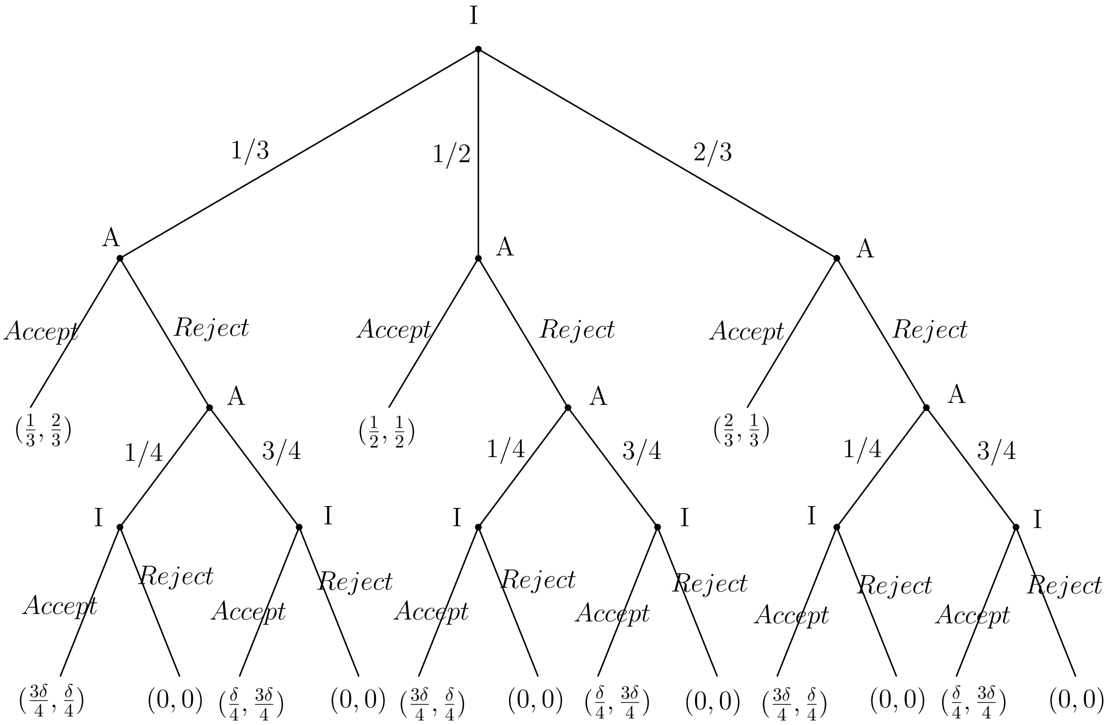
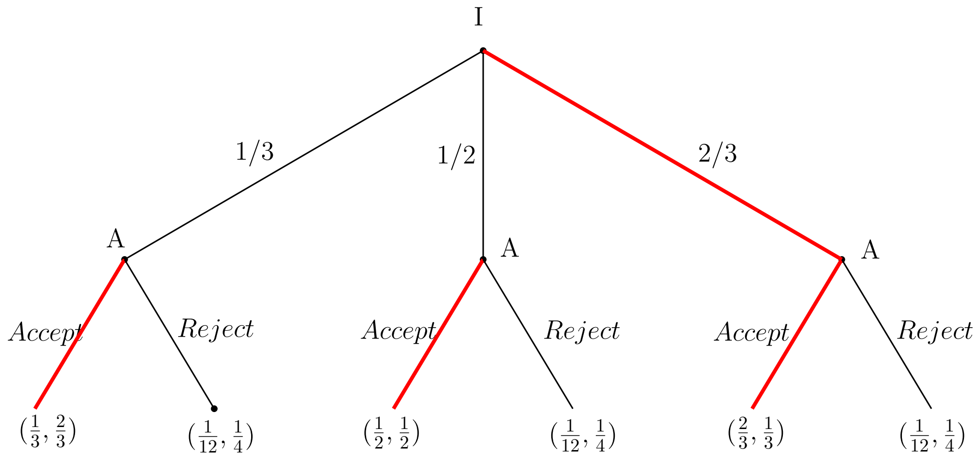
SECOND MIDTERM SOLUTIONS

1. (a) Here is the game tree;

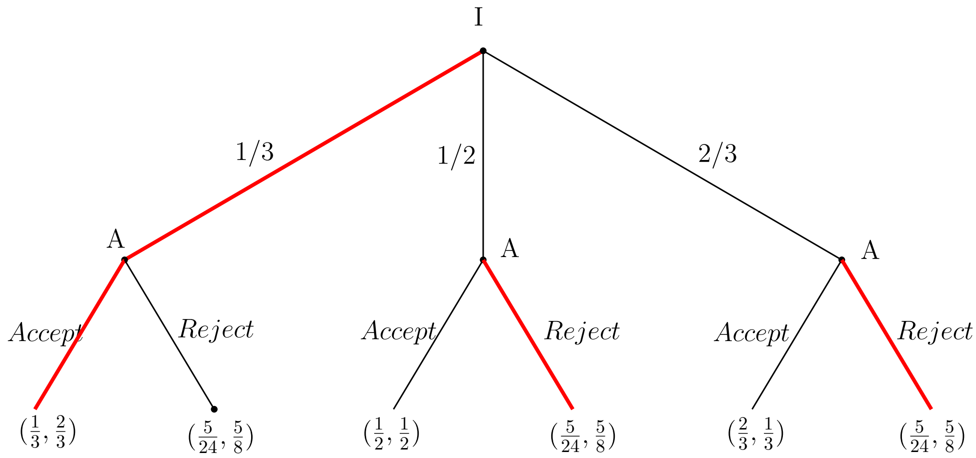


where I stands for Ingolf who is the first player, A stands for Ariel who is the second player.

(b) We determine the subgame perfect equilibrium by backward induction. At the last decision node Ingolf makes a decision on accepting or rejecting Ariel's demand. If he chooses to reject, he gets nothing, but if he accepts, he will get $\frac{\delta}{4}$ or $\frac{3\delta}{4}$ depending on Ariel's demand. In any case, it is optimal for Ingolf to accept, and knowing this Ariel demands 3/4. Hence, in any subgame that starts in Period2 (there are 3 of them), Ariel demands 3/4 and Ingolf accepts and the payoffs are $(\frac{\delta}{4}, \frac{3\delta}{4})$. Given that $\delta = \frac{1}{3}$, the payoffs are $(\frac{1}{12}, \frac{1}{4})$. Hence, we can truncate the game tree by replacing the subgames that start in Period2 with payoffs $(\frac{1}{12}, \frac{1}{4})$, and then it is easy to see that Ariel will accept any demand. Knowing this, it is optimal for Ingolf to demand 2/3. Therefore Ingolf demands 2/3 and Ariel accepts any demand in period1 and Ariel demands 3/4 and Ingolf accepts any demand in period2 form a Subgame perfect equilibrium. The illustration is given in the game tree below;



(c) We already determine that if game reaches to the second period, Ariel demands 3/4 and Ingolf accepts and the payoffs are $(\frac{\delta}{4}, \frac{3\delta}{4})$. Given that $\delta = \frac{5}{6}$, the payoffs are $(\frac{5}{24}, \frac{5}{8})$. Again truncating the game tree by replacing the subgames that start in Period2 with payoffs $(\frac{5}{24}, \frac{5}{8})$, now Ariel only accepts if Ingolf demands 1/3, and rejects if Ingolf demands 1/2 and 2/3. Knowing this, it is optimal for Ingolf to demand 1/3. Therefore Ingolf demands 1/3 and Ariel accepts if Ingolf demands 1/3, and rejects otherwise in period1 and Ariel demands 3/4 and Ingolf accepts any demand in period2 form a Subgame perfect equilibrium. The illustration is given in the game tree below;



(d) If Ariel forms a strategy in which he only accepts if Ingolf demands 1/3 and rejects otherwise, then it is optimal for Ingolf to demand 1/3. This strategy profile forms a Nash equilibrium since demanding 1/3 is best response to Ariel's strategy and accepting is best response to demanding 1/3. However, it is not a Subgame perfect equilibrium. The reason is if Ingolf demanded 1/2, then it would not be optimal for Ariel to reject it since by rejecting he would get 1/4, but by accepting 1/2. This is a typical example of incredible threat.

2. (a) The profit functions of Intendo and CCube are given below;

$$\Pi_I(x_I, x_C) = [30 - (x_I + x_C)]x_I - 9x_I$$

$$\Pi_C(x_I, x_C) = [30 - (x_I + x_C)]x_C - 3x_C$$

where I denotes Intendo and C denotes CCube.

Best response of Intendo to CCube's output is determined by

$$\max_{x_I} \Pi_I(x_I, x_C) = [30 - (x_I + x_C)]x_I - 9x_I$$

Since the profit is maximized when $\frac{\partial \Pi_I}{\partial x_I} = 0$, we obtain

$$\frac{\partial \Pi_I}{\partial x_I} = 30 - 2x_I - x_C - 9 = 0$$

$$21 - x_C = 2x_I$$

$$\boxed{x_I = 10.5 - \frac{x_C}{2}}$$

Similarly, best response of CCube to Intendo's output is determined by

$$\max_{x_C} \Pi_C(x_I, x_C) = [30 - (x_I + x_C)]x_C - 3x_C$$

Since the profit is maximized when $\frac{\partial \Pi_C}{\partial x_C} = 0$, we obtain

$$\frac{\partial \Pi_C}{\partial x_C} = 30 - x_I - 2x_C - 3 = 0$$

$$27 - x_I = 2x_C$$

$$\boxed{x_C = 13.5 - \frac{x_I}{2}}$$

(b) If Intendo is the Stackelberg leader, then Intendo knows that CCube will choose own output level as a best response to Intendo's output. Thus, knowing this Intendo can perfectly guess the output level of CCube in terms of its own output so that it can internalize it.

Thus, Intendo's profit maximization problem can be solved by replacing x_C with $13.5 - \frac{x_I}{2}$ in the profit function given below;

$$\max_{x_I} \Pi_I(x_I, x_C) = [30 - (x_I + 13.5 - \frac{x_I}{2})]x_I - 9x_I$$

Since the profit is maximized when $\frac{\partial \Pi_I}{\partial x_I} = 0$, we obtain

$$\frac{\partial \Pi_I}{\partial x_I} = 30 - 2x_I - 13.5 + x_I - 9 = 0$$

$$7.5 - x_I = 0$$

$$\boxed{x_I = 7.5}$$

Since $x_I = 7.5$ and $x_C = 13.5 - \frac{x_I}{2}$, then we obtain

$$x_C = 13.5 - \frac{7.5}{2}$$

$$\boxed{x_C = 9.75}$$

Hence, $(x_I, x_C) = (7.5, 9.75)$ is the Stackelberg equilibrium.

(c) Similarly, if CCube is the Stackelberg leader, then CCube knows that Intendo will choose own output level as a best response to CCube's output. Thus, knowing this CCube can perfectly guess the output level of Intendo in terms of its own output so that it can internalize it.

So, CCube's profit maximization problem can be solved by replacing x_I with $10.5 - \frac{x_C}{2}$ in the profit function given below;

$$\max_{x_C} \Pi_C(x_I, x_C) = [30 - (x_C + 10.5 - \frac{x_C}{2})]x_C - 3x_C$$

Since the profit is maximized when $\frac{\partial \Pi_C}{\partial x_C} = 0$, we obtain

$$\frac{\partial \Pi_C}{\partial x_C} = 30 - 2x_C - 10.5 + x_C - 3 = 0$$

$$16.5 - x_C = 0$$

$$\boxed{x_C = 16.5}$$

Since $x_C = 16.5$ and $x_I = 10.5 - \frac{x_C}{2}$, then we obtain

$$x_I = 10.5 - \frac{16.5}{2}$$

$$\boxed{x_I = 2.25}$$

Hence, $(x_I, x_C) = (2.25, 16.5)$ is the Stackelberg equilibrium.

3. (a) In order to determine the static Nash equilibria (NE) of this game, best responses are underlined on the payoff matrix which is given below;

	C	D
C	$15, 10$	$-5, \underline{50}$
D	$\underline{30}, -20$	$\underline{5}, \underline{0}$

Hence, (D, D) is the unique static NE.

(b) Each player plays D in every period irrespective of what has happened in the past. These strategies form an SPNE since a player can only hurt himself by deviating in any period. Given that they are playing the Nash Equilibrium in every period, deviation does not lead to any gain today. Moreover since the strategy in any period does not depend on what has happened in the past, deviating today cannot result in higher payoff in the future. Notice that nowhere in this argument does δ play any role. By following the strategies player 1 gets 5. By deviating today his average payoff becomes $(1-\delta)(-5)+\delta 5$, which is never greater than 5 irrespective of the value of δ. A similar argument applies for player2.

(c) Grim-trigger strategies form a Subgame Perfect Nash Equilibrium (SPNE) if there is no profitable deviation in any period for any player. So, at any period t, if player1 (row player) plays C then his average discounted payoff will be 15 (player2 also follows grim-trigger strategies and plays C unless the other player deviated in the previous period). If he chooses to deviate, then he will get 30 for the current period, and 5 in the subsequent period (because player2 will punish player1 by playing D after observing the deviation). So, his average discounted payoff is $(1-\delta)30+5\delta$. Since playing C must be optimal, then

$$15 \geq (1-\delta)30 + 5\delta$$

$$15 \geq 30 - 25\delta$$

$$\boxed{\delta \geq 0.6}$$

However, it is not sufficient to consider optimality of player1's strategy only (payoffs are not symmetric). Similarly, if player2 (column player) plays C in the current and subsequent periods, then her average discounted payoff is 10. If she chooses to deviate, then she receives a payoff 50 in the current period, but 0 in the subsequent periods. So, her average discounted payoff is $(1-\delta)50$. Then, by optimality condition,

$$10 \geq (1-\delta)50$$

$$0.2 \geq 1-\delta$$

$$\boxed{\delta \geq 0.8}$$

Hence, the common discount factor has to be greater than both 0.8 and 0.6 (otherwise, player2 chooses to deviate). Therefore, Grim-trigger strategy forms a SPNE if $\delta \geq 0.8$.