Solutions Key to
Problem Sets in Game Theory

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Problem Set 3: Decision Theory

Exercise 1: Risk Aversion in the Lab

From experimental data of Peter Boessarts and Charles Plott, individuals in the laboratory are indifferent between getting nothing for sure, and a gamble which pays: \$9.75, –\$3.00, –\$2.25 each with probability 1/3. Assume the standard approximation for the absolute risk premium p using a fixed coefficient of relative risk aversion ρ:

$$p = \frac{\rho}{x}\frac{\sigma^2}{2}$$

We have that ρ is the relative risk aversion, σ^2 the variance of the lottery, x is the wealth of the individuals. We will compute the expected value of the lottery to then compute its variance.

$$\begin{aligned} L_1 &= 9.75, -3, -2.25 \qquad p = 1/3 \\ L_0 &= 0 \qquad\qquad\qquad p = 1 \\ \mathbb{E}[L_1] &= \frac{1}{3}(9.75 - 3 - 2.25) = 1.5 \\ \sigma^2 &= \frac{1}{3}[(9.75-1.5)^2 + (-3-1.5)^2 + (-2.25-1.5)^2] = 34.125 \end{aligned}$$

Risk premium p: difference between the mean outcome of a lottery and the riskless amount that makes the agent indifferent between taking the lottery or not (certainty equivalent).

$$u\left(\mathbb{E}[L_1] - p + w\right) = \left(\mathbb{E}[L_0] + w\right)$$
$$1.5 - p = 0 \qquad\qquad p = 1.5$$

*This version builds on the solutions provided by Damiano Argan and Konuray Mutluer.

- *If wealth is $350,000, what is the coefficient of relative risk aversion?*

$$\rho = 2p\frac{x}{\sigma^2} = 2 \cdot 1.5 \frac{350000}{34.125} \approx 30769$$

- *If the coefficient of relative risk aversion is 20, what is wealth?*

$$x = \frac{\rho}{p}\frac{\sigma^2}{2} = \frac{20}{1.5}\frac{34.125}{2} \approx 227.5$$

Side note: the equity premium puzzle

We can connect these results to the equity premium puzzle. The equity premium puzzle refers to the inability of macroeconomic models to explain the excess returns of stocks over U.S. bonds. To achieve realistic values of the equity premium, these models need to be calibrated with an unrealistically high risk aversion coefficient. [1] In a sense, this is also happening int the lab. These individuals display extreme risk averse behavior even under very small stakes relative to their initial wealth. During the TA class we discussed some reasons why this could be the case: experimenter effects such as lab bias or the effects of an artificial level of wealth created in the lab.

Exercise 2: Present Bias and Quasi-hyperbolic Discounting

Suppose that you are indifferent between receiving 175 euros today and 225 euros in four weeks time. You are also indifferent between receiving 175 euros in twenty six weeks time and 180 euros in thirty weeks time. Assume that you are a quasi-hyperbolic discounter so that your objective function has the form:

$$u_0 + \theta \sum_{t=1}^{\infty} \delta^t u_t$$

You are also risk neutral. What are θ, δ?

$$\begin{cases} 175 = \theta\delta^4 225 \\ \theta\delta^{26} 175 = \theta\delta^{30} 180 \end{cases}$$

That is $\delta = 0.992$ and $\theta = 0.8$.

Exercise 3: Maurice at the Casino

Suppose that you go to the casino and at the entrance a French gentleman introduces himself as Maurice Allais and offers you a choice of either 2400 euros for sure or a gamble with a 33% chance of winning 2500 euros and a 66% chance of winning 2400 euros. You choose the 2400 euros for sure. The next night, the same gentleman again greets you and offers you a choice between 33%

[1] Mehra, Rajnish; Edward C. Prescott (1985). "The Equity Premium: A Puzzle". *Journal of Monetary Economics.* **15** (2): 145–161.

chance of winning 2500 euros and a 34% chance of winning 2400 euros. You choose the 2500 euro gamble. Prove that you are not an expected utility maximizer.

From expected utility theorem:

$$L \succsim L^{'} \quad \text{iff} \quad \sum_{n=1}^{N} u_n p_n^{\prime} =\geq \sum_{n=1}^{N'} u_{n'} p_{n'}$$

$$\begin{array}{lccc} 1^{st} \text{ night:} & u(24) & > & 0.33u(25) + 0.66u(24) + 0.01u(0) \\ 2^{nd} \text{ night:} & 0.33u(25) + 0.67u(0) & > & 0.34u(24) + 0.66u(0) \end{array}$$

$$\left.\begin{array}{lcl} 0.34u(24) & < & 0.33u(25) + 0.01u(0) \\ 0.34u(24) & > & 0.33u(25) + 0.01u(0) \end{array}\right\} \text{Contradiction}$$

There exists no increasing utility function satisfying the two inequalities above. The agent is not an expected utility maximizer.

Exercise 4: Gambling Professors

An economics professor from Big U is watching about to watch a basketball game between Big U and State U. His friend proposes a bet on the outcome – whoever loses has to purchase beer at the pub after the game. The economics professor is reluctant because he always loses whenever he bets with this friend on basketball games. The friend in a spirit of fairness offers the professor the chance to choose either team, but insists he must bet. What should the professor do and why?

This is a game between the professor and nature. The professor can use a public randomization mechanism and assign a probability q to Big U. Nature *chooses* p for Big U to win. In the statement *the professor always looses* we can read that the professor feels that nature is minimizing him.

Let me normalize the payoffs, so that the utility of winning the bet is 1, and the utility of loosing the bet is 0. The expected utility of the professor is then:

$$\begin{aligned} \mathbb{E}[u] &= pq + (1-p)(1-q) = 1 + 2pq - p - q \\ &= p(2q-1) + 1 - q \end{aligned}$$

In the last equation we isolate the term that nature can influence: $(2q - 1)$. Depending on the values that the professor assigns to this term, we can have several cases:

$$\begin{array}{lclcc} 2q-1>0 & \rightarrow & q>1/2 & \text{prof. feels that } p=0 & \mathbb{E}[u]=1-q<1/2 \\ 2q-1<0 & \rightarrow & q<1/2 & \text{prof. feels that } p=1 & \mathbb{E}[u]=q<1/2 \\ 2q-1=0 & \rightarrow & q=1/2 & \forall p \in [0,1] & \mathbb{E}[u]=q=1/2=1-q \end{array}$$

The professor should insure against a "hostile" nature by betting 1/2 of the time for Big U. and 1/2 for State U.

Exercise 5: Where to Drink?

In Berlin there are many nightclubs of varying quality $q \in (0, \infty)$. If you spend amount m at a nightclub of quality you receive utility:

$$u(m|q) = \log q - \frac{(m/q)^{1-\rho} - 1}{\rho - 1} \qquad \rho \geq 1$$

1. *If you gamble at the nightclub what is your coefficient of relative risk aversion? [here you gamble after choosing a nightclub]*

   One gambles after choosing q, so we take q as given.

$$\begin{aligned}
RRA &= -m\frac{u''}{u'} \\
\frac{\partial u}{\partial m} &= -m^{-\rho}\left(\frac{1}{\rho}\right)^{1-\rho} \\
\frac{\partial^2 u}{\partial m^2} &= \rho m^{-\rho-1}\left(\frac{1}{\rho}\right)^{1-\rho} \\
RRA &= -(-m)\frac{\rho m^{-\rho-1}}{m^{-\rho}\left(\frac{1}{\rho}\right)^{1-\rho}} = \rho
\end{aligned}$$

   As the utility function is CRRA, the risk aversion coefficient is constant.

2. *If you have a budget of to spend at the nightclub which quality should you choose? We may imagine that if you go to a nightclub that sells expensive wine you will not get much satisfaction if you have a very limited budget, while if you go to a nightclub that sells cheap beer and have a very large budget you will find that much of your entertainment budget is wasted. Does this utility function capture that idea?*

   Fixed quantity of m, choose q. Since $u(\cdot)$ is strictly increasing in m, all money will be spent in one club. Additionally, as the function is concave in q: $q = m$.

$$\begin{aligned}
\frac{\partial u}{\partial q} &= 0 \\
\frac{1}{q} &= \frac{(1-\rho)(w/q)^{-q}(-w/q^2)}{\rho - 1} \\
q &= w
\end{aligned}$$

   We will choose the quality that matches the wealth. In fact, wealth and quality need to go hand in hand. [2]

3. *If you choose between risky investments for your budget what is your coefficient of relative risk aversion for your investments? [here you gamble before choosing a nightclub]*

   We will apply backwards induction. For a fixed gamble, we will choose a quality. Recall that the individual is optimizing, so he will choose a quality equal to his budget ($q = m$). The utility function at the top of the gamble becomes:

$$u(m|q = m) = \log m$$

[2] This can also be seen from the positive cross-derivative.

And the coefficient of relative risk aversion is:

$$\rho = -m\frac{u''}{u'} = m\frac{1/m^2}{1/m} = 1$$

Note that in this part of the question, the individual is less risk averse than in *1*. This is related to timing. In question *3*, the individual is allowed to "re-optimize" the gamble with respect to the club: first choose m, then choose q. In question *1*, he is given the club before the gamble takes place.