1 The Extensive Form of a Game

First I formally define a game in extensive form, and then I look at mixed and behavior strategies.

1.1 Formal Description of a Game

Perhaps obsessively, I will avoid reference to any objects that aren’t derived explicitly from terminal nodes or players. I won’t distinguish games that differ by redundant moves, i.e., branches like this.

Formally, we begin with a finite set $Z$ of terminal nodes. Let $X \subset 2^Z$ be a family of subsets of $Z$. Call $X$ a rooted tree over $Z$ if it satisfies the following three conditions:

1. $Z \in X$ (there is a root node),
2. $\{z\} \in X$ for every $z \in Z$ (exhaustion of alternatives),
3. For any $x, y \in X$, if $x \cap y \neq \emptyset$ then either $x \subset y$ or $y \subset x$ (uniqueness of paths).\(^1\)

We say that $y$ is a successor of $x$ if $y \subset x$. The interpretation of a node $x$ is that it fully characterizes (and modulo redundant nodes is also characterized by) the set of possible terminal nodes that are reachable from $x$. This motivates defining $Z$ as the root node, and $\{z\}$ as a terminal node.

\(^1\)My convention is that $\subset$ denotes weak inclusion (so $x \subset x$).
For any two nodes $x \supset y$, the partial path from $x$ to $y$ is denoted and defined as

$$[x, y] := \{w \in X : x \supset w \supset y\}.$$ 

By definition, there is only one partial path from $x$ to $y$. In particular, every terminal node $\{z\}$ has a unique path to it from the root node $Z$. The set of nonterminal nodes is $X \setminus Z$.\(^2\)

Let $S(x) = \{y : [x, y] = \{x, y\}\}$ be the set of immediate successors of $x$.\(^3\) A nodal decision is a partial path $[x, y]$ with $y \in S(x)$. A nodal decision problem is the set of possible decisions $[x, S(x)] = \{[x, y] : y \in S(x)\}$. Hence, each node is uniquely associated with a decision problem.

A *decision structure* for $Z$ is a triple $(X, H, A)$ satisfying:

1. $X$ is a rooted tree over $Z$,
2. $H$ is a partition of the nonterminal nodes $X \setminus Z$ such that for every $h \in H$ and $x, y \in h$, $|S(x)| = |S(y)|$, that is, $x$ and $y$ have the same number of immediate successors,
3. $A$ is a correspondence mapping each information set $h \in H$ to a subset $A(h) \subset \prod_{x \in h}[x, S(x)]$ such that for any $x \in h$ and $y \in S(x)$, there is a unique $a \in A(h)$ with $a(x) = [x, y]$.

We now define players’ tastes and their turns in the game. Let $I$ be a set of players, with $i$’s utility function $u_i : Z \rightarrow \mathbb{R}$ defined on terminal nodes. It remains is to allocate information sets to players. An *allocation of turns and tastes to $I$* is a triple $(u, T, \mu)$ such that $u : I \times Z \rightarrow \mathbb{R}$ is a profile of utility functions, $T : H \rightarrow I \cup \{N\}$ is an allocation of turns\(^4\) (i.e., information sets) to players, and $\mu$ belongs to

$$\prod_{h \in T^{-1}(N)} \Delta(A(h)),$$

the product space of probability measures over each of nature’s possible moves. Let’s denote by $H_i = T^{-1}(i)$ the collection of information sets that belong to player $i$.

**Definition 1.1** An extensive form game is a tuple $\Gamma = (Z, (X, H, A), I, (u, T, \mu))$ such that

1. $(X, H, A)$ is a decision structure for $Z$,
2. $(u, T, \mu)$ is an allocation of turns and tastes to $I$.

Without further ado, let’s consider a simple game and translate it into this formal framework.

\(^2\)This is not $X \setminus \{Z\}$, which is the set of noninitial nodes.

\(^3\)By condition 2, $S(x)$ is empty if and only if $x$ is a terminal node.

\(^4\)Player $N$ is nature.
Example 1.2 Consider the following extensive-form game.

![Game Diagram]

The game in Example 1.2 is described as follows. The set of terminal nodes is

\[ Z = \{ b_1w_2, b_1x_2, a_1w_2y_1, a_1w_2z_1, a_1x_2y_1, a_1x_2z_1 \} . \]

The rooted tree \( X \subset 2^Z \) is constructed so that each node is represented by the possible terminal nodes from it. Hence,

\[ X = \{ Z , \{ a_1w_2y_1, a_1w_2z_1, a_1x_2y_1, a_1x_2z_1 \} , \{ b_1w_2, b_1x_2 \} , \{ a_1w_2y_1, a_1w_2z_1 \} , \{ a_1x_2y_1, a_1x_2z_1 \} , \{ b_1w_2 \} , \{ b_1x_2 \} , \{ a_1w_2y_1 \} , \{ a_1w_2z_1 \} , \{ a_1x_2y_1 \} , \{ a_1x_2z_1 \} \} . \]

Players 1 and 2 play this game. Since we can identify nodes on the game tree with partial paths on the tree from the root, there is no loss in denoting the set \( X \) of nodes by

\[ X = \{ \aleph, a_1, b_1, a_1w_2, a_1x_2, b_1w_2, b_1x_2, a_1w_2y_1, a_1w_2z_1, a_1x_2y_1, a_1x_2z_1 \} , \]

where \( \aleph \) stands for the root node. There are no chance nodes in this game. The information sets belonging players 1 and 2 are

\[ H_1 = \{ \aleph, \{ a_1w_2, a_1x_2 \} \} , \]
\[ H_2 = \{ \{ a_1, b_1 \} \} . \]

For each player \( i \), an element of \( H_i \) is an information set for \( i \). The action correspondence \( A \) is given by the sets of partial paths below.\(^5\)

\[ A(\aleph) = \{ a_1, b_1 \} , \]
\[ A(\{ a_1, b_1 \}) = \{ w_2, x_2 \} , \]
\[ A(\{ a_1w_2, a_1x_2 \}) = \{ y_1, z_1 \} . \]

\(^5\)Formally, \( w_2 = \{ a_1w_2, b_1w_2 \} , \text{ etc.} \)
1.2 Mixed and Behavior Strategies

There are two basic ways to think of strategic behavior in extensive form games. We will now define a mixed strategy and a behavior strategy.

**Definition 1.3** Fix a player \( i \in N \). A **mixed strategy** for player \( i \) is any \( \sigma_i \) in

\[
\Sigma_i := \Delta \left( \prod_{h_i \in H_i} A(h_i) \right). \quad \text{(Normal-form representation.)}
\]

A behavior strategy for player \( i \) is any \( \pi_i \) in

\[
\Pi_i := \prod_{h_i \in H_i} \Delta (A(h_i)). \quad \text{(Multi-agent representation.)}
\]

The set of pure strategies is the product space

\[
C_i := \prod_{h_i \in H_i} A(h_i),
\]

where each \( c_i \in C_i \) is to be interpreted as a complete contingent plan for player \( i \).

I will also write \( \Sigma_{-i}, \Pi_{-i}, \Sigma, \Pi \) to mean the obvious objects. (The product spaces across all players but \( i \) and across all players, respectively.) Let’s find \( C_1 \) and \( C_2 \) in the example.

\[
C_1 = \{a_1, b_1\} \times \{y_1, z_1\} = \{(a_1, y_1), (a_1, z_1), (b_1, y_1), (b_1, z_1)\},
\]

\[
C_2 = \{w_2, x_2\}.
\]

A mixed strategy for player \( i \) is an element of \( \Delta(C_i) \). Clearly, the mixed and behavior strategies for Player 2 coincide. An example of a mixed strategy for Player 1 is \( \left[ (a_1, y_1) \right] + \left[ (b_1, z_1) \right] \). A behavior strategy for player 1 is an element of

\[
\Delta((a_1, b_1)) \times \Delta((y_1, z_1)).
\]

An example of a behavior strategy is \( \left[ (\frac{1}{2}[a_1] + \frac{1}{2}[b_1], \frac{1}{2}[y_1] + \frac{1}{2}[z_1]) \right] \). This strategy is clearly very different from the mixed strategy above. In the mixed strategy, Player 1 doesn’t mix at all between \( y_1 \) and \( z_1 \).

Given a behavior strategy \( \pi_i \), its mixed-strategy representation is the mixed strategy \( \sigma_i \) given by

\[
\sigma_i(c_i) = \prod_{h_i \in H_i} \pi_i(c_i(h_i)).
\]

We construct the mixed-strategy representation by assuming that behavior strategies involve independent mixing across information sets. For instance, the mixed-strategy representation of the behavior strategy \( \left[ (\frac{1}{2}[a_1] + \frac{1}{2}[b_1], \frac{1}{2}[y_1] + \frac{1}{2}[z_1]) \right] \) is given by \( \frac{1}{4}[(a_1, y_1)] + \frac{1}{4}[(a_1, z_1)] + \frac{1}{4}[(b_1, y_1)] + \frac{1}{4}[(b_1, z_1)] \).
Given a mixed-strategy profile $\sigma$ and a terminal node $z \in Z$, the probability of $z$ being reached under $\sigma$ is denoted by $P(z|\sigma)$. Given an information state $h_i$, the probability of $h_i$ being reached under $\sigma$ is denoted by $P(h_i|\sigma)$.\footnote{Without chance nodes, if $c$ is a pure-strategy profile, then $P(h_i|c)$ is either 0 or 1.} Given a behavior strategy $\pi_i$, I will denote by $\hat{\sigma}_i(\cdot|\pi_i)$ the mixed representation of $\pi_i$. I will also write, for a behavior-strategy profile $\pi$, $P(x|\pi)$ to represent the probability that $x$ is reached under $\pi$. Clearly, $P(x|\pi) = P(x|\hat{\sigma}(\cdot|\pi))$. In Example 1.2, let $\sigma$ be the pure-strategy profile $((b_1, y_1), w_2)$. It is easy to verify that $P(z|\sigma) = 1$ if $z = b_1 w_2$ and 0 otherwise. Similarly, $P\{a_1, b_1\} = 1$, but $P\{a_1 w_2, a_1 x_2\} = 0$.

**Definition 1.4** For each information set $h_i \in H_i$, and action $a_i \in A(h_i)$, let

$$C_1^i(h_i) = \left\{ c_i \in C_i : \exists c_{-i} \text{ s.t. } P(h_i|c) > 0 \right\},$$

$$C_1^{**}(h_i, a_i) = \left\{ c_i \in C_1^i(h_i) : c_i(h_i) = a_i \right\}.$$

For example, I compute the following sets.

$$C_1^1 = C_1^1\{(a_1 w_2, a_1 x_2)\} = \{(a_1, y_1), (a_1, z_1)\},$$

$$C_1^{**} = C_1^{**}\{(a_1 w_2, a_1 x_2), y_1\} = \{(a_1, y_1)\}.$$

Given a mixed strategy $\sigma_i$, a behavior strategy $\pi_i$ is called a behavioral representation of $\sigma_i$ if for every $h_i \in H_i$, and every $a_i \in A(h_i)$, the following equality holds:

$$\pi_i(h_i)(a_i) \left[ \sum_{c_i \in C_1^i(h_i)} \sigma_i(c_i) \right] = \sum_{c_i \in C_1^{**}(h_i, a_i)} \sigma_i(c_i). \quad (*)$$

Clearly, if the summation on the left is zero then the summation on the right is zero, too. Notice that if we read the terms of $(*)$ as

$$P(a_i|h_i) := \pi_i(h_i)(a_i), \quad P(h_i|\sigma_i) := \sum_{c_i \in C_1^i(h_i)} \sigma_i(c_i), \quad P(h_i, a_i|\sigma_i) := \sum_{c_i \in C_1^{**}(h_i, a_i)} \sigma_i(c_i),$$

then $(*)$ is just Bayes’ rule: $P(a_i|h_i) P(h_i|\sigma_i) = P(h_i, a_i|\sigma_i)$. Hence, a behavioral representation of $\sigma_i$ is any behavior strategy $\pi_i$ that satisfies, given $h_i \in H_i$, and $a_i \in A(h_i)$,

$$\pi_i(h_i)(a_i) = \begin{cases} \frac{P(h_i, a_i|\sigma_i)}{P(h_i|\sigma_i)} & \text{if } P(h_i|\sigma_i) > 0, \\ \text{anything} & \text{otherwise,} \end{cases}$$

with the proviso that “anything” must add up to one on every $h_i$.\footnote{I.e., behavioral representations are only defined up to zero-probability events, like conditional expectation.} For example, consider the mixed strategies $\frac{1}{2}[(a_1, y_1)] + \frac{1}{2}[(b_1, z_1)]$ and $\frac{1}{2}[(a_1, y_1)] + \frac{1}{2}[(b_1, y_1)]$. They both have the same behavioral representation $\left(\frac{1}{2}[(a_1, y_1)] + \frac{1}{2}[(b_1, y_1)]\right)$, which in turn has the mixed-strategy representation $\left(\frac{1}{2}[(a_1, y_1)] + \frac{1}{2}[(b_1, y_1)]\right)$.\footnote{Without chance nodes, if $c$ is a pure-strategy profile, then $P(h_i|c)$ is either 0 or 1.}
Definition 1.5 Two mixed strategies \( \sigma_i, \tau_i \in \Delta(C_i) \) are called behaviorally equivalent if they have the same behavioral representation whenever \( P(h_i|\sigma_i)P(h_i|\tau_i) > 0 \). They are called payoff equivalent if for every player \( j \in N \) and any profile of strategies by \( i \)'s opponents \( \sigma_{-i} \),

\[
    u_j(\sigma_{-i}, \sigma_i) = u_j(\sigma_{-i}, \tau_i),
\]

where \( u_j \) is Player \( j \)'s utility function in the normal-form representation of the game.

To illustrate, consider the game in Example 1.6 below. This game doesn’t have perfect recall: player 1 forgets his move in the first node. Consider the mixed strategies \( \sigma_1 = \frac{1}{2}[(x_1, x_3)] + \frac{1}{2}[(y_1, y_3)] \) and \( \tau_1 = \frac{1}{2}[(x_1, y_3)] + \frac{1}{2}[(y_1, x_3)] \). It is clear that they are behaviorally equivalent, with behavioral representation \( (\frac{1}{2}[x_1] + \frac{1}{2}[y_1], \frac{1}{2}[x_3] + \frac{1}{2}[y_3]) \). However, they’re not payoff equivalent. For instance, given \([x_2]\), \( u_1(\sigma_1, [x_2]) = 0 \) yet \( u_1(\tau_1, [x_2]) = -1 \).

Example 1.6 A game with imperfect recall.

We will generally restrict ourselves to games with perfect recall, for the following reason.

Theorem 1.7 (Kuhn, 1953) In every game with perfect recall, any two mixed strategies that are behaviorally equivalent are also payoff equivalent.

See Myerson’s book for a proof.

2 Equilibrium in Extensive Form Games

This section follows pretty closely Chapter 4 of Myerson’s book.
2.1 Drawbacks in the Normal Form and Multi Agent Representations

In Section 1.2 we defined the normal-form (NFR) and multi-agent (MAR) representations of a game. Restricted to Nash equilibrium, one is left with the dilemma of too many equilibria in the NFR and unintuitive ones in the MAR. To illustrate, consider the game in Example 1.2.

\[w_2\]
\[x_2\]
\[a_1y_1\]
\[3, 2| 2, 3\]
\[a_1z_1\]
\[0, 5| 4, 1\]
\[b_1y_1\]
\[2, 3| 3, 2\]
\[b_1z_1\]
\[2, 3| 3, 2\]

In this NFR, there are no pure Nash equilibria. However, there are many mixed Nash equilibria. For example, \((\frac{1}{2}[a_1y_1] + \frac{1}{2}(p[b_1y_1] + (1 - p)[b_1z_1]), \frac{1}{2}[w_2] + \frac{1}{2}[x_2])\) is a mixed NE if \(0 \leq p \leq 1\). Trivially, there are too many (behaviorally equivalent) equilibria that lead to the same behavior strategy: \(((\frac{1}{2}[a_1] + \frac{1}{2}[b_1], [y_1]), \frac{1}{2}[w_2] + \frac{1}{2}[x_2])\). On the other hand, MAR leads to a slightly different problem with Nash equilibrium. Working our way back from the end, let player 1 be the player choosing between \(y_1\) and \(z_1\), player 2 be player 2, and player 3 be the player at the root node.

\[y_1\]
\[3, 2, 3| 2, 3, 2\]
\[z_1\]
\[0, 5, 0| 4, 1, 4\]
\[a_1\]
\[b_1\]
\[w_2\]
\[x_2\]
\[w_2\]
\[x_2\]
\[2, 3| 2, 3\]
\[2, 3| 2, 3\]

The profile \((y_1, x_2, b_1)\) is a pure NE. However, it seems to reflect a miscoordination between the two player 1’s at different information sets. In this sense, the NE is unintuitive. This motivates the following definition.

**Definition 2.1** Let \(\Gamma\) be an extensive form game, with NFR \(\Gamma_N\) and MAR \(\Gamma_M\). A behavior strategy profile \(\pi\) is called an equilibrium of \(\Gamma\) if \(\pi\) is a Nash equilibrium of \(\Gamma_M\) and the mixed representation \(\hat{\sigma}\) of \(\pi\) is a Nash equilibrium of \(\Gamma_N\).

There is only one equilibrium in Example 1.2: \((\frac{1}{2}[a_1] + \frac{1}{2}[b_1], [y_1]), \frac{1}{2}[w_2] + \frac{1}{2}[x_2])\).

**Proposition 2.2** In games \(\Gamma\) with perfect recall, if \(\sigma\) is a NE of \(\Gamma_N\) then any behavior representation \(\hat{\pi}\) of \(\sigma\) is a NE of \(\Gamma_M\).

Therefore, in games with perfect recall, an equilibrium always exists.\(^9\)

\(^9\)The perfect recall condition is necessary: in Example 1.6, \(\Gamma_N\) has only one NE, \((\frac{1}{2}[x_1x_3] + \frac{1}{2}[y_1y_3], \frac{1}{2}[x_2] + \frac{1}{2}[y_2])\). However, there is no behavioral representation of this profile that makes it a NE of \(\Gamma_M\).
2.2 Sequential Rationality

Informally, a behavior strategy profile \( \pi \) is \textit{sequentially rational for player } \( i \) \textit{at history } \( h_i \) if \( \pi_i(h_i) \in \Delta(A(h_i)) \) is optimal amongst all alternatives that only change \( \pi_i(h_i) \). Formally, we require that

\[
\pi_i(h_i) \in \arg \max_{d_i(h_i)} \left \{ u_i([\pi : d_i(h_i)], \pi_{-i}|h_i) : d_i(h_i) \in \Delta(A(h_i)) \right \},
\]

where \([\pi : d_i(h_i)]\) is the strategy that only replaces \( \pi_i(h_i) \) with \( d_i(h_i) \), and \(([\pi : d_i(h_i)], \pi_{-i}|h_i)\) is the profile of conditional probabilities (over terminal nodes) that result from applying Bayes’ rule to \(([\pi : d_i(h_i)], \pi_{-i})\) conditional on \( h_i \).

For now, we assume that \( h_i \) has \textit{positive probability under } \( \pi \). The caveat of positive probability constitutes a serious drawback. While in (statistical) decision theory zero-probability events are usually unimportant and ignored, in game theory zero-probability events are crucially important. We will address this issue shortly.

From the profile \( \pi \), we may infer, for each terminal node \( z \), the probability that \( z \) is reached under \( \pi \), \( P(z|\pi) \). For any nonterminal node history \( h_i \) and nonterminal node \( x \in h_i \), since \( P(A \cup B) = P(A) + P(B) \) when \( A \cap B = \emptyset \), it follows that

\[
P(x|\pi) = \sum_{z \in x} P(z|\pi), \quad P(h_i|\pi) = \sum_{x \in h_i} P(x|\pi).
\]

Therefore, we may induce from \( \pi \) player \( i \)'s “beliefs” regarding the probability of being a node \( x \in h_i \) conditional on reaching \( h_i \):

\[
P_{\pi}(x|h_i) := \frac{P(x|\pi)}{P(h_i|\pi)}.
\]

Conversely, we may want to \textit{begin} with players’ beliefs over nodes \( x \in h_i \) given \( h_i \), and then ask whether or not a strategy is sequentially rational given those beliefs. For any player \( i \), a \textit{vector of beliefs} is a \( \beta_i \) in

\[
\prod_{h_i \in H_i} \Delta(h_i);
\]

given a profile \( \pi \), \( \beta_i \) is called \textit{weakly consistent with } \( \pi \) if given \( h_i \) and \( x \in h_i \),

\[
\beta_i(x|h_i)P(h_i|\pi) = P(x|\pi).
\]

Weak consistency has no bite off the equilibrium path.\(^{10}\) The profile \( \pi \) is \textit{sequentially rational for } \( i \) \textit{with beliefs } \( \beta_i(\cdot|h_i) \in \Delta(A(h_i)) \) at \( h_i \) if \( \beta_i \) is weakly consistent with \( \pi \) and

\[
\pi_i(h_i) \in \arg \max_{d_i(h_i)} \left \{ \sum_{x \in h_i} \beta_i(x|h_i)u_i([\pi : d_i(h_i)], \pi_{-i}|x) : d_i(h_i) \in \Delta(A(h_i)) \right \}.
\]

In games with perfect recall, equilibrium strategies are sequentially rational on the path of play.

\(^{10}\)E.g., if \( P(h_i|\pi) = 0 \) then \( P(x|\pi) = 0 \) and \( \beta_i(h_i) \) can be anything that adds up to one.
Proposition 2.3 Let $\pi$ be an equilibrium (in the sense of Definition 8) of a game with perfect recall. For any player $i$ and history $h_i$, if $P(h_i|\pi) > 0$ then $\pi$ is sequentially rational for player $i$ at $h_i$ with beliefs $P_i(\cdot|h_i)$.

For instance, in Example 1.2, $\pi = ((b_1, y_1), w_2)$ is not sequentially rational for player 1 at the root node. What beliefs would make $\pi$ sequentially rational for player 1 at the information set $h_1 = \{a_1w_2, a_1x_2\}$? If player 1 believed that the conditional probability of player 2 having played $w_2$ given $h_1$ was $p$ then

$$E_{\beta_1(h_1)}[u_1(y_1)|h_1] = 3p + 2(1-p),$$
$$E_{\beta_1(h_1)}[u_1(z_1)|h_1] = 4(1-p),$$

rendering $y_1$ conditionally optimal if $p \geq \frac{2}{3}$. It remains to check that such beliefs are weakly consistent. But since $P(a_1|\pi) = 0$ and any beliefs off the equilibrium path are weakly consistent, it follows that $\pi$ is sequentially rational for player 1 at $h_1$ with beliefs $p \geq \frac{2}{3}$.

Definition 2.4 A strategy profile $\pi$ is sequentially rational with beliefs $\beta$ if $\pi$ is sequentially rational for every player $i$ and every history $h_i$ with beliefs $\beta(\cdot|h_i)$.

2.3 Consistent Beliefs and Conditional Expectation

The inability of conditional expectation to prescribe beliefs off the equilibrium path is a problem. Consider the following related examples.

Example 2.5 Consistency of sequentially rational profiles.

The game on the left differs from that on the right in that the turns for players 3 and 4 are switched. In the left game, the profile $\pi = ([\ell_1], [\ell_2], \frac{1}{2}[\ell_3] + \frac{1}{2}[r_3], [\ell_4])$ is an equilibrium, yet there do not exist
beliefs for player 3 that would make $\pi$ sequentially rational for him given those beliefs.\footnote{By Proposition 10, it follows that $P(h_3|\pi) = 0$.} Indeed, according to $\pi$, player 4 plays $\ell_4$. Since sequential rationality only allows “deviations” by one player at a time, player 3 strictly prefers to play $\ell_3$ rather than $\pi_3$ (since $2 > 1$). Thus, $\pi_3$ is not a best response.

This suggests that sequential rationality as defined in Section 2.2 can usefully refine the equilibrium notion of Definition 8, since $\pi$ is an equilibrium that fails the test of sequential rationality. Alas, by making the apparently innocuous change from the left tree to the right tree, it turns out that not only is $\pi$ an equilibrium of the right game, but it also passes the test of sequential rationality.

To see this, let $x$ and $y$ be the two nodes in player 3’s information set, $h_3$, and let his beliefs be given by $\beta_3(x|h_3) = \beta_3(y|h_3) = \frac{1}{2}$. Since $h_3$ is reached with probability zero under $\pi$ (player 1 plays $\ell_1$), it is immediate that $\beta_3(\cdot|h_3)$ is weakly consistent with $\pi$. But given such beliefs, player 3 is indifferent between $\ell_3$ and $r_3$ (each leads to an expected payoff of 1 miserable util). Therefore, it is a best response at $h_3$ to randomize between $\ell_3$ and $r_3$. If player 3 plays $\frac{1}{2}[\ell_1] + \frac{1}{2}[r_1]$ then player 4 optimally plays $\ell_4$, etc. The problem with this equilibrium is that it seems hard to justify player 3’s beliefs when player 4 is playing the pure strategy $\ell_4$.

Kreps and Wilson (1982) developed a stronger notion of consistency that eliminates both equilibria. Let $\Pi^\circ$ be the set of strategy profiles that assign positive probability to every node. For any such $\pi^\circ$ every history has positive probability, so is on the equilibrium path. Say that $\beta^\circ$ is consistent with $\pi^\circ$ if $\beta^\circ$ is weakly consistent with $\pi^\circ$.\footnote{Of course, since every node has positive probability under $\pi^\circ$, $\beta^\circ$ is uniquely determined by Bayes’ rule in the condition for weak consistency.}

The set of all consistent pairs $(\pi^\circ, \beta^\circ)$ with $\pi^\circ \in \Pi^\circ$ is denoted by $C^\circ$. To generally define beliefs $\beta$ that are consistent with $\pi$ even if some histories occur with zero probability under $\pi$, we will approximate $\pi$ by a sequence of pairs $(\pi^\circ_n, \beta^\circ_n) \in C^\circ$. Formally, for any profile $\pi$, a beliefs vector $\beta$ is called consistent with $\pi$ if there is a sequence $(\pi^\circ_n, \beta^\circ_n) \in C^\circ$ such that
\[
\pi_i(h_i)(a_i) = \lim_{n \to \infty} \pi^\circ_{i,n}(h_i)(a_i), \quad \forall i, h_i, a_i;
\]
\[
\beta_i(x|h_i) = \lim_{n \to \infty} \beta^\circ_{i,n}(x|h_i), \quad \forall i, h_i, x \in h_i.
\]
Since $\beta^\circ$ is consistent with $\pi^\circ$, $\beta^\circ_{i,n}(x|h_i) = \frac{P(x|h_i)/P(h_i|\pi^\circ_n)}{P(h_i|\pi^\circ_n)} \leq 1$ for each $n$, so $\beta_i(x|h_i) \leq 1$. Let $C$ be the set of all such consistent pairs. (Clearly, $C \supset C^\circ$.) We are now ready to define a sequential equilibrium. Let $\pi$ be any strategy profile and $\beta$ any beliefs vector.

**Definition 2.6** The assessment $(\pi, \beta)$ is called a sequential equilibrium if

1. $\pi$ is sequentially rational with $\beta$,
2. $\beta$ is consistent with $\pi$.

Let’s try some examples now.
2.4 Examples of Sequential Equilibria

For either game in Example 2.5, the unique sequential equilibrium is \( \pi = ( [\ell_1], [\ell_2], \frac{1}{2}[\ell_3] + \frac{1}{2}[\ell_4], \frac{1}{2}[r_3] + \frac{1}{2}[r_4] ) \) together with beliefs \( \beta_1 = (\frac{1}{2}, \frac{1}{2}) \) in the left game and \( \beta_3 = (\frac{1}{2}, \frac{1}{2}) \) in the right game. The remaining players' beliefs are trivial since everyone else's information sets are singletons.

**Example 2.7** Another example of sequential equilibrium.

![Game Diagram]

Notice that \( \pi = (a_1, a_2, a_3) \) is an equilibrium of this game. Let’s find out whether or not there are beliefs \( \beta \) that turn \((\pi, \beta)\) into a sequential equilibrium. Consider the following completely mixed perturbation \( \pi' \) of \( \pi \).

\[
\pi' = \left( (1 - \epsilon_0 - \epsilon_1)[a_1] + \epsilon_0[b_1] + \epsilon_1[c_1], (1 - \epsilon_2)[a_2] + \epsilon_2[b_2], 1 - \epsilon_3[a_3] + \epsilon_3[b_3] \right).
\]

As \( \epsilon \to 0 \), i.e., as \( \epsilon_i \to 0 \) for every \( i \), it is clear that \( \pi' \to \pi \). Applying Bayes’ rule yields the following consistent beliefs: \( \beta_1(\emptyset|\emptyset) = 1 \) by a trivial application. Since mixing is independent and the only thing that could ever be learnt by Bayes’ rule is that \( a_1 \) wasn’t played,

\[
\begin{align*}
\beta_2^\epsilon(b_1|h_2) &= \frac{\epsilon_0}{\epsilon_0 + \epsilon_1} = 1 - \beta_2(c_1|h_2), \\
\beta_3^\epsilon(b_1,a_2|h_3) &= (1 - \epsilon_2)\beta_3^\epsilon(b_1|h_2), \\
\beta_3^\epsilon(c_1,a_2|h_3) &= (1 - \epsilon_2)\beta_3^\epsilon(c_1|h_2), \\
\beta_3^\epsilon(b_1,b_2|h_3) &= \epsilon_2\beta_3^\epsilon(b_1|h_2), \\
\beta_3^\epsilon(c_1,b_2|h_3) &= \epsilon_2\beta_3^\epsilon(c_1|h_2).
\end{align*}
\]

As \( \epsilon \to 0 \), beliefs converge to

\[
\begin{align*}
\beta_2(b_1|h_2) &= p = 1 - \beta_2(c_1|h_2), \\
\beta_3(b_1,a_2|h_3) &= p = 1 - \beta_3(c_1a_2|h_3), \\
\beta_3(b_1,b_2|h_3) &= 0 = \beta_3(c_1b_2|h_3).
\end{align*}
\]
where \( p = \lim \epsilon_0 / (\epsilon_0 + \epsilon_1) \) could be any number between 0 and 1, depending on the rate of convergence of \( \epsilon \). For these beliefs to be consistent, it must be the case that

\[
\beta_3(b_1a_2|h_3) + \beta_3(b_1b_2|h_3) = \beta_2(b_1|h_2),
\]

\[
\beta_3(b_1a_2|h_3) + \beta_3(c_1a_2|h_3) = \pi_2(a_2);
\]

both conditions are satisfied. Therefore \( \beta \) is consistent with \( \pi \) for any \( p \). However, \( \pi \) is not sequentially rational with \( \beta \). Indeed, if player 3 plays \( a_3 \) then with these beliefs player 2 would get an expected utility of \( 3(1 - p) \) if he played \( b_2 \) and 1 if he played \( a_2 \). Hence \( p \geq 2/3 \) is necessary for sequential rationality. Likewise, player 3’s expected utility from playing \( a_3 \) is 1 whereas the expected utility from playing \( b_3 \) is \( 3p \). Therefore, sequential rationality requires that \( p \leq 1/3 \), a contradiction.

It follows that \((a_1, a_2, a_3)\) cannot be supported by beliefs that make it a sequential equilibrium. However, the profile \((c_1, b_2, b_3)\) can be supported in sequential equilibrium (with the obvious beliefs).

**Example 2.8** The beer-quiche game.

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
   & 2,1 & q_{1s} & 1_s & b_{1s} & u_{2b} & 3,1 \\
0,0 & d_{2q} & & & & & & \\
 . & . & & & & & & \\
2 & 0.9 & N & 2; & & & & \\
. & . & & & & & & \\
3,1 & u_{2q} & q_{1w} & 1_w & b_{1w} & u_{2b} & 2,1 \\
1,2 & d_{2q} & & & & & & \\
 . & . & & & & & & \\
\end{array}
\]

Let’s show that the profile \( \pi = ((b_{1s}, b_{1w}), (d_{2q}, u_{2b})) \) is a sequential equilibrium with some consistent beliefs. On the path of play, player 2 must have beliefs consistent with Bayes’ rule. Therefore, \( \beta_2(1s|b_1) = .9 \), i.e., since both types of player 1 adopt the same strategy, player 2 doesn’t learn from observing \( b_1 \). Therefore \( u_{2b} \) is sequentially rational for player 2, since player 2’s expected payoff from playing \( u_{2b} \) is 1, which exceeds the payoff from playing \( d_{2b} \) \((2(0.1) = .2 \text{ utils})\).

Suppose that player 2’s beliefs about player 1’s type conditional on observing quiche are given by \( \beta_2(q_{1s}|q_1) = p \) and \( \beta_2(q_{1w}|q_1) = 1 - p \). The action \( d_{2q} \) is sequentially rational given these beliefs if \( 2(1 - p) \geq 1 \). In words, given that player 2 observes that player 1 played \( q_1 \), playing \( d_2 \) will be a best response only when player 2 assigns sufficiently high probability to the event that player 1’s type is \( 1_w \). But if player 2 optimally plays \( d_{2q} \) then players \( 1_s \) and \( 1_w \) will have no incentive to deviate from \( b_1 \). Hence \( \pi \) is sequentially rational with these beliefs. Furthermore, such beliefs are consistent with \( \pi \). Let \( \pi_e = ((1 - \epsilon_s)[b_{1s}] + \epsilon_s[q_{1s}], (1 - \epsilon_w)[b_{1w}] + \epsilon_s[q_{1w}]) \), with \( \epsilon = (\epsilon_s, \epsilon_w) \rightarrow 0 \). The beliefs \( \beta_1(q_{1s}|q_1) = .9\epsilon_s/(.9\epsilon_s + .1\epsilon_w) \) are consistent with \( \pi_e \), and by picking the ratio \( \epsilon_s/\epsilon_w \) appropriately we can make them converge to any \( p \) in \([0, 1]\). We’re done.
3 Self Confirming Equilibrium

First I will define the flora and fauna of self confirming equilibrium concepts and comment a little bit on them and then I’ll introduce some examples.

3.1 Equilibrium Beliefs about Opponents’ Play

Self-confirming equilibrium (SCE) is motivated by the idea that Nash equilibrium assumes players know their opponents’ strategies. The object of SCE is to relax that assumption. To begin with, players are assumed to only know the structure of the extensive form, the distribution of nature’s moves, and their own payoffs. The additional requirement for SCE is that players’ beliefs about their opponents’ play be correct on the equilibrium path, but not necessarily off the path. The space of players’ beliefs is defined in terms of behavior strategy profiles. For any player $i$, $i$’s beliefs about his opponents’ play is any probability measure $\mu_i$ in

$$\Omega_i := \Delta (\Pi_i).$$

Player $i$’s beliefs $\mu_i$ reflect his uncertainty about opponents’ play and therefore induce probabilistic beliefs on terminal nodes. If $i$ plays the pure strategy $c_i$ then his subjective probability that the terminal node $z$ will be reached is simply the expected value of the conditional probability of reaching $z$ given $(c_i, \pi_{-i})$, thus:

$$P_i(z|c_i, \mu_i) = \mathbb{E}_{\mu_i}[P(z|c_i, \tilde{\pi}_{-i})] = \int_{\Pi_{-i}} P(z|c_i, \pi_{-i})d\mu_i(\pi_{-i}).$$

Although behavior strategy profiles assume independent mixing between opponents and information sets, beliefs may be correlated. To illustrate, suppose that there are three players, and that players 2 and 3 simultaneously choose between the strategies $U$ and $D$. Let

$$\mu_1(\pi_{-1}) = \begin{cases} \frac{1}{4} & \text{if } \pi_2(U) = \pi_3(U) = 1, \\ \frac{3}{4} & \text{if } \pi_2(U) = \pi_3(U) = \frac{1}{2}. \end{cases}$$

In this case, $P_1(U, U) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}, P_1(U, D) = P_1(D, U) = P_1(D, D) = \frac{3}{16}$.

Player $i$’s utility function is defined over terminal nodes. We define the induced expected utility over belief-action pairs $(c_i, \mu_i)$ as

$$u_i(c_i, \mu_i) = \sum_{z \in Z} u_i(z)P_i(z|c_i, \mu_i).$$

Given a mixed strategy $\sigma_j \in \Sigma_j$, let me denote by $\hat{\pi}_j(\cdot|\sigma_j)$ its behavioral representation. The event

$$E_i(h_j) := \{\pi_{-i} : \pi_j(h_j) = \hat{\pi}_j(h_j|\sigma_j)\}$$

is the set of behavior strategy profiles for $i$’s opponents that are consistent with player $j$ playing according to $\sigma_j$ at information state $h_j$. Notice that $i$ and $j$ are fixed. Therefore, $E_i(h_j)$ contains
all behavior profiles \( \pi_{-i} \) whose jth entry \( \pi_j \) coincides with \( \tilde{\pi}_j(h_j|\sigma_j) \) at history \( h_j \). The strategy \( \pi_j \) could be anything outside \( h_j \), as could the strategies of all other players.

We are now ready to define a Nash equilibrium and a SCE.

**Definition 3.1** A profile \( \sigma \) is a Nash equilibrium if for any \( i \in I \) and \( c_i \in C_i \) with \( \sigma_i(c_i) > 0 \) there exists \( \mu_i \in \Omega_i \) such that

1. \( c_i \in \arg \max u_i(\cdot, \mu_i) \),
2. \( \mu_i(E_i(h_j)) = 1 \) for every \( h_j \) and every \( j \neq i \).

The first condition requires that \( c_i \) be an optimum strategy given beliefs \( \mu_i \), and the second condition requires that every player’s beliefs are correct regarding every behavior strategy of every opponent in the sense that \( \mu_i(E_i(h_j)) = 1 \), i.e., that the subjective probability for player \( i \) that player \( j \) is playing \( \tilde{\pi}_j(h_j|\sigma_j) \) at \( h_j \) is one for every \( h_j \) and every \( j \).

**Definition 3.2** A profile \( \sigma \) is a self-confirming equilibrium if for any \( i \in I \) and \( c_i \in C_i \) with \( \sigma_i(c_i) > 0 \) there exists \( \mu_i \in \Omega_i \) such that

1. \( c_i \in \arg \max u_i(\cdot, \mu_i) \),
2. \( \mu_i(E_i(h_j)) = 1 \) for every \( h_j \) such that \( P(h_j|\sigma) > 0 \) and every \( j \neq i \).

The only difference between Nash equilibrium and SCE is that Nash equilibrium requires beliefs to be correct everywhere, whereas SCE only requires beliefs to be correct on information sets that have positive probability under \( \sigma \), allowing for wrong beliefs about opponents’ behavior strategies off the equilibrium path (i.e., on information sets \( h \) with \( P(h|\sigma) = 0 \)).

Notice that in both definitions, potentially different \( \mu_i \)'s are allowed to justify different \( c_i \)'s. In the definition of NE, this flexibility is vacuous, since beliefs—always having to be correct—must be unique (two different beliefs cannot both be correct). The flexibility matters once beliefs are allowed to be wrong. This suggests the following refinement, which requires that for every player \( i \), if two strategies \( c_i \) and \( c'_i \) have positive probability of being played, then there must exist some beliefs that make both \( c_i \) and \( c'_i \) best responses simultaneously.

**Definition 3.3** A profile \( \sigma \) is a unitary self-confirming equilibrium (USCE) if given \( i \in I \) there exists \( \mu_i \in \Omega_i \) such that for any \( c_i \in C_i \) with \( \sigma_i(c_i) > 0 \)

1. \( c_i \in \arg \max u_i(\cdot, \mu_i) \),
2. \( \mu_i(E_i(h_j)) = 1 \) for every \( h_j \) such that \( P(h_j|\sigma) > 0 \) and every \( j \neq i \).
Fudenberg and Levine (1993b) formalize the motivation of Nash equilibrium as the steady state of a learning process. They construct a dynamic environment where players repeatedly play an extensive-form game and, by playing, they learn about their opponents’ strategies through Bayesian-updating. Intuitively, every player $i$ considers trying different strategies to observe the play of his opponents, thus learning about his opponents’ strategy profile, which is potentially beneficial to him. On the other hand $i$ incurs an opportunity cost from experimenting by trying different strategies because some may lead to perhaps suboptimal strategies being played, depending on the actual profile of $i$’s opponents’ strategies.

Players have a common discount factor, $\delta$. Formally, Fudenberg and Levine show that as $\delta \to 1$, the path of play that will arise in a steady state of the learning process is characterized by the set of Nash-equilibrium outcomes. Moreover, if $\delta < 1$, then players will generally play a self confirming equilibrium. The intuition behind this result is that as $\delta \to 1$, the opportunity cost of (optimum) experimentation tends to zero, implying that the benefits of learning exceed the zero cost, whence players will experiment until they know with probability 1 the strategy profile of their opponents. However, when $\delta < 1$, such opportunity cost is strictly positive, therefore it may be optimal to forego learning about opponents’ strategies to some extent. Of course, on the equilibrium path, play by opponents is necessarily observed, so beliefs ought to be correct there.

So far, it may be argued, this intuitive story only captures the notion of unitary self confirming equilibrium. True enough. Fudenberg and Levine consider games with player roles. In such games, there is a large population of players who are randomly matched from $|I|$ many pools and whose roles $i \in I$ are allocated according to a player’s pool once matched. Furthermore, there are many different kinds of player types in the population. (Some will be commitment types.) In this larger game, a player from pool $i$ will play the game with opponents that are not necessarily the same players from period to period.

In the long run, a player from pool $i$ will have played with all sorts of types of opponents. In particular, he will have observed the behavior of all sorts of commitment types. If the size of the population of commitment types is small relative to the “rational” types, a player may use the law of large numbers to distinguish between the play of commitment types and the play of rational types. Furthermore, a player in pool $i$ will be able to discern the play of his “rational” opponents on all the information sets that could be reached by some profile of commitment and “rational” types of opponents when the player in pool $i$ is playing some strategy $c_i$. Intuitively, a player “observes” the experimentation of his opponents (reflected in different types of opponents) but not of other types of players in the same pool. The concept of consistent self-confirming equilibrium naturally characterizes the equilibria that are consistent with this story. For any player $i \in I$ and pure strategy $c_i$, let

$$H(c_i) = \{h \in H : \exists c_{-i} \in C_{-i} \text{ s.t. } P(h|c) > 0\}.$$ 

The set $H(c_i)$ consists of those information sets where player $i$ may be restricted to have correct beliefs about opponents’ play if he gets to observe the experimentation of opponents. It is the histories that are reachable by some strategy profile of $i$’s opponents, $c_{-i}$, when $i$ plays $c_i$. 

15
**Definition 3.4** A profile $\sigma$ is a consistent self-confirming equilibrium (CSCE) if for any $i \in I$ and $c_i \in C_i$ with $\sigma_i(c_i) > 0$ there exists $\mu_i \in \Omega_i$ such that

1. $c_i \in \arg \max u_i(\cdot, \mu_i)$,
2. $\mu_i(E_i(h_j)) = 1$ for every $h_j \in H(c_i)$ and every $j \neq i$.

It’s about time to frame these ideas in specific games.

### 3.2 Examples of Self Confirming Equilibria

Here I present some examples to illustrate the different equilibrium concepts. The first example is perhaps the most important, originally presented by Fudenberg and Kreps (1988). This example shows that an SCE need not be Nash.

**Example 3.5** In the game below, $(a_1, a_2)$ is a self-confirming equilibrium outcome.

To show that $(a_1, a_2)$ is the outcome of a SCE, suppose that player 1 believes that player 2 will play $a_2$ and player 3 will play $r$ with probability 1. Then, player 1’s best strategy is to play $a_1$. If player 2 believes that player 3 will play $\ell$ with probability 1, then player 2’s best strategy will be to play $a_2$. Whatever player 3’s strategy may actually be, he never gets to play it if players 1 and 2 play $(a_1, a_2)$, whence the beliefs of players 1 and 2 about player 3’s strategy never get disconfirmed. Since player 1’s beliefs about player 2’s strategy are correct, we arrive at a SCE.

However, this is not a Nash equilibrium, because players 1 and 2 have different beliefs about player 3’s strategy and NE assumes that players have the same correct beliefs about everyone’s strategy. Furthermore, there is no Nash equilibrium of this game that supports $(a_1, a_2)$ as an outcome, for if players 1 and 2 both believed that player 3 would play $\ell$ with some probability $p$, then at least one of them would optimally play $d_i$.\(^{13}\)

\(^{13}\)If $p \leq \frac{2}{3}$ then player 2 would optimally play $d_2$, and if $p \geq \frac{1}{3}$ then player 1 would optimally play $d_1$. 

16
Since players only have one information set each, this SCE is automatically unitary. Finally, notice that this is SCE is not consistent. To see this, notice that

\[ H(a_1) = H(a_2) = \{h_1, h_2, h_3\}, \]

where \( h_i \) is player \( i \)’s only information set. Therefore, here the consistency refinement requires that both players 1 and 2 have correct beliefs about player 3’s play. By the previous argument, \((a_1, a_2)\) must therefore be suboptimal for at least one player.

One way to think about why this is happening is to notice that the game fails to have observed deviators. For any mixed-strategy profile \( \sigma \), let

\[ H(\sigma) = \{h \in H : P(h|\sigma) > 0\} \]

be the family of information sets that are reached with positive probability when players play \( \sigma \).

**Definition 3.6** A game has observed deviators if given a strategy profile \( c \), any player \( i \) and possible deviation \( d_i \neq c_i \), \( h \in H(d_i, c_{-i}) \setminus H(c) \) implies that there is no \( d_{-i} \) such that \( h \in H(c_i, d_{-i}) \).

This simply says that any history that may occur if \( i \) deviates from \( c_i \) to \( d_i \) cannot arise from player \( i \) playing \( c_i \) and some other player deviating. I.e., if player \( i \) deviates then somebody will find out that indeed it was player \( i \) who deviated. Games of perfect information and repeated games with observed actions have observed deviators. Also, two-player games with perfect recall satisfy this property, too. Fudenberg and Levine (1993a) prove the following classification of SCE in games with observed deviators.

**Theorem 3.7** In games with observed deviators, every self-confirming equilibrium is consistent.

It follows that every SCE is a CSCE in two-player games with perfect recall.

**Example 3.8** The convexifying effect of self-confirming equilibria that are not unitary.

In the game above, there are two Nash equilibrium outcomes: \( d_1 \) and \((r_1, r_2)\). The mixed profile \( \left( \frac{1}{2}[d_1] + \frac{1}{2}[r_1, r_2] \right) \) is a SCE. However, it isn’t unitary. Suppose that player 1 plays \( d_1 \) when he believes that player 2 will play \( d_2 \) with probability 1, and that player 1 plays \( r_1 \) when he believes that player 2 will play \( r_2 \) with probability 1. Then, when player 1 plays \( d_1 \), his beliefs about player 2’s play are not disconfirmed. The mixed strategy by player 1 is somehow related more to random
beliefs than to random behavior. It's as if a coin is flipped and if heads, player 1 will play \( d_1 \) because he believes that player 2 will play \( d_2 \), and if tails, player 1 will play \( r_1 \) because he believes that player 2 will play \( r_2 \).

Finally, it is easy to verify that the outcomes from non-unitary self confirming equilibria convexify the set of Nash equilibrium outcomes.

The last example reflects the possibility that correlated beliefs about opponents' play may lead to SCE that are not Nash outcomes.

**Example 3.9** Consider the following game with three players. Player 1 has the first move. He may choose one of four actions: \( A, L_1, M_1, R_1 \). If player 1 plays \( A \), then the game ends and payoffs are \( (1, 0, 0) \). If player 1 plays \( L_1, M_1, \) or \( R_1 \), then players 2 and 3 play the corresponding game:

\[
\begin{array}{c|cc}
U_2 & L_3 & R_3 \\
\hline
L_3 & 4, 1, -1 & 0, -1, 1 \\
D_2 & 0, -1, 1 & -4, 1, -1
\end{array}
\quad
\begin{array}{c|cc}
L_3 & R_3 \\
\hline
L_3 & -4, 1, -1 & 0, -1, 1 \\
L_3 & 0, 1, -1 & 3, -1, 1
\end{array}
\quad
\begin{array}{c|cc}
L_3 & R_3 \\
\hline
L_3 & 0, 1, -1 & 3, -1, 1 \\
R_3 & 3, -1, 1 & 0, 1, -1
\end{array}
\]

However, neither player 2 nor player 3 observes player 1’s action.

In this example, the strategy \( A \) is a best response by player 1 to the correlated beliefs \( P_1(U_2, L_3) = P_1(D_2, R_3) = \frac{1}{2} \), since for each possible strategy in \{\( L_1, M_1, R_1 \)\}, its expected payoff to player 1 given these beliefs is zero, and the payoff from \( A \) is 1 util. However, \( A \) is never a best response to any strategy profile of players 2 and 3. For proof, suppose that player 2 plays \( U_2 \) with probability \( p_2 \) and that player 3 plays \( L_3 \) with probability \( p_3 \). In order for \( A \) to be a best response for player 1, it must be the case that

\[
\begin{align*}
L_1 : & \quad 4p_2p_3 - 4(1 - p_2)(1 - p_3) \leq 1 \quad \Rightarrow \quad p_2 + p_3 \leq 5/4, \\
M_1 : & \quad -4p_2p_3 + 4(1 - p_2)(1 - p_3) \leq 1 \quad \Rightarrow \quad p_2 + p_3 \geq 3/4, \\
R_1 : & \quad 3p_2(1 - p_3) + 3(1 - p_2)p_3 \leq 1.
\end{align*}
\]

If \( p_2 \leq \frac{1}{7} \), then the lowest that \( p_3 \) could be whilst satisfying \((R_1)\) is \( p_3 = \frac{3}{4} - p_2 \). Substituting this into \((R_1)\) leads to the quadratic \( 2p_2^2 - \frac{3}{2}p_2 + \frac{3}{4} \), which is minimized with respect to \( p_2 \) when \( p_2 = 3/8 \). The implied value of \( p_3 \) is 3/8, too. But when \( p_2 = p_3 = 3/8 \), it is easy to check that the left-hand side of \((R_1)\) is 30/64 > 1/3, implying that \((R_1)\) is violated. The proof if \( p_2 > \frac{1}{7} \) is the same.

Let me underline that player 1 need not believe that players 2 and 3 correlate their play (indeed player 1 does not), it is precisely that player 1’s beliefs are correlated.