Economics 211: Dynamic Games

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Basic Concepts of Game Theory and Equilibrium

Course Slides

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A Finite Game

an N player game $i = 1 \ldots N$

$P(S)$ are probability measure on S

finite strategy spaces

$\sigma_i \in \Sigma_i \equiv P(S_i)$ are mixed strategies

$s \in S \equiv \times_{i=1}^{N} S_i$ are the strategy profiles

$\sigma \in \Sigma \equiv \times_{i=1}^{N} \Sigma_i$

other useful notation $s_{-i} \in S_{-i} \equiv \times_{j \neq i} S_j$

$\sigma_{-i} \in \Sigma_{-i} \equiv \times_{j \neq i} \Sigma_j$

$u_i(s)$ payoff or utility

$u_i(\sigma) \equiv \sum_{s \in S} u_i(s) \prod_{j=1}^{N} \sigma_j(s_j)$ is expected utility

Dominant Strategies

σ_i weakly (strongly) dominates σ'_i if
$u_i(\sigma_i, s_{-i}) \geq (>) u_i(\sigma'_i, s_{-i})$ with at least one strict

Nash Equilibrium

players can anticipate on another's strategies

σ is a *Nash equilibrium* profile if for each
$i \in 1,\ldots N \;\; u_i(\sigma) = \max_{\sigma'_i} u_i(\sigma'_i, \sigma_{-i})$

Theorem: a Nash equilibrium exists in a finite game

this is more or less why Kakutani's fixed point theorem was invented

$B_i(\sigma)$ is the set of best responses of i to σ_{-i}, and is UHC convex valued

This theorem fails in pure strategies: consider matching pennies

Some Classic Simultaneous Move Games

Coordination Game

	R	L
U	1,1	0,0
D	0,0	1,1

three equilibria (U,R) (D,L) (.5U,.5L)

too many equilibria??

Coordination Game

	R	L
U	2,2	-10,0
D	0,-10	1,1

risk dominance:

indifference between U,D

$$2p_2 - 10(1 - p_2) = (1 - p_2)$$

$$13p_2 = 11, p_2 = 11/13$$

if U,R opponent must play equilibrium w/ 11/13

if D,L opponent must play equilibrium w/ 2/13

½ dominance: if each player puts weight of at least ½ on equilibrium strategy, then it is optimal for everyone to keep playing equilibrium (same as risk dominance in 2x2 games)

Prisoner's Dilemma Game

	R	L
U	2,2	0,3
D	3,0	1,1

a unique dominant strategy equilibrium (D,L)
this is Pareto dominated by (U,R) does it really occur??

discuss repeated version

time average with grim strategies

this leads to a coordination problem

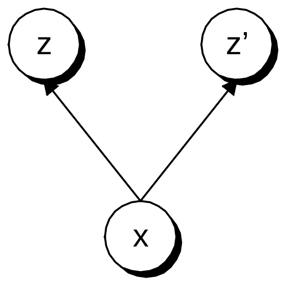
Next: dynamic (extensive form) games

Extensive Form Games

a finite game tree X with nodes $x \in X$

nodes are partially ordered and have a single root (minimal element)

terminal nodes are $z \in Z$ (maximal elements)



player 0 is nature

information sets $h \in H$ are a partition of $X \setminus Z$
each node in an information set must have exactly the same number of immediate followers

each information set is associated with a unique player who "has the move" at that information set

$H_i \subset H$ information sets where *i* has the move

More Extensive Form Notation

information sets belonging to nature $h \in H_0$ are singletons

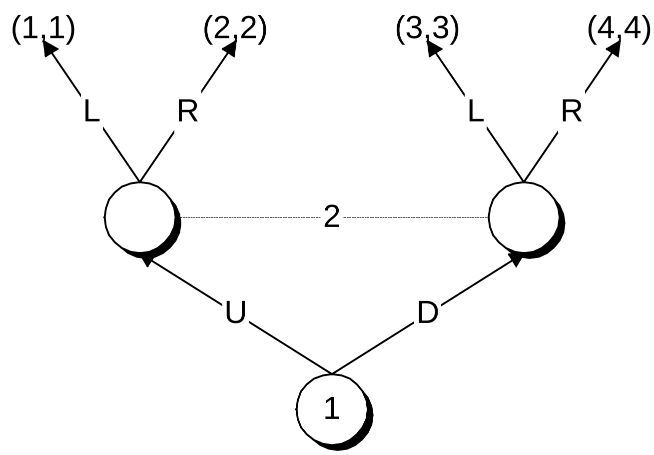
$A(h)$ feasible actions at $h \in H$

each action and node $a \in A(h), x \in h$ is associated with a unique node that immediately follows x on the tree

each terminal node has a payoff $r_i(z)$ for each player

by convention we designate terminal nodes in the diagram by their payoffs

Example: a simple simultaneous move game



Behavior Strategies

a *pure strategy* is a map from information sets to feasible actions $s_i(h_i) \in A(h_i)$

a *behavior strategy* is a map from information sets to probability distributions over feasible actions $\pi_i(h_i) \in P(A(h_i))$

Nature's move is a behavior strategy for Nature and is a fixed part of the description of the game

We may now define $u_i(\pi)$

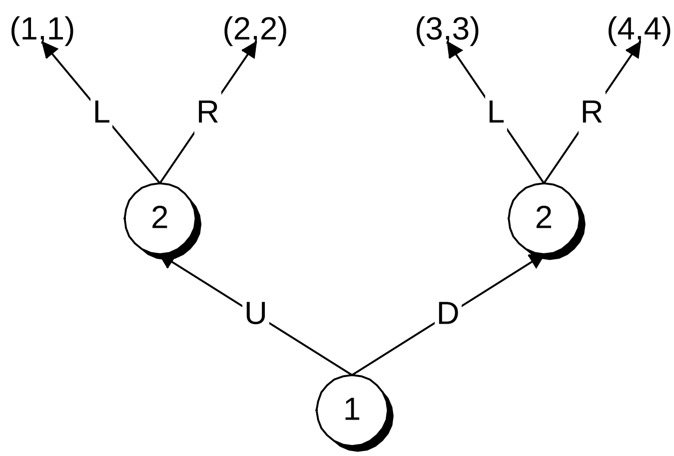
normal form are the payoffs $u_i(s)$ derived from the game tree

	L	R
U	1,1	2,2
D	3,3	4,4

Kuhn's Theorem:

every mixed strategy gives rise to a unique behavior strategy

The converse is NOT true



1 plays .5 U
behavior: 2 plays .5L at U; .5L at R
mixed: 2 plays .5(LL),.5(RR)
2 plays .25(LL),.25(RL),.25(LR),.25(RR)

however: if two mixed strategies give rise to the same behavior strategy, they are *equivalent*, that is they yield the same payoff vector for each opponents profile $u(\sigma_i, s_{-i}) = u(\sigma'_i, s_{-i})$

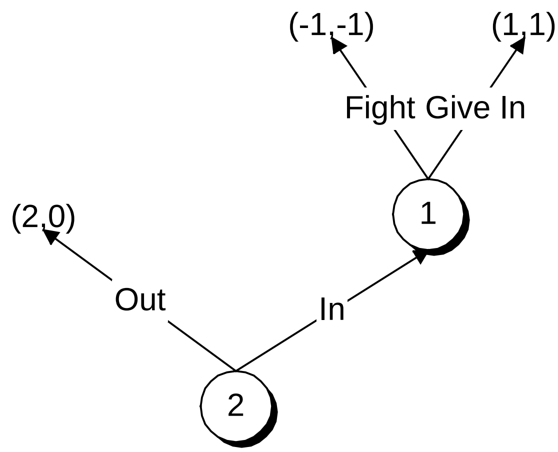
Refinements of Nash Equilibrium

some games seem to have too many Nash equilibria

Ultimatum Bargaining
player 1 proposes how to divide $10 in pennies
player 2 may accept or reject

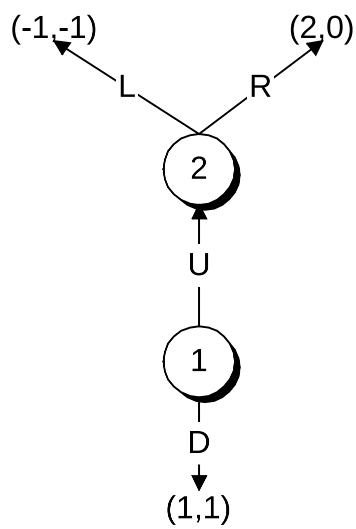
Nash: any proposal by player 1 with all poorer proposals rejected and equal or better proposals accepted

Chain Store



Subgame Perfection

Selten Game



	L	R
U	-1,-1	2,0
D	1,1	1,1

Define subgame perfection equilibria:
UR is subgame perfect
D and .5 or more L is Nash but not subgame perfect

Application to Bargaining

the pie division game: there is one unit of pie;
player 1 demands p_1
player 2 accepts or rejects
if player 2 rejects one period elapses, then the roles are reversed, with player 2 demanding p_2
common discount factor $0 < \delta < 1$

Nash equilibrium: player 1 gets all pie, rejects all positive demands by player 2; player 2 indifferent and demands nothing

conversely: player 2 gets all the pie

wait 13 periods then split the pie 50-50; if anyone makes a positive offer during this waiting period, reject then revert to the equilibrium where the waiting player gets all the pie

subgame perfection: one player getting all pie is not an equilibrium: if your opponent must wait a period to collect all pie, he will necessarily accept demand of $1-\delta-\varepsilon$ today, since this give him $\delta+\varepsilon$ in present value, rather than δ the present value of waiting a period

Rubinstein's Theorem:

there is a unique subgame perfect equilibrium

players always make the same demands, and if they demand no more than the equilibrium level their demands are accepted

to compute the unique equilibrium observe that a player may reject an offer, wait a period, make the equilibrium demand of *p* and have it accepted, thus getting δp today; this means the opposing player may demand up to $1-\delta p$ and have the demand accepted; the equilibrium condition is

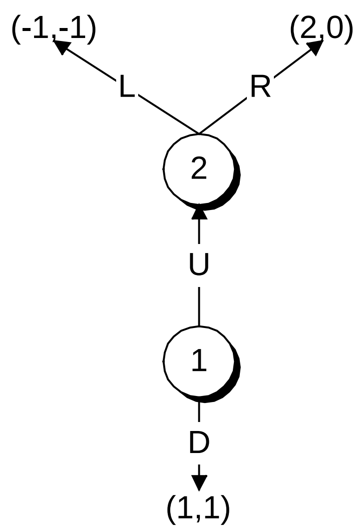
$p = 1-\delta p$ or $p = \frac{1}{1+\delta}$

notice that the player moving second gets

$\frac{\delta}{1+\delta}$ and that as $\delta \to 1$ the equilibrium converges to a 50-50 split a problem: if offers are in pennies, subgame perfect equilibrium is not unique

More on Refinements

Selten Game



	L	R
U	-1,-1	2,0
D	1,1	1,1

subgame perfect
equilibria:
UR is subgame perfect
D and .5 or more L is Nash but not subgame
perfect

can also solve by weak dominance
or by trembling hand perfection

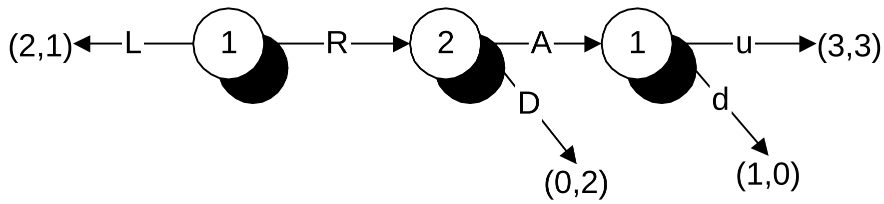
Summary of Refinements

- subgame perfection (backwards induction)
- iterated dominance (forwards induction)
- trembling hand perfection
- extensive form trembling hand perfection
- sequentiality

definition of trembling hand perfection

σ is trembling hand perfect if there is a sequence $\sigma^n >> 0, \sigma^n \to \sigma$ such that if $\sigma^i(s^i) > 0$ then s^i is a best response to σ^n

Example of Trembling Hand not Subgame Perfect



	A	D	
Lu=Ld	2,1	2,1	(n-2)/n
Ru	3,3	0,2	1/n
Fd	1,0	0,2	1/n
	1/n	(n-1)/2	

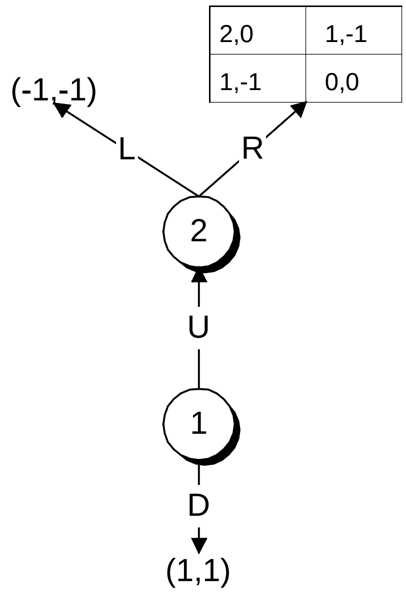
Here Ld,D is trembling hand perfect but not subgame perfect

definition of the agent normal form
each information set is treated as a different player, e.g. 1a, 1b if player 1 has two information sets; players 1a and 1b have the same payoffs as player 1

extensive form trembling hand perfection is trembling hand perfection in the agent normal form

Iterated Dominance

example of iterated weak dominance



	L	R-l	R-r
U-u	-1,-1	2,0	1,1
U-d	-1,-1	1,-1	0,0
D	1,1	1,1	1,1

Eliminate U-d
Eliminate R-r

example of order dependent iterated weak dominance

3,2	2,2
1,1	0,0

eliminate BOTTOM then everything is OK for 2
eliminate LEFT then BOTTOM and only (3,2) left

2 players + iterated dominance + Nash implies subgame perfect
n-players + weak rationalizability + Nash implies subgame perfect

a strategy not weakly dominated by anything is a best response to some correlated opponent strategies

rationalizability vs. dominance

-8	0
0	0
-3	-3

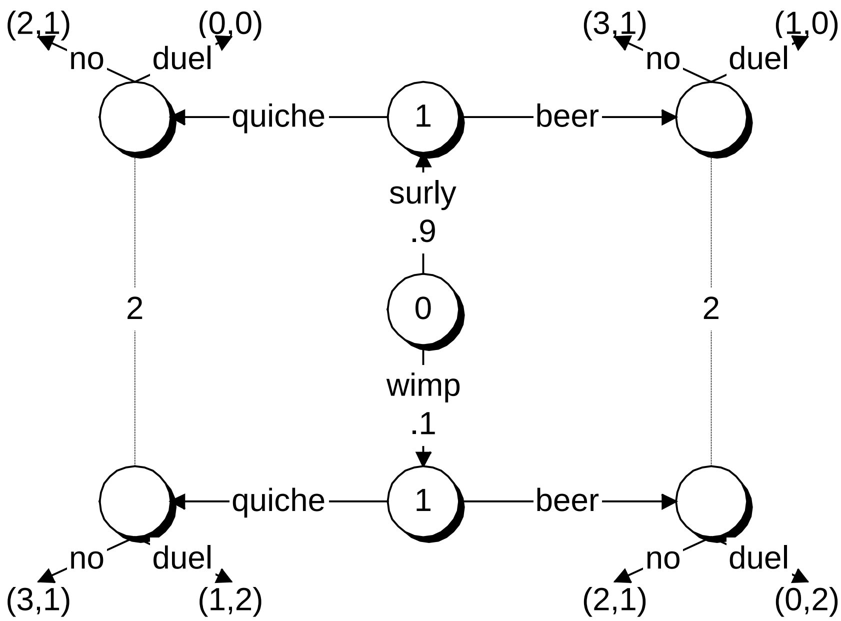
0	0
0	-8
-3	-3

player 1 choosing bottom gives him -3

bottom is not dominated
if opponents correlate so as to randomize 50-50 between UU and DD then top or middle yields -4

bottom is not rationalizable
50-50 between up and middle guarantees -2 against any opponent uncorrelated strategies

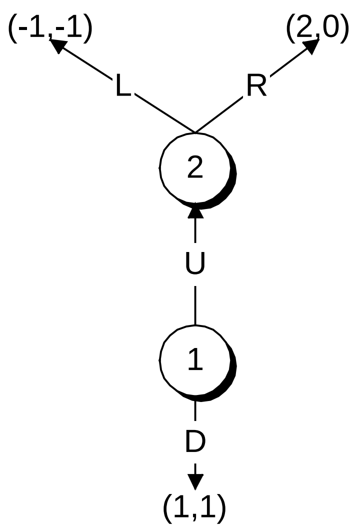
Signaling

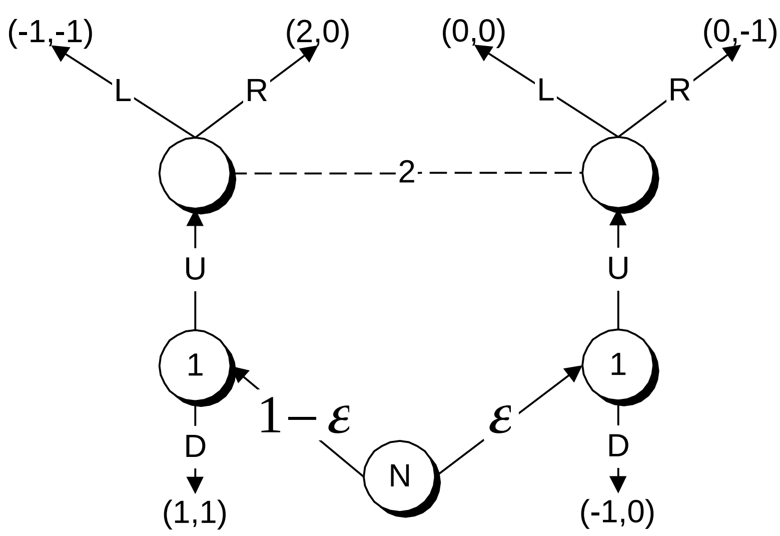


sequential vs. trembling hand perfect
pooling and separating

Robustness

genericity in normal form games
example of Selten extensive form game





elaborated Selten game

normal form of elaborated Selten game

	L	R
$D_L D_R$	$1-2\varepsilon, 1-\varepsilon$	$1-2\varepsilon, 1-\varepsilon$
$D_L U_R$	$1-\varepsilon, 1-\varepsilon$ **	$1-\varepsilon, 1-2\varepsilon$
$U_L D_R$	$-1, -1+\varepsilon$	$2-3\varepsilon, 0$
$U_L U_R$	$-1+\varepsilon, -1+\varepsilon$	$2-2\varepsilon, -\varepsilon$

Approximate Equilibria and Near Equilibria

Approximate Equilibrium

- exact: $u_i(s_i|\mu_i) \geq u_i(s_i'|\mu_i)$
  approximate: $u_i(s_i|\mu_i) + \varepsilon \geq u_i(s_i'|\mu_i)$

- Approximate equilibrium can be very different from exact equilibrium

Radner’s work on finite repeated PD
gang of four on reputation

upper and lower hemi-continuity

A small portion of the population playing "non-optimally" may significantly change the incentives for other players causing a large shift in equilibrium behavior.

Correlated Equilibrium

Chicken

6,6	2,7
7,2	0,0

three Nash equilibria (2,7), (7,2) and mixed equilibrium w/ probabilities (2/3,1/3) and payoffs (4 2/3, 4 2/3)

1/3	1/3
1/3	0

is a correlated equilibrium giving utility (5,5)

Extensive Form Correlated Equilibrium

Public randomization only
Sequential public randomization = sunspot

Extensive form correlated equilibrium

One that is not correlated

Stage 1

	L	M	R
U	13,15	13,14	13,11
D	12,11	12,14	12,15

Stage 2

R	P
0,0	-10,-10

Stage 1 private signal to 1 is 50-50 between U,D while 2 plays M
Stage 2 private signal to 1 is revealed to 2, if 1 did as required play R, else play P

Notice that in correlated equilibrium 1 must randomize to get 2 to play M, and is not indifferent between U and D, so must expect P when U with positive probability. But can't happen with positive probability