## Carlsson and van Damme

<table>
<thead>
<tr>
<th></th>
<th>Invest</th>
<th>NotInvest</th>
</tr>
</thead>
<tbody>
<tr>
<td>Invest</td>
<td>$\theta,\theta$</td>
<td>$\theta - 1, 0$</td>
</tr>
<tr>
<td>NotInvest</td>
<td>$0, \theta - 1$</td>
<td>$0, 0$</td>
</tr>
</tbody>
</table>

Three cases

- $\theta > 1$ dominant strategy to invest
- $\theta \in [0,1]$ two pure equilibria – coordination problem
- $\theta < 1$ dominant strategy not to invest
incomplete information about $\theta$

each player observes a noisy signal $x_i = \theta + \sigma \varepsilon_i$

where $\varepsilon_i$ are independent normal random variables with zero mean and unit variance

improper uniform prior over $\theta$

each player sees $\theta$ as normal with mean $x_i$ and variance $\sigma^2$; each sees their opponents signal as the sum of this normal and an independent normal with mean zero and variance $\sigma^2$, that is, a normal with mean $x_i$ and variance $2\sigma^2$
expected utility gain from investing if probability of opponent notInvesting is $q(x_i)$ is

$$E[\theta|x_i] - q(x_i)$$ so best response is invest if this is non-negative; since $E[\theta|x_i] = x_i$, this can be written as $x_i - q(x_i)$

Suppose you believe your opponent invests for $x_i > b$. Then

$$q(x_i) \leq \Phi\left(-\frac{b-x_i}{2^{1/2} \sigma}\right)$$, hence you must invest if

$$x_i > \Phi\left(-\frac{b-x_i}{2^{1/2} \sigma}\right)$$

Suppose you believe your opponent notInvests for $x_i < b$. Then

$$q(x_i) \geq \Phi\left(-\frac{b-x_i}{2^{1/2} \sigma}\right)$$ hence you notInvest if

$$x_i < \Phi\left(-\frac{b-x_i}{2^{1/2} \sigma}\right)$$
Implicitly define the function $b(k) = \Phi\left( -\left( b(k) - k \right) / \left( 2^{1/2} \sigma \right) \right)$

this has a unique solution because lhs strictly increasing in $b$ and rhs strictly decreasing in $b$

since rhs strictly increasing in $k$, $b(k)$ is strictly increasing

$b(k)$ has a unique fixed point at $\frac{1}{2}$

why?? substitute $b(k) = k$ and the RHS becomes $\Phi(0) = \frac{1}{2}$. 
Figure 2.1: Function $b(k)$
any strategy that is not dominated must satisfy

\[ s(x) = \begin{cases} 
  \text{Invest} & x > 1 \\
  \text{NotInvest} & x < 0 
\end{cases} \]

suppose you know your opponent will choose NotInvest for \( x < k \)
dominance implies you should choose NotInvest for \( x < b(k) \)

suppose you know your opponent will choose Invest for \( x > k \)
dominance implies you should choose Invest for \( x > b(k) \)

so after \( n \) round of iterated dominance

\[ s(x) = \begin{cases} 
  \text{Invest} & x > b^n(1) \\
  \text{NotInvest} & x < b^n(0) 
\end{cases} \]
Since $b(k)$ strictly increasing and has a unique fixed point at $\frac{1}{2}$

$$\lim_{n \to \infty} b^n(0), b^n(1) = \frac{1}{2}$$ (see the diagram)

so the only thing to survive iterated weak dominance is the cutpoint strategy

$$s(x) = \begin{cases} 
\text{Invest} & x > \frac{1}{2} \\
\text{NotInvest} & x \leq \frac{1}{2} 
\end{cases}$$

and this is a best response to itself since $b(\frac{1}{2}) = \frac{1}{2}$ so it is and equilibrium as well as the only thing to survive iterated dominance (weak or strong dominance?)
conditional on $\theta$ the choice of the two players is independent and the probability of investment is

$$\Phi\left(\frac{1}{2}-\theta \right)/\sigma$$
also a continuum of players result:

payoff to investing $\theta - 1 + l$ where $l$ is fraction of players investing

iterated deletion of dominated strategies leaves only: Invest when you get a signal greater than $1/2$. 
relationship to common knowledge