Question 2-Repeated Games

	A	B
A	4,4	1,6*
B	5*,1	2*,2*

The static Nash equilibrium in this game is (B,B).We would like to find strategies and a discount factor such that (A,A), giving the payoff (4,4), can be sustained in the infinitely repeated game.

We can use the following "grim" strategies to sustain (A,A):
For player 1: Start with playing A. Play A as long as (A,A) has been observed in every previous period, otherwise play B.
For player 2: Start with playing A. Play A as long as (A,A) has been observed in every previous period, otherwise play B.

For the above strategies to constitute an equilibrium, no player should have an incentive to deviate. The conditions for this are given below:

For Player 1:
$(1-\delta)5+\delta(2) \leq 4$ (Average payoff to deviating should be smaller than the average payoff to sticking to the grim strategy)
$\Leftrightarrow 5-5\delta+2\delta \leq 4$
$\Leftrightarrow 3\delta \geq 1 \Leftrightarrow \delta \geq 1/3$

For Player 2, the relevant condition is:
$(1-\delta)6+\delta(2) \leq 4$
$\Leftrightarrow 4\delta \geq 2 \Leftrightarrow \delta \geq 1/2$

So, players should have a discount factor of at least ½ for the proposed strategies to be a Nash equilibrium of the repeated game.

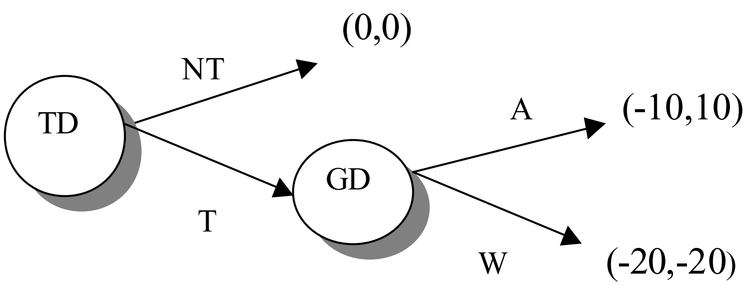
Yes, this equilibrium is subgame perfect, because the grim strategies constitute a Nash equilibrium in every subgame of the repeated game. This is so because we use the static Nash equilibrium (which is subgame perfect in the repeated game) as a punishment. In subgames where previous play has always been (A,A), the prescribed strategy is (A,A), which we showed is a Nash equilibrium. In subgames where in at least one previous period (A,A) has not been observed, the static Nash is played forever, which is again a Nash equilibrium of the repeated game. So, the grim strategies are subgame perfect.

The Folk Theorem tells us that any payoff in the socially feasible individually rational set can be sustained in a subgame perfect equilibrium of the repeated game, for sufficiently high discount factors (discount factors close enough to 1).

(Notice that the minmax payoff to each player is 2 in this game. This implies that the SFIR region would include socially feasible payoff pairs greater than or equal to (2,2)).

3. Long Run versus Short Run

a, b) The extensive and normal form are given below.(Notice that the gentle democracy is player 1)



	NT	T
A	0*,0	-10*,10*
W	0*,0*	-20, -20

Nash: (A,T) and (W,NT) (stars denote best responses)
Subgame perfect: (A,T) (marked with thicker lines)
Note that (W,NT) is Nash, but not plausible in the stage game, since the democracy wouldn't go to war if threat takes place.

c) If 1 commits himself to play A: 2 plays T, 1 gets –10.
If 1 commits himself to play W: 2 plays NT, 1 gets 0.
Thus the Stackelberg equilibrium is (W,NT)

d) GD: Play W in the first period, keep playing W as long as you have played W in the past, play A if you have played A at least once.

TD: Play NT in the first period, Play NT as long as the democracy has played W, play T if the democracy has played A at least once.

The short run player always plays optimally. Since (A,T) is SPE of the stage game, it's SPE of the repeated game. After a failure to go to war, the equilibrium is the subgame perfect (A,T) equilibrium. For above strategies to be subgame perfect, we must find the value of δ such that it's optimal for the democracy to go to war if the dictator threatens (If he does so, the dictators won't wish to enter).

$$-20 + 0\delta + 0\delta^2 + \dots \geq -10 - 10\delta + 10\delta^2 + \dots$$

where left-hand side is the payoff to the democracy if threat takes place (in that case the democracy will get –20 upon threat but gets zero thereafter, since the short run players would not threaten seeing that the democracy has fought) and right-hand side is the payoff from deviation, that is from appeasing every period (if A is played once, we go to (A,T)).

Simple algebra yields,

$-20 \geq -10/(1-\delta)$

$-20/(1-\delta) \geq -10$

$\delta \geq 1/2$

Note that this makes sense, since the democracy is willing to fight in the repeated game (with sufficiently high discount factor), since the penalty is threat forever afterwards when he does not.

e) Reputation matters here. If threat occurs, the democracy will fight (by part d) and thus has the opportunity of building reputation, and this will keep short run players out, i.e. they won't threaten.

5. Cournot with Uncertain Cost

$p = 17$ ¡ X

	L^2	H^2
L^1	:1	:1
H^1	:1	:7

Note that we have to work out the conditional probabilities …rst.

$$P(L^2{=}L^1) = \frac{P(L^2 \setminus L^1)}{P(L^1)} = \frac{:1}{:1 + :1} = \frac{1}{2}$$

$$P(H^2{=}L^1) = 1 \text{ ¡ } P(L^2{=}L^1) = \frac{1}{2}$$

$$P(L^2{=}H^1) = \frac{P(L^2 \setminus H^1)}{P(H^1)} = \frac{:1}{:1 + :7} = \frac{1}{8}$$

$$P(H^2{=}H^1) = 1 \text{ ¡ } P(L^2{=}H^1) = \frac{7}{8}$$

Firm 1's problem, when it is low-cost:

$$\underset{x_L^1}{\text{Max}}\ ¦_L^1 = (17 \text{ ¡ } (x_L^1 + \frac{1}{2}x_L^2 + \frac{1}{2}x_H^2))x_L^1 \text{ ¡ } x_L^1$$

First-order condition yields,

$$16 = 2x_L^1 + \frac{1}{2}x_L^2 + \frac{1}{2}x_H^2 \qquad (1)$$

Firm 1's problem, when it is high-cost:

$$\underset{x_H^1}{\text{Max}}\ ¦_H^1 = (17 \text{ ¡ } (x_H^1 + \frac{1}{8}x_L^2 + \frac{7}{8}x_H^2))x_H^1 \text{ ¡ } 3x_H^1$$ (Note that we use the relevant conditional probabilities and marginal cost)

First-order condition yields,

$$14 = 2x_H^1 + \frac{1}{8}x_L^2 + \frac{7}{8}x_H^2 \qquad (2)$$

In principal, we have four unknowns, so we need four equations. But given the cost structure is the same (1 or 3 for both …rms) AND the probability matrix is symmetric, …rm two has the same conditional probabilities and …rm two's maximization problems (one for low-cost, one for high-cost) are the same as firm one's, with 1 and 2 switched around. In other words, we can impose symmetry at this point and say

$$x_L^1 = x_L^2 = x_L$$

$$x_H^1 = x_H^2 = x_H$$

(This is analogous to imposing symmetry in the complete information case when marginal costs are the same)

Now, we have two equations and two unknowns. The solution to this system of equations will be the Bayesian-Nash equilibrium.

With symmetry, (1) yields,

$$\frac{5}{2}x_L + \frac{x_H}{2} = 16$$

$$5x_L + x_H = 32$$

$$x_H = 32 - 5x_L \qquad (3)$$

Similarly, rewriting (2) gives,

$$\frac{23}{8}x_H + \frac{1}{8}x_L = 14$$

$$23x_H + x_L = 112 \qquad (4)$$

Substituting (3) into (4) gives,

$$112 - x_L = 736 - 115x_L$$

thus $x_L = \frac{104}{19}$

Plug this back into (3) to get $x_H = \frac{88}{19}$

The total industry output is simply the sum of individual outputs in each case and is summarized below

	L^2	H^2
L^1	$\frac{208}{19}$	$\frac{192}{19}$
H^1	$\frac{192}{19}$	$\frac{176}{19}$

Complete information:

There are four cases:

1) Both are low-cost
Firm 1's problem:

$$\underset{x^1}{Max}\ \pi^1 = (17 - (x^1 + x^2))x^1 - x^1$$

Differentiating gives,

$$16 = 2x^1 + x^2 \text{ or } x^1 = 8 - \frac{x^2}{2}$$

By symmetry, $x^1 = x^2 = x$ and from the reaction function above we can get $x^1 = x^2 = \frac{16}{3}$

2) Both are high-cost
Firm 1's problem:

$$\underset{x^1}{Max}\ \pi^1 = (17 - (x^1 + x^2))x^1 - 3x^1$$

$$14 = 2x^1 + x^2 \text{ or } x^1 = 7 - \frac{x^2}{2}$$

By symmetry, $x^1 = x^2 = x$ and from the reaction function above we can get $x^1 = x^2 = \frac{14}{3}$

3) Firm 1 is low-cost, firm 2 is high-cost
Firm 1's problem is the same as case 1 so we have

$$x^1 = 8 - \frac{x^2}{2} \qquad (5)$$

But now we have asymmetric costs and hence symmetry cannot be imposed. Instead we should work out firm 2's problem.

$$\underset{x^2}{Max}\ \pi^2 = (17 - (x^1 + x^2))x^2 - 3x^2$$

Differentiating and rewriting gives the reaction function of firm 2.

$$x^2 = 7 - \frac{x^1}{2} \qquad (6)$$

solving (5) and (6) gives $x^1 = 6; x^2 = 4:$

4) Firm 1 is high-cost, ...rm 2 is low-cost
That will give us $x^1 = 4; x^2 = 6$:

The total industry output is simply the sum of individual outputs in each case and is summarized below

	L^2	H^2
L^1	$\frac{32}{3}$	10
H^1	10	$\frac{28}{3}$

Comparing this to the uncertain cost case by subtracting the second matrix from the ...rst one gives,

	L^2	H^2
L^1	$\frac{16}{57}$	$\frac{2}{19}$
H^1	$\frac{2}{19}$	i $\frac{4}{57}$

The comparison reveals two things. First, the total outputs are pretty close, second, the uncertain cost structure yields higher output in all cases but when both ...rms are high-cost.

4. Bayes Law

The probabilities that are directly and indirectly given in the description,

$\Pr(-/TT) = 0.1$ — the probability given you tell the truth however the lie-detector shows lying signal

$\Pr(+/TT) = 1 - \Pr(-/TT) = 0.9$

Pr (L) =0.2 — deceptive employees percentage,

$\Pr(TT) = 1 - P(L) = 0.8$

$\Pr(-) = 0.2$ — the probability that the lie-detector shows lying signal

(1) calculate $\Pr(-/L)$

From Bayes rule:

$$\Pr(-) = \Pr(-/TT) * \Pr(TT) + \Pr(-/L) * \Pr(L)$$
$$0.2 = 0.1 * 0.8 + \Pr(-/L) * 0.2$$
$$\Pr(-/L) = 0.6$$

(2) $\Pr(-/spy) = 1/2 * 0.6 = 0.3$ $\Pr(+/spy) = 1 - \Pr(-/spy) = 0.7$

(3) Pr (spy) = 0.001 Pr (~spy) = 1 – Pr (spy) = 0.999
$\Pr(-/\sim spy) = \Pr(-/TT) = 0.1$ (from description)
$\Pr(+/\sim spy) = \Pr(+/TT) = 1 - \Pr(-/TT) = 0.9$

$$\Pr(spy/-) = \frac{\Pr(-/spy) * \Pr(spy)}{\Pr(-/spy) * \Pr(spy) + \Pr(-/\sim spy) * \Pr(\sim spy)}$$

$$= \frac{0.3 * 0.001}{0.3 * 0.001 + 0.1 * 0.999}$$

$$\approx 0.00299 = 0.299\%$$

$$\Pr(spy/+) = \frac{\Pr(+/spy) * \Pr(spy)}{\Pr(+/spy) * \Pr(spy) + \Pr(+/\sim spy) * \Pr(\sim spy)}$$

$$= \frac{0.7 * 0.001}{0.7 * 0.001 + 0.9 * 0.999}$$

$$\approx 0.000778 = 0.0778 \%$$