Econ 4011 12/05/2011
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PROBLEM SET #7 SOLUTIONS

Mechanism Design

1. (a) The objective function of the board of directors is the expected profit of the firm when CEO performs a high effort, and it is given as follows;

$$\begin{aligned}\Pi(w_H, e_H) &= P(success|e_H)(4v) + P(bankrupcy|e_H)(0) - w_H \\ &= (\frac{3}{4})(4v) + (\frac{1}{4})(0) - w_H \\ &= 3v - w_H \qquad (1)\end{aligned}$$

where w_H is the wage that is paid in case of high effort. Similarly, the objective function of the board of directors when the effort is low is

$$\begin{aligned}\Pi(w_L, e_L) &= P(success|e_L)(4v) + P(bankrupcy|e_L)(0) - w_L \\ &= (\frac{1}{4})(4v) + (\frac{3}{4})(0) - w_L \\ &= v - w_L \qquad (2)\end{aligned}$$

where w_L is the wage that is paid in case of low effort.

(b) Assume $w_L = 0$. Now, CEO provides high effort if the utility that he gains providing high effort exceeds the utility that he gains providing low effort. Formally,

$$u(w_H, e_H) \geq u(w_L, e_L)$$

Since $w_L = 0$, then $u(0, e_L) = log(1+0) = 0$. Thus, if we rearrange the above inequality, we obtain

$$log(1 + w_H) - log3 \geq 0$$

$$1 + w_H \geq 3$$

$$w_H \geq 2$$

Hence, if board wants CEO to provide high effort, then it must pay at least 2. In fact, board will pay exactly 2 since CEO's wage is the cost in objective function and must be chosen as small as possible.

(c) Board wants CEO to provide high effort if the net expected profit under high effort is greater than the net expected profit under low effort. Formally,

$$\Pi(w_H, e_H) \geq \Pi(w_L, e_L)$$

Since w_H is chosen 2, then by rearranging the above inequality we obtain

$$3v - 2 \geq v$$

$$v \geq 1$$

Hence, if v is greater than or equal to 1, board prefer to induce the CEO to provide high effort.

(d) In this case, board cannot observe the CEO's effort. However, two different objection functions can be written depending on the effort of CEO; when the effort is low, with probability 1/4 the firm will be successful and the profit of the firm is $4v - w_v$, but with probability 3/4 the firm will go bankrupt and the profit of the firm is $0 - w_0$. Thus, the objective function when the effort is low is

$$\begin{aligned}
\Pi(w_v, w_0, e_L) &= P(success|e_L)(4v - w_v) + P(bankrupcy|e_L)(0 - w_0) \\
&= (\frac{1}{4})(4v - w_v) + (\frac{3}{4})(-w_0) \\
&= v - \frac{w_v}{4} - \frac{3w_0}{4} \qquad (3)
\end{aligned}$$

Similarly, the objection function under the high effort is

$$\begin{aligned}
\Pi(w_v, w_0, e_H) &= P(success|e_H)(4v - w_v) + P(bankrupcy|e_H)(0 - w_0) \\
&= (\frac{3}{4})(4v - w_v) + (\frac{1}{4})(-w_0) \\
&= 3v - \frac{3w_v}{4} - \frac{w_0}{4} \qquad (4)
\end{aligned}$$

(e) Assume $w_0 = 0$. CEO only provides high effort if the expected utility of providing high effort exceeds the expected utility of providing low effort. Formally,

$$\begin{aligned}
Eu(w, e_H) &\geq Eu(w, e_L) \\
P(success|e_H)u(w_v, e_H) + P(bankrupcy|e_H)u(w_0, e_H) &\geq P(success|e_L)u(w_v, e_L) \\
&\quad +P(bankrupcy|e_L)u(w_0, e_L) \\
\frac{3}{4}[log(1 + w_v) - log3] + \frac{1}{4}[log(1 + w_0) - log3] &\geq \frac{1}{4}[log(1 + w_v)] + \frac{3}{4}[log(1 + w_0)] \\
\frac{1}{2}log(1 + w_v) - log3 &\geq 0 \\
log(1 + w_v) &\geq 2log3 \\
log(1 + w_v) &\geq log9 \\
w_v &\geq 8
\end{aligned}$$

Hence, if board wants CEO to provide high effort, then it must pay at least 8. In fact, board will pay exactly 8 since CEO's wage is the cost in objective function and must be chosen as small as possible.

(f) Board wants CEO to provide high effort if the net expected profit under high effort (equation 4) is greater than the net expected profit under low effort (equation 3). Formally,

$$\Pi(w_v, w_0, e_H) \geq \Pi(w_v, w_0, e_L)$$

Since $w_v = 8$ and $w_0 = 0$, then by rearranging the above inequality we obtain

$$3v - \frac{24}{4} \geq v - \frac{8}{4}$$

$$2v \geq 4$$

$$v \geq 2$$

Hence, if v is greater than or equal to 2, board prefer to induce the CEO to provide high effort.

2. There are two possible types for a consumer. The low-type's valuation is 1 and high type's valuation is 3 per unit of the good. Since the seller is only able to sell 1 unit or 2 units of the good, then the seller's problem is organized as a mechanism design problem as follows;

$$\begin{array}{lll} \max_{x^h, x^l} & \frac{1}{2}p^h x^h + \frac{1}{2}p^l x^l & \\ & & \\ \text{s.t} & (1 - p^i)x^i \geq 0 & \text{(IR)} \\ & (v^i - p^i)x^i \geq (v^i - p^{-i})x^{-i} & \text{(IC)} \\ & x^i \in \{1, 2\} & \end{array}$$

where $i \in \{h, l\}$. Notice that there are 2 IR constraints and 2 IC constraints in the above problem. We guess that individual rationality constraint of low type and incentive constraint of high type is binding. i.e.,

$$(v^l - p^l)x^l = 0 \tag{5}$$

$$(v^h - p^h)x^h = (v^h - p^l)x^l \tag{6}$$

Since $v^l = 1$ and $x^l \in \{1, 2\}$, then from equation (5) we obtain

$$\boxed{p^l = 1} \tag{7}$$

By combining equation (6) and (7), and given that $v^h = 3$, we obtain

$$\begin{aligned} (3 - p^h)x^h &= (3-1)x^l \\ 3 - p^h &= \frac{2x^l}{x^h} \\ p^h &= 3 - \frac{2x^l}{x^h} \end{aligned} \tag{8}$$

If we pluq equations (7) and (8) into the objective function, we obtain

$$\max_{x^h, x^l} \quad \tfrac{1}{2}(3 - \tfrac{2x^l}{x^h})x^h + \tfrac{1}{2}x^l$$

$$\text{s.t.} \qquad x^l, x^h \in \{1, 2\}$$

By rearranging the terms in objective function, we obtain

$$\max_{x^h, x^l} \quad \tfrac{3}{2}x^h - \tfrac{1}{2}x^l$$

$$\text{s.t.} \qquad x^l, x^h \in \{1, 2\}$$

Hence, seller chooses x^l as low as possible which is equal to 1 and chooses x^h as high as possible which is equal to 2. Therefore, seller offers two options;
(i) one unit of good for a price 1
(ii) two units of good for a price 2 each.

Demand Theory

1. The maximization problem is

$$\max_{x_1, x_2} \quad log(x_1) + log(x_2)$$

$$\text{s.t.} \qquad p_1x_1 + p_2x_2 \leq m$$

In order to determine the demand for good x_1 and x_2, we need to write down the Lagrangian and find the first order conditions (FOC).

$$\max_{x_1, x_2} L(x_1, x_2) = log(x_1) + log(x_2) + \lambda(m - p_1x_1 - p_2x_2)$$

Then, we obtain the FOCs by taking the derivative of L with respect to x_1 and x_2 and set them equal to 0. So,

$$\frac{1}{x_1} - \lambda p_1 = 0$$

$$\frac{1}{x_2} - \lambda p_2 = 0$$

If we rearrange and divide side by side, we obtain

$$\frac{x_2}{x_1} = \frac{\lambda p_1}{\lambda p_2}$$

$$\boxed{p_1 x_1 = p_2 x_2}$$

If we plug it into the budget constraint, we obtain

$$p_1 x_1 + p_1 x_1 = m$$

$$x_1(p_1, p_2, m) = \frac{m}{2p_1} \tag{9}$$

Similarly,

$$x_2(p_1, p_2, m) = \frac{m}{2p_2} \tag{10}$$

If the price of x_1 rises by 10%, then the new price becomes $1.1p_1$. So, the new demand is

$$x_1' = \frac{m}{2(1.1)p_1} = \frac{x_1}{1.1} \approx 0.91x_1$$

Hence, the demand falls approximately 9%. The demand for flour (see equation (10)) is not affected by the changes in price of flounder.

2. A demand function must satisfy the following two properties;
(i) Homogeneous of degree 0
(ii) Budget constraint

(a) Let's first check the homogeneity of degree 0;

$$x_1(\lambda p_1, \lambda p_2, \lambda m) = \frac{(\lambda m)(\lambda p_2)}{\lambda p_1} = \lambda \frac{m p_2}{p_1} = \lambda x_1(p_1, p_2, m)$$

Hence, the function x_1 is not homogeneous of degree 0. Therefore, it cannot be a demand function.

(b) Let's first check the homogeneity of degree 0;

$$x_1(\lambda p_1, \lambda p_2, \lambda m) = \frac{\lambda m}{\lambda p_2 + \lambda p_1} = \frac{\lambda m}{\lambda(p_2 + p_1)} = \frac{m}{p_2 + p_1} = x_1(p_1, p_2, m)$$

Hence, the functions x_1 and x_2 are homogeneous of degree 0. Now, let's check whether they satisfy the budget constraint;

$$p_1 x_1 + p_2 x_2 = p_1(\frac{m}{p_2 + p_1}) + p_2(\frac{m}{p_2 + p_1}) = \frac{m p_1 + m p_2}{p_2 + p_1} = m$$

Hence, x_1 and x_2 are also satisfy the budget constraint. Therefore, they can be demand functions.

Partial Equilibrium

1. (a) The utility maximization of the consumer is given as follows;

$$\max_{x_1,x_2} \quad x_1 - (x_2 - 12)^2$$

$$\text{s.t.} \qquad p_1x_1 + p_2x_2 \leq I$$
$$x_1 \geq 0, x_2 \geq 0$$

Observe that consumer wants to have 12 units of good 2 in order to maximize his/her utility. However, depending on the prices having that amount of good 2 may not be feasible, so in order to determine the demand for good x_1 and x_2, again we need to write down the Lagrangian and find the first order conditions (FOC).

$$\max_{x_1,x_2} L(x_1, x_2) = x_1 - (x_2 - 12)^2 + \lambda(I - p_1x_1 - p_2x_2)$$

Then, we obtain the FOCs by taking the derivative of L with respect to x_1 and x_2 and set them equal to 0. So,

$$1 - \lambda p_1 = 0$$

$$-2(x_2 - 12) - \lambda p_2 = 0$$

If we rearrange and combine them, we obtain

$$x_2 - 12 = -\frac{p_2}{2p_1}$$

$$x_2 = 12 - \frac{p_2}{2p_1}$$

Since x_2 must be greater than or equal to 0, then

$$12 - \frac{p_2}{2p_1} \geq 0$$
$$24p_1 \geq p_2$$

Otherwise, if $24p_1 < p_2$, then this means that the price of good 2 is so high that consumer does not spend any money on good 2 and demands only good 1, so x_2 would be 0. Hence, the demand for good 2 is

$$x_2(p_1, p_2, m) = \begin{cases} 12 - \frac{p_2}{2p_1} & \text{if} \quad 24p_1 \geq p_2 \\ 0 & \text{if} \quad 24p_1 < p_2 \end{cases}$$

If we plug it into the budget constraint, we obtain the demand for good 1

$$p_1x_1 + p_2(12 - \frac{p_2}{2p_1}) = I$$

$$p_1 x_1 = I - 12p_2 + \frac{p_2^2}{2p_1}$$

$$x_1(p_1, p_2, m) = \begin{cases} \frac{I}{p_1} - 12\frac{p_2}{p_1} + \frac{1}{2}(\frac{p_2}{p_1})^2 & \text{if} \quad 24p_1 \geq p_2 \\ \frac{I}{p_1} & \text{if} \quad 24p_1 < p_2 \end{cases}$$

(b) Let's first check the homogeneity of degree 0 of the demand function for good 1. If $24p_1 \geq p_2$, then

$$x_1(\lambda p_1, \lambda p_2, \lambda m) = \frac{\lambda I}{\lambda p_1} - 12\frac{\lambda p_2}{\lambda p_1} + \frac{1}{2}(\frac{\lambda p_2}{\lambda p_1})^2 = \frac{I}{p_1} - 12\frac{p_2}{p_1} + \frac{1}{2}(\frac{p_2}{p_1})^2 = x_1(p_1, p_2, m)$$

and if $24p_1 < p_2$, then

$$x_1(\lambda p_1, \lambda p_2, \lambda m) = \frac{\lambda I}{\lambda p_1} = \frac{I}{p_1} = x_1(p_1, p_2, m)$$

Hence, the function x_1 is homogeneous of degree 0.

Now, let's check the homogeneity of degree 0 of the demand function for good 2. If $24p_1 \geq p_2$, then

$$x_2(\lambda p_1, \lambda p_2, \lambda m) = 12 - \frac{\lambda p_2}{2\lambda p_1} = 12 - \frac{p_2}{2p_1} = x_2(p_1, p_2, m)$$

and if $24p_1 < p_2$, then

$$x_2(\lambda p_1, \lambda p_2, \lambda m) = 0 = x_2(p_1, p_2, m)$$

Hence, the function x_2 is also homogeneous of degree 0.

(c) If $x_2 > 12$, then the consumer only consumes 12 unit exactly and throws away the remaining.

(d) The firm's profit maximization problem is

$$\max_{x_2} p_2 x_2 - c x_2$$

In the competitive equilibrium, firm makes zero profit, otherwise other firms would enter the market and cut prices until it is equal to the marginal cost. Thus, $p_2 = c$. If $24p_1 \geq c$, then the consumer demands $12 - (c/2p_1)$ units of good 2 and firm will produce exactly this amount. If $24p_1 < c$, then consumer demands 0 unit implying that no production will occur.

(e) Monopolist decides how much to charge for good 2 by taking into account the demand of consumer. So, monopolist's profit maximization problem is

$$\max_{x_2} \quad (p_2 - c)(12 - \tfrac{p_2}{2p_1})$$

$$\text{s.t} \qquad p_2 \leq 24p_1$$

If we take the derivative of this profit function with respect to p_2 and set it equal to 0, we obtain

$$(12 - \frac{p_2^*}{2p_1}) + (p_2^* - c)(\frac{-1}{2p_1}) = 0$$
$$24p_1 - p_2^* - p_2^* + c = 0$$
$$p_2^* = \frac{24p_1 + c}{2}$$

Then, the consumer demand will be

$$\hat{x}_2 = 12 - \frac{24p_1 + c}{4p_1} = 6 - \frac{c}{4p_1}$$

This only holds if $24p_1 \geq c$. Otherwise, no production will take place. Notice that monopolist's price higher than competitive equilibrium price.

General Equilibrium

1. Let's denote Orca by O and Dorca by D. Then, the endowments are $\bar{x}^O = (0,1)$ and $\bar{x}^D = (1,0)$. First, we need to find the demands. The maximization problem is

$$\max_{x_1,x_2} \quad x_1^{1/2}x_2^{1/2}$$

$$\text{s.t.} \qquad p_1x_1 + p_2x_2 \leq p_1\bar{x}_1 + p_2\bar{x}_2$$

Then, the Lagrangian is

$$\max_{x_1,x_2} L(x_1,x_2) = x_1^{1/2}x_2^{1/2} + \lambda(p_1\bar{x}_1 + p_2\bar{x}_2 - p_1x_1 - p_2x_2)$$

Then, we obtain the FOCs by taking the derivative of L with respect to x_1 and x_2 and set them equal to 0. So,

$$\frac{1}{2}x_1^{-1/2}x_2^{1/2} - \lambda p_1 = 0$$
$$\frac{1}{2}x_1^{1/2}x_2^{-1/2} - \lambda p_2 = 0$$

If we rearrange and divide side by side, we obtain

$$\frac{x_2}{x_1} = \frac{\lambda p_1}{\lambda p_2}$$

$$\boxed{p_1x_1 = p_2x_2}$$

If we plug it into the budget constraint, we obtain

$$p_1x_1 + p_1x_1 = p_1\bar{x}_1 + p_2\bar{x}_2$$

$$x_1(p_1, p_2, \bar{x}_1, \bar{x}_2) = \frac{p_1\bar{x}_1 + p_2\bar{x}_2}{2p_1} \tag{11}$$

Similarly,

$$x_2(p_1, p_2, \bar{x}_1, \bar{x}_2) = \frac{p_1\bar{x}_1 + p_2\bar{x}_2}{2p_2} \tag{12}$$

So, given demand functions, individual demands are

$$x_1^O = \tfrac{p_2}{2p_1} \quad x_2^O = \tfrac{1}{2}$$
$$x_1^D = \tfrac{1}{2} \quad x_2^D = \tfrac{p_1}{2p_2}$$

In competitive equilibrium markets clear, so the following equations must hold;

$$x_1^O + x_1^D = \bar{x}_1^O + \bar{x}_1^D \tag{13}$$

$$x_2^O + x_2^D = \bar{x}_2^O + \bar{x}_2^D \tag{14}$$

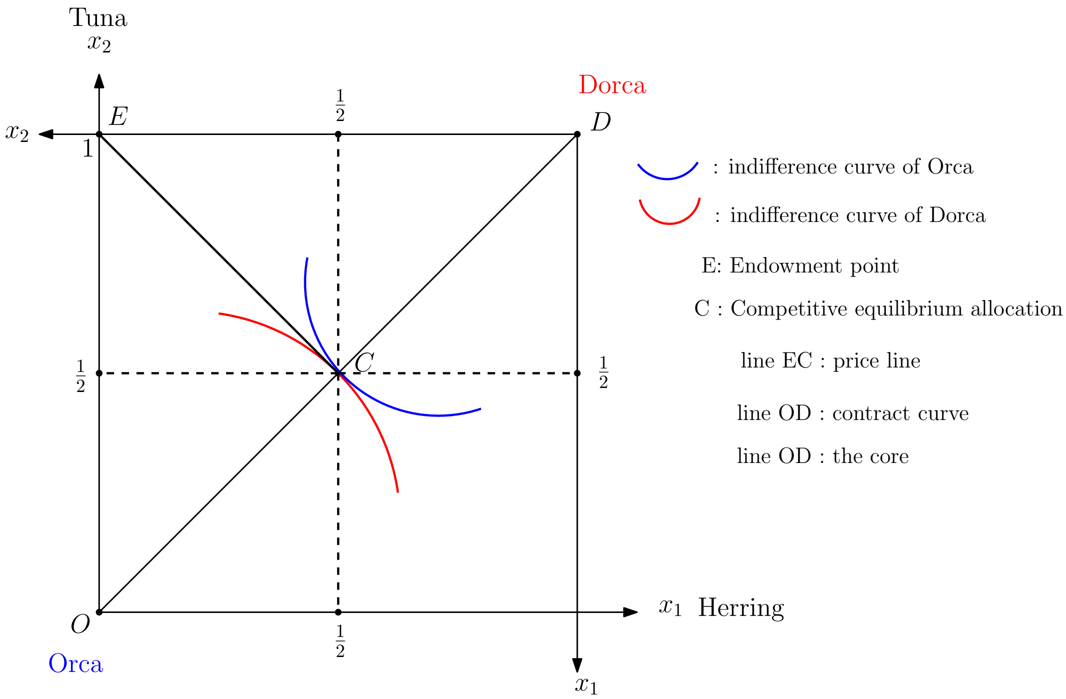
If we plug the values of individual demands and endowments into equation (13) and (14), we obtain

$$\boxed{\hat{p}_1 = \hat{p}_2}$$

Thus, in the equilibrium the prices must be the same. Moreover, the equilibrium demands (allocation) are

$$\hat{x}^O = (\hat{x}_1^O, \hat{x}_2^O) = (\frac{1}{2}, \frac{1}{2})$$
$$\hat{x}^D = (\hat{x}_1^D, \hat{x}_2^D) = (\frac{1}{2}, \frac{1}{2})$$

Here is the Edgeworth Box that shows the endowment point, competitive equilibrium allocation, price line, contract curve and the core;



2. Now their endowments are $\bar{x}^O = (\bar{x}_1^1, 0)$ and $\bar{x}^D = (0, \bar{x}_2^2)$. Since the utility function is different, first we need to determine demand functions;

$$\max_{x_1,x_2} \quad x_1^\alpha + x_2^\alpha$$

$$\text{s.t.} \qquad p_1x_1 + p_2x_2 \leq p_1\bar{x}_1 + p_2\bar{x}_2$$

In this case, both Orca and Dorca value herring and tuna as substitutes in the sense that they don't care how much they consume tuna or herring (they are equally good), they only care the total amount they consume. So, they always demand the cheaper good, and if the prices of herring and tuna are the same, then any division between two goods are equally good (they are indifferent). Formally,

$$x_1(p_1, p_2, \bar{x}) = \begin{cases} 0 & \text{if} \quad p_1 > p_2 \\ \frac{p_1\bar{x}_1+p_2\bar{x}_2}{p_1} & \text{if} \quad p_1 < p_2 \\ 0 \leq x_1 \leq \bar{x}_1 + \bar{x}_2 & \text{if} \quad p_1 = p_2 \end{cases}$$

and

$$x_2(p_1, p_2, \bar{x}) = \begin{cases} \frac{p_1\bar{x}_1+p_2\bar{x}_2}{p_1} & \text{if} \quad p_1 > p_2 \\ 0 & \text{if} \quad p_1 < p_2 \\ \bar{x}_1 + \bar{x}_2 - x_1 & \text{if} \quad p_1 = p_2 \end{cases}$$

(a) Let's write down the individual demands first;

$$\begin{matrix} x_1^O = 0 & x_2^O = \frac{p_1\bar{x}_1^1}{p_2} \\ x_1^D = 0 & x_2^O = \bar{x}_2^2 \end{matrix} \quad \text{if} \quad p_1 > p_2$$

$$\begin{matrix} x_1^O = \bar{x}_1^1 & x_2^O = 0 \\ x_1^D = \frac{p_1\bar{x}_1^1}{p_2} & x_2^O = 0 \end{matrix} \quad \text{if} \quad p_1 < p_2$$

$$\begin{matrix} x_1^O + x_2^O = \bar{x}_1^1 \\ x_1^D + x_2^D = \bar{x}_2^2 \end{matrix} \quad \text{if} \quad p_1 = p_2$$

Individual excess demand is the difference between the individual demand and endowment, and they are given below;

$$\begin{array}{ll} z_1^O = -\bar{x}_1^1 & z_2^O = \frac{p_1 \bar{x}_1^1}{p_2} \\ z_1^D = 0 & z_2^O = 0 \end{array} \quad \text{if} \quad p_1 > p_2$$

$$\begin{array}{ll} z_1^O = 0 & z_2^O = 0 \\ z_1^D = \frac{p_1 \bar{x}_1^1}{p_2} & z_2^O = -\bar{x}_2^2 \end{array} \quad \text{if} \quad p_1 < p_2$$

$$\begin{array}{ll} z_1^O = -x_2^O & z_2^O = \bar{x}_1^1 - x_1^O \\ z_1^D = \bar{x}_2^2 - x_2^D & z_2^D = -x_1^D \end{array} \quad \text{if} \quad p_1 = p_2$$

(b) It is easy to see that the market does not clear when the prices are not equal implying that no competitive equilibrium exist. Markets only clear when $p_1 = p_2$. Hence, the following are satisfied;

$$\hat{x}_1^O + \hat{x}_1^D = \bar{x}_1^1$$

$$\hat{x}_2^O + \hat{x}_2^D = \bar{x}_2^2$$

$$\hat{x}_1^O + \hat{x}_2^O = \bar{x}_1^1$$

$$\hat{x}_1^D + \hat{x}_2^D = \bar{x}_2^2$$

Notice that there are more than one allocation that satisfies above equations. If the price of good 1 is 1, then the price of good 2 is 1 as well. Hence, increase in $\bar{x}_2^2$ does not affect p_2.

3. (a) If Mr. Yuppie does not donate, his utility would be

$$u_{MrY}(14, 14, 8) = (14)^2(14)(8) = 21952$$

If Mr. Yuppie donates a crystal to the student, his utility would be

$$u_{MrY}(13, 14, 9) = (13)^2(14)(9) = 21294$$

Hence, he will not donate.
Similarly, if Ms. Yuppie does not donate, her utility would be

$$u_{MsY}(14, 14, 8) = (14)(14)^2(8) = 21952$$

If Ms. Yuppie donates a crystal to the student, her utility would be

$$u_{MsY}(14, 13, 9) = (14)(13)^2(9) = 21294$$

Hence, she will not donate.

(b) Before donation Mr. Yuppie and Ms. Yuppie's utilities are 21952, and the student's utility is 12544. If both Mr. Yuppie and Ms. Yuppie donate a crystal to the student, then the utilities are

$$u_{MrY}(13, 13, 10) = (13)^2(13)(10) = 21970$$

$$u_{MsY}(13, 13, 10) = (13)(13)^2(10) = 21970$$

$$u_S(13, 13, 10) = (13)(13)(10)^2 = 16900$$

Hence, everyone will be better off. This implies that competitive equilibrium is not efficient.

(c) It is obvious that neither Mr. Yuppie nor Ms. Yuppie will be better off by donating more than one crystal to the student by himself or herself alone. Moreover, they will not be better off by donating the same amount (more than one) of crystal to the student (why?). Then, the problem is whether Ms. Yuppie will choose to donate a crystal. So, if Mr. Yuppie decides to donate a crystal to the student and if Ms. Yuppie matches his donation, then his utility will be 21970 implying that both will be better off. However, Ms. Yuppie has an incentive to deviate since she will be better off by not matching Mr. Yuppie's donation,

$$u_{MsY}(13, 14, 9) = (13)(14)^2(9) = 22932 > 21970 = u_{MsY}(13, 13, 10)$$

Hence, knowing this Mr. Yuppie will not donate.