(U, \sigma_L) = 1 - 3\sigma_L \\
&u(M, \sigma_L) = 3\sigma_L - 2 \\
&u(D, \sigma_L) = 0 \\
&\min_{\sigma_L} \max \{1 - 3\sigma_L, 3\sigma_L - 2\}
\end{align*}
$$

The min max is achieved when the utilities from $U$ and $M$ are equal, which happens when $\sigma_L = 1/2$ and $u(U, \sigma_L = 1/2) = u(M, \sigma_L = 1/2) = -1/2$. However, player 1 can simply decide to play $D$, in which case he always gets a zero utility regardless of the choice of $\sigma_L$. Hence, zero is player 1’s min max payoff.

As for player 2, we have a similar reasoning. Call $\sigma_U$ and $\sigma_M$ the probabilities that player 1 plays $U$ and $M$ respectively, then player 2’s payoffs are:

$$
\begin{align*}
&u(\sigma^1, L) = 1 + \sigma_U - 3\sigma_M \\
&u(\sigma^1, R) = 1 + \sigma_M - 3\sigma_U \\
&\min_{\sigma_U, \sigma_M} \max \{1 + \sigma_U - 3\sigma_M, 1 + \sigma_M - 3\sigma_U\}
\end{align*}
$$

Once again, this expression is minimized when $U, M$ are played with a $1/2$ probability, meaning that player 1 never plays $D$.

Note the importance of considering mixed strategies when calculating the min max of a game. If we had exclusively focused on pure strategies, we would have identified $(1, 1)$ as the min max value. The min max if we only allow for pure strategies is at least as high as the min max using mixed strategies, as we are minimizing over a smaller set.

To plot the socially feasible region, mark the payoffs achievable under pure strategies and join them creating a convex set.

---

20If you have trouble seeing this, plot the utilities as a function of $\sigma_L$. 

5 Stackelberg equilibrium

A Stackelberg game is a two player sequential game with perfect information in which a “leader” chooses an action from a set $A_1$ and a “follower”, informed of the leader’s choice chooses an action from a set $A_2$. The solution usually applied is that of subgame perfect equilibrium (SPE). Some (but not all) SPE of a Stackelberg game correspond to solutions of the maximization problem:

$$\max_{(a_1, a_2) \in A_1 \times A_2} u_1(a_1, a_2) \quad \text{st} \quad a_2 \in \arg \max_{a'_2 \in A_2} u_2(a_1, a'_2)$$

where $u_i$ is a payoff function that represents player $i$’s preferences. If the set $A_i$ of actions of each player $i$ is compact and the payoff functions $u_i$ are continuous, then this maximization problem has a solution.

The leader can (pre-)commit in two ways:

- Pure pre-commitment means that players can use only pure strategies.
- Mixed pre-commitment means they can use mixed strategies.

When we have a player that cares about the long-run, we can relate the payoffs from our discussion from the Folk Theorem and the possible Stackelberg commitment strategies as follows:

Note that the set of dynamic equilibria is more restricted than that predicted by the Folk Theorem. This happens because the Folk Theorem is applicable only to games with two players who are both patient, while the “dynamic equilibria” relates to games with a long-run and a short-run player. It is the case that a long-run player may do better facing a long-run opponent than a short-run opponent. A long-run opponent can be forced to make concessions by threats of future punishments that a short-run opponent would not care about.

\[22\text{From OR 6.2}\]
5. STACKELBERG EQUILIBRIUM

- $\max u'(a)$
- mixed precommitment/Stackelberg
- $\nu^1$ best dynamic equilibrium
- pure precommitment/Stackelberg
- **Set of dynamic equilibria**
- static Nash
- $\nu^1$ worst dynamic equilibrium
- minmax
- $\min u'(a)$
3. Uncertainty

1 Expected utility theory

Uncertain or risky alternatives can be described by the possible outcomes and associated probabilities (lotteries). The decision maker has preferences over such lotteries. This leads to very convenient representation of preferences (von Neumann-Morgenstern expected utility model) and allows to develop basic properties of preference and choice with uncertainty, such as measures of risk aversion. We will assume probabilities are objective and hence the decision maker knows them. This is what is called ‘objective risk’.

The decision maker faces a choice among risky alternatives. An alternative delivers one of \( N \) possible outcomes in the (finite) set \( A = \{a_1, \ldots, a_n\} \). This could be consumption bundles, or just money. The outcomes are realized according to an objective probability distribution.

We call each of these alternatives a simple lottery \( L = (p_1, \ldots, p_N) \) where \( p_n \geq 0 \) is the probability that the \( n \)-th outcome realizes and \( \sum_n p_n = 1 \). A lottery space \( \mathcal{L} \) is the set of available lotteries over the specific set of outcomes \( A \).

As usual, the decision maker will have preferences over consequences. However, possible consequences are fixed here and what could be chosen/affected is the uncertainty of prospects. Hence we discuss a binary preference relations \( \succeq \) over lotteries, which allows the individual to compare any two lotteries in \( \mathcal{L} \).

Independence axiom

The preference relation \( \succeq \) on the space of simple lotteries \( \mathcal{L} \) satisfies the independence axiom if for all \( L, L', L'' \in \mathcal{L} \) and \( \alpha \in (0, 1) \), we have:

\[
L \succeq L' \iff \alpha L + (1 - \alpha) L'' \succeq \alpha L' + (1 - \alpha) L''
\]

Expected Utility Theorem

Suppose that the rational preference relation \( \succeq \) on the space of lotteries \( \mathcal{L} \) satisfies the continuity and independence axioms. Then, \( \succeq \) admits a utility representation of the expected utility form. That is, we can assign a number \( u_n \) to each outcome \( i = 1, \ldots, N \) in such a manner that for any two lotteries \( L = (p_1, \ldots, p_n) \) and \( L' = (p'_1, \ldots, p'_n) \) we have:

\[
U : \mathcal{L} \to \mathbb{R} \quad U(L) = u_1 p_1 + \cdots + u_n p_n
\]

\[
L \succeq L' \iff \sum_{i=1}^n u_i p_i \geq \sum_{i=1}^n u_i p'_i
\]

\(^1\)From MWG 6B
\(^2\)This material should be familiar to you from the first course of the microeconomics sequence.
where \( u = (u_1, \ldots, u_n) \) are called Bernoulli utilities of outcomes.

The theorem provides a simple and convenient representation of lotteries, as they are linear in probability.

\[
U \left( \sum_{k=1}^{K} \alpha_k L_k \right) = \sum_{k=1}^{K} \alpha_k U(L_k)
\]

**The independence axiom and the EU theorem**

There is an equivalence in implications between the EU theorem and the independence axiom.

The EU form implies, from the linearity of probabilities, that:

\[
U(L) = U(L') \leftrightarrow U(\alpha L + (1 - \alpha) L') = \alpha U(L) + (1 - \alpha) U(L')
\]

Hence in the simplex \( U \) is constant on the line connecting \( L \) and \( L' \), so indifference curves are linear and parallel shifts.

The independence axiom implies that:

\[
L \sim L' \leftrightarrow L \sim \alpha L + (1 - \alpha) L' \sim L'
\]

So that indifference curves are parallel.

2 Money lotteries and risk aversion

Instead of general outcomes, let’s refine and restrict the only possible outcome as an amount of money \( x \in \mathbb{R}_+ \). Our lottery space \( \mathcal{L} \) is now the set of all distribution functions over non-negative amounts of money, ie, over the interval \([a, \infty)\).

An equivalent EU Theorem states that (under the same conditions) a preference relation for the final distribution of money \( F(\cdot) \) over distribution functions \( F(x) \) can be represented with:

\[
U(F) = \mathbb{E}[u(x)]
\]

---

3From MWG 6C
2. MONEY LOTERIES AND RISK AVERSION

Where $U$ is the vNM utility function and $u$ is the Bernoulli utility function over sure amounts. Note that we assume that money is continuous, the expectation is calculated with an integral. If we are dealing with discrete sums of money, the valid expression is simply the sum over outcomes times their probability.

It is useful to define for any function $F$ the degenerate $\bar{F}$ that delivers the mean with certainty. $\bar{F} = 0 \quad \forall x < \mathbb{E}[x]$ and 1 otherwise.

**Attitudes towards risk**

**Risk averse**
A risk averse individual prefers the sure amount to the gamble. Under vNM expected utilities:

$$\mathbb{E}[u(x)] \leq u(\mathbb{E}[x])$$

The individual trades expected return for certainty.

**Risk loving**
For any possible outcome, the individual prefers being exposed to risk rather than uncertainty:

$$\mathbb{E}[u(x)] \geq u(\mathbb{E}[x])$$

**Risk neutral**
Both risk averse and risk loving:

$$\mathbb{E}[u(x)] = u(\mathbb{E}[x])$$

**Certainty equivalent:**
The $CE(F, u)$ is the (risk-free) amount of money that makes an individual indifferent between the gamble $F(x)$ and the certain (risk-free) amount $CE(F, u)$.

$$u(CE(F, u)) = \mathbb{E}[u(x)] = U(F)$$

---

4Note that this is Jensen’s inequality to define concave functions. $f(\alpha x + (1 - \alpha)x') \leq \alpha f(x) + (1 - \alpha)f(x')$. 

Probability premium:
The probability premium \( \pi(x, \epsilon, u) \) is implicitly defined by the additional wining probability over fair odds that makes the individual indifferent between the certain outcome \( x \) and a gamble \( x \pm \epsilon \). \(^6\)

\[
\mathbb{E}[u] = u(x) = \left( \frac{1}{2} + \pi \right) \times u(x + \epsilon) + \left( \frac{1}{2} - \pi \right) \times u(x - \epsilon)
\]

For example, for risk averse individuals, more probability needs to be placed on the positive outcome to compensate for the additional risk.

Let derive the formula for the risk premium. \(^7\)

Consider a random variable \( x \), with mean \( r_f \) and variance \( \sigma^2 \). Note that we can rewrite \( x = r_f + y \) where \( \mathbb{E}[y] = 0 \) and \( Var(y) = \sigma^2 \).

\[
x = r_f + y \quad x \sim (r_f, \sigma^2)
\]
\[
y \sim (0, \sigma^2)
\]
\[
\pi(x): u(\mathbb{E}[x] - \pi) = u(\mathbb{E}[x]) = u(x)
\]

We perform a Taylor approximation around the risk free point \( r_f \). \(^8\)

\[
u(\mathbb{E}[x] - \pi) = u(r_f - \pi)
\]
\[
u(r_f - \pi) \approx u(r_f) + (r_f - \pi - r_f)u'(r_f)
\]

\[
\mathbb{E}[u(x)] = \mathbb{E}[u(r_f + y)]
\]
\[
\mathbb{E}[u(r_f + y)] \approx \mathbb{E}
\left[
  u(r_f) + (r_f + y - r_f)u'(r_f) + \frac{(r_f + y - r_f)^2}{2}u''(r_f)
\right]
\]
\[
= u(r_f) + \mathbb{E}[y]u'(r_f) + \frac{\mathbb{E}[y^2]}{2}u''(r_f)
\]

\(^6\)Another way to say it: the risk premium is the minimum amount of money by which the expected return on a risky asset must exceed the certain return on a risk-free asset to make the individual indifferent between investing in the risk-free and the risky asset.

\(^7\)Because it is interesting, but this is not asked in the problem sets.

\(^8\)Taylor approximation around \( x_0 \):

\[
\pi(x) \approx \pi(x_0) + (x - x_0)\pi'(x_0) + \frac{(x - x_0)^2}{2}\pi''(x_0)
\]
\[ u(\mathbb{E}[x] - \pi) = \mathbb{E}[u(x)] \]
\[ u(r_f) - \pi u'(r_f) = u(r_f) + \frac{\sigma^2}{2} u''(r_f) \]
\[ \pi = -\frac{\sigma^2}{2} u''(r_f) = \frac{\sigma^2}{2} \rho \]

where \( \rho \) is the coefficient of relative risk aversion.

**Arrow-Pratt measure of absolute risk aversion (ARA):**

Suppose \( \succsim \) is a preference relation represented by the twice differentiable Bernoulli index \( u(\cdot) \). ARA is defined as:

\[ r_A(x) = -\frac{u''(x)}{u'(x)} \]

Although concavity of \( u \) reflects risk aversion, \( u'' \) is not robust to strictly increasing linear transformations of \( u \), which instead are compatible with preferences representations. However, \( u''/u' \) is.

The normalization considers the curvature of \( u \), since \( u'' \) may depend on the size of \( u' \).

An alternative interpretation defines ARA as the rate at which the probability premium increases at certainty with the small risk measured by \( \epsilon \).

Comparing across identical individuals who only differ in their income, their risk aversion will differ: \( u(\cdot) \) implies decreasing absolute risk aversion if \( r_A(x) \) decreasing in \( x \), or constant or increasing. The richer you are, the lower aversion towards risk.

**Example:** Constant absolute risk aversion.

\[ u(x) = \alpha \exp^{-ax} + \beta, \] often with \( \beta = -\alpha = 1 \) such that \( r_A(x) = a \) does not depend on \( x \).

**Relative risk aversion**

We use the relative measure of risk aversion (RRA) \( r_r(x) = x \times r_A(x) \) when we measure proportional/percentage changes instead of absolute changes (ARA) around initial wealth \( x \): begin with \( x \) and the ‘proportional gamble’ changes it to \( tx \).

Set \( \tilde{u}(t) = u(tx) \). Then, the ARA on \( \tilde{u} \) at \( t=1 \) is:

\[ \rho \equiv r_A(t = 1, \tilde{u}) = -\frac{\tilde{u}''(1)}{\tilde{u}'(1)} = -\frac{x^2 u''(x)}{xu'(x)} = x \times r_A(x, u) \]

Generally, we assume non-increasing RRA. Note that risk averse individual with non-decreasing RRA implies decreasing risk version, but the converse is not true.

There are two ways to express the coefficient of relative risk aversion \( \rho \):

\[ \rho = \frac{w 2\pi}{\sigma^2} \quad \text{or} \quad \rho = -\frac{u''(\cdot)}{u'(\cdot)} \]

Typically, the coefficient of relative risk aversion is set between one and five. For example, if an individual has a coefficient of relative risk aversion of two, then she is indifferent between sure consumption of $66,667 and a 50/50 gamble between $50,000 and $100,000 of consumption.
Our preferences for the long run tend to conflict with our short-run behavior. When planning for the long run, we intend to meet our deadlines, exercise regularly, and eat healthfully. But in the short run, we have little interest in revising manuscripts, jogging on the StairMaster, and skipping the chocolate souffle a la mode. Delay of gratification is a nice long-term goal, but instant gratification is disconcertingly tempting.

Economists usually assume that discount functions are exponential. Specifically, a util delayed $\tau$ periods is worth $\delta^\tau$ as much as a util enjoyed immediately ($\tau = 0$). Typically, economists assume $\delta < 1$, capturing the fact that future utils are worth less than current utils. However, the experimental evidence implies that the actual discount function declines at a greater rate in the short run than in the long run. In other words, delaying a short-run reward by a few days reduces the value of the reward more in percentage terms than delaying a long-run reward by a few days. Consumers have both a short-run preference for instantaneous gratification and a long-run preference to act patiently. This combination of time preferences is usually called “hyperbolic” discounting, since generalized hyperbolas were first used in the 60s to capture such intertemporal preferences.

When researchers estimate the shape of the discount function based on choices by experimental subjects, the estimates are better approximated by generalized hyperbolic functions than by exponential functions. In the original psychology literature, researchers used hyperbolic discount functions like $1/\tau$ and $1/(1 + \alpha \tau)$, with $\alpha > 0$. The most general hyperbolic discount function weights events $\tau$ periods away with factor $1/(1 + \alpha \tau)^{-\gamma/\alpha}$ with $\alpha, \gamma > 0$.

To capture declines at a faster rate in the short run than in the long run, Laibson (1997a) adopted a discrete-time discount function, $\{1, \beta \delta, \beta \delta^2, \beta \delta^3, \ldots\}$ which had previously been used to model intergenerational time preferences. This “quasi-hyperbolic function” reflects the sharp short-run drop in valuation measured in the experimental time preference data and has been adopted as a research tool because of its analytical tractability. The quasi-hyperbolic discount function is only “hyperbolic” in the sense that it captures the key qualitative property of the hyperbolic functions: a faster rate of decline in the short run than in the long run. Laibson (1997a) adopted the phrase “quasi-hyperbolic” to emphasize the connection to the “hyperbolic discounting” literature in psychology. Other researchers call them “present biased” or “quasi-geometric.”
Let’s compare a hyperbolic discounter and an exponential discounter.

\[
U^h(x; \beta, \delta) = u(x_0) + \beta \sum_{\tau=1}^{T} \delta^\tau u(x_\tau) \quad \text{and} \quad U^e(x; \delta) = \sum_{\tau=0}^{T} \delta^\tau u(x_\tau)
\]

The figure below plots a particular parameterization of the quasi-hyperbolic discount function: \(\beta = 0.7\) and \(\delta = 0.957\). Using annual periods, these parameter values roughly match experimentally measured discounting patterns. Delaying an immediate reward by a year reduces the value of that reward by approximately \(1/3 \approx (1 - \beta \delta)\). By contrast, delaying a distant reward by an additional year reduces the value of that reward by a much smaller percentage: \((1 - \delta)\).
4. Bayesian games

1 Weak perfect Bayesian equilibrium

1.1 Beliefs and sequential rationality

Sometimes subgame perfect Nash equilibrium is not enough to rule out some “unreasonable” equilibria. Take as an example the following game:

There are 2 pure NE: (out, fight if entry) and (in₁, accommodate if entry). (out, fight if entry), however, is based on what we have called an empty threat, that is if the entrant decides to enter, accommodate is a dominant strategy for the incumbent. Subgame perfection is of no use here as the only subgame is the game as a whole: both PSNE are subgame perfect. That is why we introduce the concept of weak perfect Bayesian equilibrium (WPBE), which extends the principle of sequential rationality by formally introducing the notion of beliefs.

First, we have to define the concept of system of beliefs and sequential rationality of strategies, two critical components of WPBE.

System of beliefs:
A system of beliefs $\mu$ in an extensive form game $\Gamma_E$ is a specification of a probability $\mu(x) \in [0, 1]$ for each decision node $x$ in $\Gamma_E$ such that, for all information sets $H$:

$$\sum_{x \in H} \mu(x) = 1$$

A system of beliefs specifies, for each information set, a probabilistic assessment by the player who moves at that set of the relative likelihoods of being at each information set’s various decision nodes, conditional upon play having reached that information set.

1From MWG 9C
To define sequential rationality, it is useful to first describe $E[u_i|H, \mu, \sigma_i, \sigma_{-i}]$, which is player $i$’s expected utility starting at her information set $H$ if her beliefs regarding the conditional probabilities of being at the various nodes in $H$ are given by $\mu$, and if she follows strategies $\sigma_i$ while her rivals use strategies $\sigma_{-i}$.

**Sequential rationality**

A strategy profile $\sigma$ in extensive form game $\Gamma_E$ is sequentially rational at information set $H$ given a system of beliefs $\mu$ if, denoting by $\iota(H)$ the player who moves at information set $H$, we have:

$$E\left[u_{\iota(H)}|H, \mu, \sigma_{\iota(H)}, \sigma_{-\iota(H)}\right] \geq E\left[u_{\iota(H)}|H, \mu, \sigma'_{\iota(H)}, \sigma_{-\iota(H)}\right]$$

$\forall \sigma'_{\iota(H)} \in \Delta(S_{\iota(H)})$. If strategy profile $\sigma$ satisfies this condition for all information sets $H$, then we say that $\sigma$ is sequentially rational given belief system $\mu$.

In words, a strategy profile $\sigma$ is sequentially rational if no player finds it worthwhile, once one of her information sets has been reached, to revise her strategy given her beliefs about what has already occurred (as embodied in $\mu$) and her rivals’ strategies.

### 1.2 Weak perfect Bayesian equilibrium and sequential equilibrium

**Weak perfect Bayesian equilibrium**

A profile of strategies and system of beliefs $(\sigma, \mu)$ is a WPBE in an extensive form game $\Gamma_E$ if it has the following properties:

1. The strategy profile $\sigma$ is sequentially rational given belief system $\mu$.

2. The system of beliefs $\mu$ is derived from strategy profile $\sigma$ through Bayes’ rule whenever possible. That is, for any information set $H$ such that $\Pr(H|\sigma) \geq 0$ (on the equilibrium path, the probability of reaching information set $H$ is positive under strategy $\sigma$) we must have:

$$\mu(x) = \frac{\Pr(x|\sigma)}{\Pr(H|\sigma)} \quad \forall x \in H$$

It is important to note that the definition of WPBE refers to the pair strategy-beliefs $(\sigma, \mu)$. The definition, furthermore, requires (1) rationality and (2) consistency of beliefs. In addition note that WPBE is a subset of NE, where we require sequential rationality only on the equilibrium path.

Let’s come back to our previous example. $(\text{out}, \text{fight if entry})$ cannot be a WPBE, as for any belief of firm $I$, accommodate is a dominant strategy: $(\text{fight if entry})$ is not sequentially rational for any belief. In turn, $(\text{in}_1, \text{accommodate if entry})$ is a WPBE. First, note that as beliefs have to be derived through Bayes’ rule, the beliefs have to be such that beliefs of being at node $\text{in}_1$ for the Incumbent has to be one. Then accommodate is a dominant strategy for any belief. So, the WPBE is $(\text{in}_1, \text{accommodate if entry}, \mu_I(\text{in}_1) = 1)$

Let’s now try with a different example.
In this case, firm $I$ is willing to fight only if she believes that she is at $in_1$ and firm $E$’s optimal strategy depends on firm $I$’s behavior. To solve the game we look for a fixed point at which the behavior generated by beliefs is consistent with these beliefs.

Call $\sigma_F$ the probability that firm $I$ fights after entry and $\mu_1$ be firm $I$’s belief of being at node $in_1$ and call $\sigma_0, \sigma_1, \sigma_2$ the probability of out, $in_1,in_2$ generated by firm $E$’s actual strategy. First, we will compute for which beliefs $I$ is willing to play fight, that is:

$$u(F|\mu_1) = -1 \geq -2\mu_1 + (1 - \mu_1) = u(A|\mu_1) \Rightarrow \mu_1 \geq \frac{2}{3}$$

Let’s check each possible candidate for equilibrium:

1. Suppose $\mu_1 > \frac{2}{3}$. Then $I$ plays fight for sure. But then best response of $E$ is $in_2$. Note that this alternative does not comply with the consistency requirement: this strategy leads us to node $in_2$ and if player $I$ is to have consistent beliefs they should be $\mu_1 = 0$, a contradiction. This cannot be a WPBE.

2. Now take $\mu_1 < \frac{2}{3}$, then firm $I$ prefers to play $Acc$, but the BR of firm $E$ is $in_1$ for sure. Again, the beliefs has to be as such that $\mu_1 = 1$ to be consistent. This is a contradiction, and hence again this is not a WPBE.

3. Finally, take $\mu_1 = \frac{2}{3}$. Then player $I$ is indifferent and randomizes. In order to have consistent beliefs, we know that $E$ has to randomize and in particular must play $in_1$ with twice the probability of $in_2$. As $E$ randomizes, he has to be made indifferent by $I$’s strategy that is:

$$-1\sigma_F + 3(1 - \sigma_F) = \gamma\sigma_F + 2(1 - \sigma_F) \quad \Leftrightarrow \quad \sigma_F = \frac{1}{\gamma + 2}$$

Firm $E$’s payoff from playing $in_1$ or $in_2$ is then $\frac{3\gamma + 2}{\gamma + 2} > 0$ and so firm $E$ plays out with zero probability.

So the unique WPBE is:

$$\left(\sigma, \mu\right) = \left(\sigma_E = \left(\frac{2}{3}, \frac{1}{3}\right), \sigma_I = \left(\frac{1}{\gamma + 2}\right); \mu_1 = \frac{2}{3}\right)$$
Note that the only requirement for beliefs, other than that they specify nonnegative probabilities which add up to 1 within each information set, is that they are consistent with the equilibrium strategies on the equilibrium path. No restrictions at all are placed on beliefs off the equilibrium path. This means that a WPBE need not be subgame perfect.

Let’s look at an example:

Here there is a WPBE that is \((\sigma, \mu) = (\text{out, accommodate if in}, \text{fight if in}); \mu_I(F) = 1\). Note that these beliefs are permitted under WPBE because they are off-path, since player E plays out. This WPBE is not SPE and in order to rule that kind of equilibria out we introduce the concept of sequential equilibrium.

To work around this weakness of WPBE, we will restrict our attention to equilibria that induce a WPBE in every subgame. This will insure subgame perfection.

**Sequential equilibrium**

A strategy profile and system of beliefs \((\sigma, \mu)\) is a sequential equilibrium of extensive form game \(\Gamma_E\) if it has the following properties:

1. Strategy profile \(\sigma\) is sequentially rational given belief system \(\mu\).
2. There exist a sequence of completely mixed strategies \((\sigma^k)_{k=1}^\infty\) with \(\lim_{k \to \infty} \sigma^k = \sigma\), such that \(\mu = \lim_{k \to \infty} \mu^k\), where \(\mu^k\) denotes the beliefs derived from strategy profile \(\sigma^k\) using Bayes rule.

The idea is that beliefs be justifiable as coming from a fully mixed strategy close to the equilibrium strategy. \(^3\) Note that sequential equilibrium is a subset of WPBE and of SPNE.

\(^3\)Note the resemblance to trembling hand perfection.
Going back to the previous example, let me check that (out, accommodate if in ) (fight if firm E plays in) cannot be a sequential equilibrium. Consider any totally mixed strategy $\sigma_m$ and any node $x$ in firm $I$’s information set $H_I$. Call $z$ firm E’s decision node following entry, and $\mu_{\sigma_m}$ the belief associated with the fully mixed strategy at $H_I$. Note that $\Pr(H_I|\sigma_m, z) = 1$.

$\mu_{\sigma_m} = \Pr(x|\sigma_m) = \frac{\Pr(x|\sigma_m, z) \Pr(z|\sigma_m)}{\Pr(H_I|\sigma_m, z) \Pr(z|\sigma_m)} = \Pr(x|\sigma_m, z)$

$\Pr(x|\sigma_m, z)$ is the probability of reaching node $x$ under the fully mixed strategy conditional on having reached $z$ and it is exactly the probability that firm $E$ plays the action that leads to node $x$ in the fully mixed strategy. Thus, any sequence of totally mixed strategies $\{\sigma^E_m\}_k^{\infty}$ that converge to $\sigma_m$ must generate limiting beliefs for firm $I$ that coincide with the play at node $z$ specified in firm $E$’s actual strategy. Note that this means that the strategy in any sequential equilibrium must specify NE behavior in the post entry game and so constitute a SPNE.

**Proposition:**
In every sequential equilibrium $(\sigma, \mu)$ of an extensive form game $\Gamma_E$, the equilibrium strategy profile $\sigma$ constitutes a SPNE of $\Gamma_E$.

**To summarize and some tips**

In every SE $(\sigma, \mu)$ of an extensive form game, the equilibrium profile $\sigma$ is a SPNE of the game. The SE is both WPBE and SPNE. If a game has no WPBE then it has no SE. If there is only one information set, we can state that the WPBE is the only sequential equilibrium.

The strategy to find SE is to first look for WPBE and then check if they are sequential using the converging sequence to verify the beliefs off-path.
References


