Definition of Bayes Equilibrium

Harsanyi [1967]

- What happens when players do not know one another’s payoffs?
- Games of “incomplete information” versus games of “imperfect information”
- Harsanyi’s notion of “types” encapsulating “private information”
- Nature moves first and assigns each player a type; player’s know their own types but not their opponents’ types
- Players do have a common prior belief about opponents’ types
Bayesian Games

There are a finite number of types $\theta_i \in \Theta_i$

There is a common prior $p(\theta)$ shared by all players

$p(\theta_{-i}|\theta_i)$ is the conditional probability a player places on opponents’ types given his own type

The stage game has finite action spaces $a_i \in A_i$ and has utility function $u^i(a, \theta)$
Bayesian Equilibrium

A Bayesian Equilibrium is a Nash equilibrium of the game in which the strategies are maps from types $s_i : \Theta_i \rightarrow A_i$ to stage game actions $A_i$.

This is equivalent to each player having a strategy as a function of his type $s_i(\theta_i)$ that maximizes conditional on his own type $\theta_i$ (for each type that has positive probability)

$$\max_{s_i} \sum_{\theta_{-i}} u_i(s_i, s_{-i}(\theta_{-i}), \theta_i, \theta_{-i})p(\theta_{-i}|\theta_i)$$
Cournot Model with Types

- A duopoly with demand given by $p = 17 - x$
- A firm’s type is its cost, known only to that firm: each firm has a 50-50 chance of cost constant marginal cost 1 or 3.

Profits of a representative firm

$$\pi_i(c_i, x) = \left[17 - c_i - (x_i + x_{-i})\right] x_i$$

Let us look for the symmetric pure strategy equilibrium
Finding the Bayes-Nash Equilibrium

$x^1, x^3$ will be the output chosen in response to cost

$$\pi_i(x_i, c_i) = 0.5 \left[ 17 - c_i - (x_i + x^1) \right] x_i + 0.5 \left[ 17 - c_i - (x_i + x^3) \right] x_i$$

maximize with respect to $x_i$ and solve to find

$x^1 = 11/2, x^3 = 9/2$
Industry Output

probability ¼ 11
probability ½ 10
probability ¼ 9

Suppose by contrast costs are known

If both costs are 1 then competitive output is 16 and Cournot output is 2/3rds this amount 10 2/3

If both costs are 3 then competitive output is 14 and Cournot output is 9 1/3

If one cost is 1 and one cost is 3 Cournot output is 10

With known costs, mean industry output is the same as with private costs, but there is less variation in output
Sequentiality

Kreps-Wilson [1982]

Subforms

Beliefs: assessment $a_i$ for player $i$ probability distribution over nodes at each of his information sets; belief for player $i$ is a pair $b_i = (a_i, \pi^i_{-i})$ consisting of $i$’s assessment over nodes $a_i$, and $i$’s expectations of opponents’ strategies $\pi^i_{-i} = (\pi^i_j)_{j \neq i}$

Beliefs come from strictly positive perturbations of strategies

belief $b_i \equiv (a_i, \pi^i_{-i})$ is consistent (Kreps and Wilson) if where $a^n_i$

$a_i = \lim_{n \to \infty} a^n_i$ obtained using Bayes rule on a sequence of strictly positive strategy profiles of the opponents, $\pi^i_{-i} \to \pi_{-i}$
Sequential Optimality

given beliefs we have a well-defined decision problem at each information set; can define optimality at each information set

A sequential equilibrium is a behavior strategy profile $\pi$ and an assessment $a_i$ for each player such that $(a_i, \pi^i_{-i})$ is consistent and each player optimizes at each information set
Cho-Kreps [1987]

Signaling

 Cho-Kreps [1987]
Types of Equilibrium

sequential vs. trembling hand perfect
pooling and separating
**Chain Store Paradox**
Kreps-Wilson [1982], Milgrom-Roberts [1982]

finitely repeated model with long-run versus short-run
Reputational Model

two types of long-run player $\omega \in \Omega$

“rational type” and “committed type”

“committed type” will fight no matter what

types are privately known to long-run player, not known to short run player

Kreps-Wilson; Milgrom-Roberts

Solve for the sequential equilibrium; show that at the time-horizon grows long we get no entry until near the end of the game

“triumph of sequentiality”
The Holdup Problem

- Chari-Jones, the pollution problem
- problem of too many small monopolies

\( \rho \) is the profit generated by an invention with a monopoly with a patent, drawn from a uniform distribution on \([0, 1]\), private to the inventor

\( \varphi^F \) is the fraction of this profit that can be earned without a patent

To create the invention requires as input \( N \) other existing inventions

It costs \( \epsilon/N \) to make copies of each of these other inventions, where \( \epsilon < 1/2 \) and \( \epsilon/\varphi^F < 1 \)
Case 1: Competition

if $\varphi^F \rho \geq \epsilon$ the new invention is created, probability is $1 - \epsilon / \varphi^F$. 
**Case 2: Patent**

Each owner of the existing inventions must decide a price $p_i$ at which to license their invention; $\varphi N$ current inventions are still under patent.

Subgame Perfection/Sequentiality implies that the new invention is created when $\varphi \rho \geq \sum_i p_i$

Profit of a preexisting owner $(1 - \frac{(\varphi N - 1) \rho + p_i}{\varphi}) p_i$

FOC $1 - \frac{(\varphi N - 1) \rho + 2p_i}{\varphi} = 0$

unique symmetric equilibrium $p = \frac{\varphi}{(\varphi N + 1)}$ ; $\sum_i p_i / \varphi = \frac{\varphi N p}{\varphi}$

corresponding probability of invention is $1/(\varphi N + 1)$
Robustness

genericity in normal form games
example of Selten extensive form game
Fudenberg, Kreps, Levine [1988]

\[
\begin{align*}
(-1, -1) & \quad \text{L} \\
(1, 1) & \quad \text{U} \\
(2, 0) & \quad \text{R}
\end{align*}
\]
elaborated Selten game
normal form of elaborated Selten game

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Mechanism Design: An “auction” problem

- Single seller has a single item
- Seller does not value item
- Two buyers with independent valuations

\[ 0 \leq v^l < v^h \] low and high valuations

\[ \pi^l + \pi^h = 1 \] probabilities of low and high valuations
what is the best way to sell the object

- Auction
- Fixed price
- Other
The Revelation Principle

Design a game for the buyers to play

- Auction game
- Poker game
- Etc.

Design the game so that there is a Nash equilibrium that yields highest possible revenue to the seller

The revelation principle says that it is enough to consider a special game

- strategies are “announcements” of types
- the game has a “truthful revelation” equilibrium
In the Auction Environment

Fudenberg and Tirole section 7.1.2

$q^l, q^h$ probability of getting item when low and high

$p^l, p^h$ expected payment when low and high
individual rationality constraint

(IR) \[ q^i v^i - p^i \geq 0 \]

- if you announce truthfully, you get at least the utility from not playing the game

incentive compatibility constraint

(IC) \[ q^i v^i - p^i \geq q^{-i} v^i - p^{-i} \]

- you gain no benefit from lying about your type

the incentive compatibility constraint is the key to equilibrium
Other constraints

$q^l, q^h$ probability of getting item when low and high

they can’t be anything at all:

probability constraints

(1) $0 \leq q^i \leq \pi^{-i} + \pi^i / 2$

(win against other type, 50% chance of winning against self)

(2) $\pi^l q^l + \pi^h q^h \leq 1 / 2$

(probability of getting the good before knowing type less than 50%)
**Seller Problem**

Maximize seller utility \( U = \pi^l p^l + \pi^h p^h \)

Subject to IC and IR

To solve the problem we make a guess:

IR binds for low value
\[ q^l v^l - p^l = 0 \]

IC binds for high value
\[ q^h v^h - p^h = q^l v^h - p^l \]
The solution

\( p^l = q^l v^l \) from low IR

substitute into high IC

\( p^h = (q^h - q^l) v^h + q^l v^l \)

plug into utility of seller

\[
U = \pi^l q^l v^l + \pi^h ((q^h - q^l) v^h + q^l v^l)
\]

\[
U = q^l (\pi^l v^l - \pi^h v^h + \pi^h v^l) + \pi^h q^h v^h
\]

\( \pi^l + \pi^h = 1 \) so

\[
U = q^l (v^l - \pi^h v^h) + \pi^h q^h v^h
\]
**Case 1:** $v^l > \pi^h v^h$

$$U = q^l(v^l - \pi^h v^h) + \pi^h q^h v^h$$

(1) $0 \leq q^i \leq \pi^{-i} + \pi^i / 2$

(2) $\pi^l q^l + \pi^h q^h \leq 1/2$

Make $q^l, q^h$ large as possible so $\pi^l q^l + \pi^h q^h = 1/2$

$$U = \frac{1/2 - \pi^h q^h}{\pi^l} (v^l - \pi^h v^h) + \pi^h q^h v^h$$

$$U = \frac{1}{2\pi^l} (v^l - \pi^h v^h) + q^h \frac{\pi^h}{\pi^l} (v^h - v^l)$$
Finish of Case 1

so $q^h$ should be as large as possible

$$q^h = \pi^l + \pi^h / 2$$

plug back into (2) to find

$$q^l = \pi^l / 2$$

expected payments

$$p^l = q^l v^l, \quad p^h = (q^h - q^l)v^h + q^l v^l$$

$$p^l = v^l \pi^l / 2, \quad p^h = v^h / 2 + \pi^l v^l / 2$$
Implementation of Case 1

modified auction: each player announces their value
the highest announced value wins; if there is a tie, flip a coin
if the low value wins, he pays his value; if the high value wins he pays

\[
\frac{p^h}{q^h} = \frac{v^h/2 + \pi^l v^l/2}{\pi^l + \pi^h/2}
\]

under these rules
probability that high type wins is \(q^h = \pi^l + \pi^h/2\)
probability that low type wins is \(q^l = \pi^l/2\)
just as in the optimal mechanism;
this means the expected payments are the same too
Case 2: $v^l < \pi^h v^h$

$$U = q^l (v^l - \pi^h v^h) + \pi^h q^h v^h$$

(1) $0 \leq q^i \leq \pi^{i-1} + \pi^i / 2$

(2) $\pi^l q^l + \pi^h q^h \leq 1/2$

Make $q^h$ large as possible, $q^l$ as small as possible

$q^h = \pi^l + \pi^h / 2$

$q^l = 0$
expected payments

\[ p^l = q^l v^l, \quad p^h = (q^h - q^l)v^h + q^l v^l \]

\[ p^l = 0 \]

\[ p^h = (\pi^l + \pi^h / 2)v^h \]
Implementation of Case 2

set a fixed price equal to the highest valuation

\[ p^h = \frac{\pi^l + \pi^h / 2}{\pi^l + \pi^h / 2} \]
Macro Mechanism Design: The Insurance Problem

Kehoe, Levine and Prescott [2000]
continuum of traders ex ante identical
two goods \( j = 1, 2 \)
\( c_j \) consumption of good \( j \)
utility is given by \( \tilde{u}_1(c_1) + \tilde{u}_2(c_2) \)
each household has an independent 50% chance of being in one of two states, \( s = 1, 2 \)
endowment of good 1 is state dependent
\( \omega_1(2) > \omega_1(1) \)
endowment of good 2 fixed at \( \omega_2 \).
In the aggregate: after state is realized half of the population has high endowment half low endowment
**Gains to Trade**

after state is realized

low endowment types purchase good 1 and sell good 2

before state is realized

traders wish to purchase insurance against bad state

unique first best allocation

all traders consume \( \frac{\omega_1(1) + \omega_1(2)}{2} \) of good 1, and \( \omega_2 \) of good 2.
Private Information

idiosyncratic realization private information known only to the household

first best solution is not incentive compatible

low endowment types receive payment

\[
\frac{(\omega_1(2) - \omega_1(1))}{2}
\]

high endowment types make payment of same amount

high endowment types misrepresent type to receive rather than make payment
Incomplete Markets

prohibit trading insurance contracts

consider only trading ex post after state realized

resulting competitive equilibrium

- equalization of marginal rates of substitution between the two goods for the two types

- low endowment type less utility than the high endowment type
Mechanism Design

purchase $x_1(1) > 0$ in exchange for $x_1(2) < 0$

no trader allowed to buy a contract that would later lead him to misrepresent his state

assume endowment may be revealed voluntarily, so low endowment may not imitate high endowment

incentive constraint for high endowment

$$\tilde{u}_1(\omega_1(2) + x_1(2)) + \tilde{u}_2(\omega_2 + x_2(2))$$

$$\geq \tilde{u}_1(\omega_1(2) + x_1(1)) + \tilde{u}_2(\omega_2 + x_2(1))$$

- Pareto improvement over incomplete market equilibrium possible since high endowment strictly satisfies this constraint at IM equilibrium

Need to monitor transactions
**Lotteries and Incentive Constraints**

one approach: $X$ space of triples of net trades satisfying incentive constraint

use this as consumption set

enrich the commodity space by allowing sunspot contracts (or lotteries)
1) $X$ may fail to be convex
2) incentive constraints can be weakened - they need only hold on average

\[
E \left|_2 \tilde{u}_1(\omega_1(2) + x_1(2)) + \tilde{u}_2(\omega_2 + x_2(2)) \right. \\
\geq E \left|_1 \tilde{u}_1(\omega_1(2) + x_1(1)) + \tilde{u}_2(\omega_2 + x_2(1)) \right.
\]
Other Applications of Mechanism Design

- general equilibrium theory
- public goods
- taxation
- price discrimination
**Common versus Individual Punishment**

\[ N \]

choose \( a_i \in \{0, 1\} \) effort to contribute to a public good (equals cost)

no effort, no input \( y_i = 0 \)

effort, probability \( 1 - \pi \) of input

let \( M \) be the number who contribute, then contributors get

\[
(M(1 - \pi)/N)V - 1
\]

non-contributors get

\[
(M(1 - \pi)/N)V
\]

where \( V(1 - \pi) > 1 \)

suppose also that \( V(1 - \pi)/N < 1 \) so no voluntary contributions
Crime and Punishment

A punishment $P$ 

common punishment: if the punishment occurs everyone is punished (will cancel future public goods projects...)

individual punishment: each individual may be punished or not separately
Common Punishment under Certainty

\[ \pi = 0 \]
if everyone contributes no punishment
otherwise punishments
incentive compatibility
\[ V - 1 \geq ((N - 1)/N)V - P \]
or
\[ P \geq 1 - V/N \]
expected cost of the punishment is zero
Common Punishment under Uncertainty

Probability someone doesn’t contribute is $1 - \pi^N$

incentive compatibility

$$V(1 - \pi) - 1 - (1 - \pi^{N-1})P \geq \left(\frac{(N - 1)}{N}\right)(1 - \pi)V - P$$

or

$$P \geq \frac{(1 - V(1 - \pi)/N) / \pi^{N-1}}{\pi^{N}}$$

expected cost of punishment

$$(1 - \pi^N)(1 - V(1 - \pi)/N) / \pi^N$$

goes to infinity as $N \to \infty$
Theorem (Fudenberg, Levine and Pesendorfer)

People are always trying to figure a way around this (perpetual motion machine of economics)

Suppose that $P$ is bounded above

for any mechanism public good production goes to zero as $N \to \infty$
Individual Punishment

Punish if \( y_i = 0 \)

incentive constraint

\[
V(1 - \pi) - 1 - \pi P \geq ((N - 1)/N)V(1 - \pi) - P
\]

or

\[
P \geq (1 - V(1 - \pi)/N)/(1 - \pi)
\]

with expected cost of punishment

\[
\pi(1 - V(1 - \pi)/N)/(1 - \pi)
\]

less than \( V(1 - \pi) - 1 \) then produce the public good