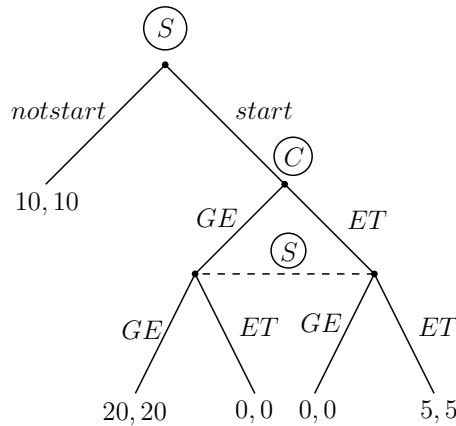


PROBLEM SET #3 SOLUTIONS

1. -Draw the extensive form of this game.

Here is the game tree;



where S stands for Stephen J. Seagull who is the first player, C stands for Clod VandeCamp who is the second player, GE stands for the action of choosing George Spellbinder as the director, and ET stands for the action of choosing Ed Tree as the director.

-Find the normal form.

The normal form representation is given as follows;

	<i>GE</i>	<i>ET</i>
<i>startGE</i>	<u>20, 20</u>	0, 0
<i>startET</i>	0, 0	5, <u>5</u>
<i>notstartGE</i>	10, <u>10</u>	<u>10, 10</u>
<i>notstartET</i>	10, <u>10</u>	<u>10, 10</u>

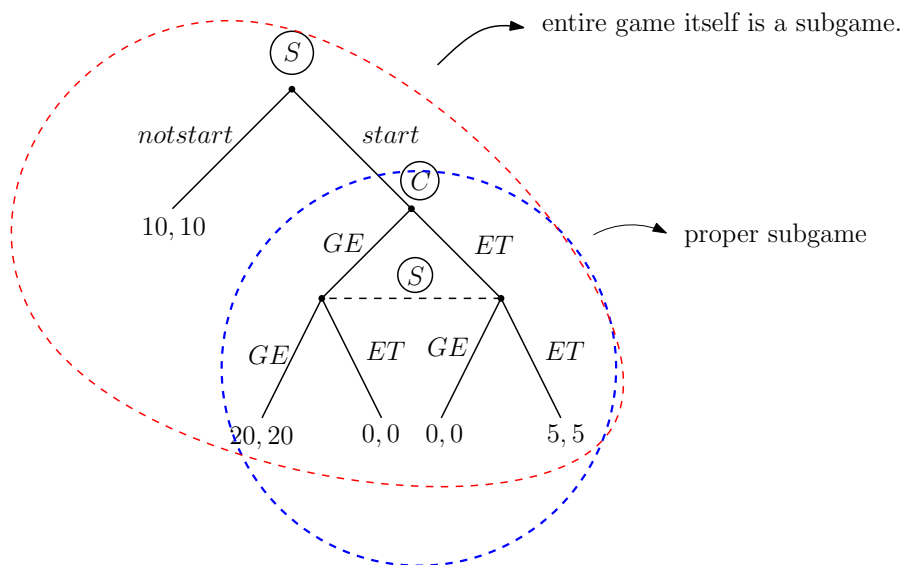
where player S has 4 strategies and each strategy consists of two actions; one for the first information set which consist of root node, and one for the second information set which consists of two decision nodes.

-Find all the Nash equilibria.

The best responses are underlined in the payoff matrix above. Thus, there are three Nash equilibria; $(startGE, GE)$, $(notstartET, ET)$ and $(notstartGE, ET)$.

-Find all the subgame perfect equilibria.

In this game, we need to be careful while using backward induction for subgame perfection since even though there are four nodes, in the two nodes where player S moves, S is not perfectly informed about the choice of player C. So it is better to determine subgames first. There are two subgames of this game shown below; one starts with the decision node of player C and the other is the entire game itself.



Thus, we need to determine the Nash equilibria of the proper subgame, and it is shown by underlining the best responses on the normal form representation of it which is given below;

	<i>GE</i>	<i>ET</i>
<i>GE</i>	<u>20, 20</u>	0, 0
<i>ET</i>	0, 0	<u>5, 5</u>

Now, we can check whether any Nash equilibria of the entire game is Subgame Perfect or not. Since we know that SPNE requires to be a Nash equilibrium in each subgame, let's first consider the strategy profile $(startGE, GE)$. Since (GE, GE) is a Nash equilibrium in the proper subgame, then $(startGE, GE)$ is a Subgame Perfect Nash Equilibrium (SPNE). Similarly, $(notstartET, ET)$ is SPNE since (ET, ET) is a Nash equilibrium in the proper subgame. However, $(notstartGE, ET)$ is not a SPNE since (GE, ET) is not a Nash equilibrium in the proper subgame.

-Apply the theory of iterated elimination of weakly dominated strategies and state its prediction. Since $startET$ is strictly dominated by $notstartET$ and $notstartGE$, then we can eliminate $startET$ and we obtain the following smaller matrix;

	GE	ET
$startGE$	20, 20	0, 0
$notstartGE$	10, 10	10, 10
$notstartET$	10, 10	10, 10

Now, ET is weakly dominated by GE implying that ET is eliminated, and we obtain the following payoff matrix;

	GE
$startGE$	20, 20
$notstartGE$	10, 10
$notstartET$	10, 10

Observe that both $notstartET$ and $notstartGE$ is strictly dominated by $startGE$ implying that both of them are eliminated and hence the outcome of iterated elimination of weakly dominated strategies is $(startGE, GE)$.

2. The profit functions of Savannah and Frontier are given below;

$$\Pi_S(x_S, x_F) = [17 - (x_S + x_F)]x_S - 3x_S$$

$$\Pi_F(x_S, x_F) = [17 - (x_S + x_F)]x_F - x_F$$

where S denotes Savannah and F denotes Frontier.

-What is the Stackelberg Equilibrium if Savannah is the Stackelberg leader?

If Savannah is the Stackelberg leader, then Savannah knows that Frontier will choose own output level as a best response to Savannah's output. Thus, knowing this Savannah can perfectly guess the output level of Frontier in terms of its own output so that it can internalize it.

Formally, best response of Frontier to Savannah's output is determined by

$$\max_{x_F} \Pi_F(x_S, x_F) = [17 - (x_S + x_F)]x_F - x_F$$

Since the profit is maximized when $\frac{\partial \Pi_F}{\partial x_F} = 0$, we obtain

$$\frac{\partial \Pi_F}{\partial x_F} = 17 - x_S - 2x_F - 1 = 0$$

$$16 - x_S = 2x_F$$

$$\boxed{x_F = 8 - \frac{x_S}{2}}$$

Now, Savannah's profit maximization problem can be solved by replacing x_F with $8 - \frac{x_S}{2}$ in the profit function given below;

$$\max_{x_S} \Pi_S(x_S, x_F) = [17 - (x_S + 8 - \frac{x_S}{2})]x_S - 3x_S$$

Since the profit is maximized when $\frac{\partial \Pi_S}{\partial x_S} = 0$, we obtain

$$\frac{\partial \Pi_S}{\partial x_S} = 17 - 2x_S - 8 + x_S - 3 = 0$$

$$6 - x_S = 0$$

$$\boxed{x_S = 6}$$

Since $x_S = 6$ and $x_F = 8 - \frac{x_S}{2}$, then we obtain

$$x_F = 8 - \frac{6}{2}$$

$$\boxed{x_F = 5}$$

Hence, $(x_S, x_F) = (6, 5)$ is the Stackelberg equilibrium.

-What is the Stackelberg Equilibrium if Frontier is the Stackelberg leader?

Similarly, if Frontier is the Stackelberg leader, then Frontier knows that Savannah will choose own output level as a best response to Frontier's output. Thus, knowing this Frontier can perfectly guess the output level of Savannah in terms of its own output so that it can internalize it.

Formally, best response of Savannah to Frontier's output is determined by

$$\max_{x_S} \Pi_S(x_S, x_F) = [17 - (x_S + x_F)]x_S - 3x_S$$

Since the profit is maximized when $\frac{\partial \Pi_S}{\partial x_S} = 0$, we obtain

$$\frac{\partial \Pi_S}{\partial x_S} = 17 - 2x_S - x_F - 3 = 0$$

$$14 - x_F = 2x_S$$

$$\boxed{x_S = 7 - \frac{x_F}{2}}$$

Now, Frontier's profit maximization problem can be solved by replacing x_S with $7 - \frac{x_F}{2}$ in the profit function given below;

$$\max_{x_F} \Pi_F(x_S, x_F) = [17 - (x_F + 7 - \frac{x_F}{2})]x_F - x_F$$

Since the profit is maximized when $\frac{\partial \Pi_F}{\partial x_F} = 0$, we obtain

$$\frac{\partial \Pi_F}{\partial x_F} = 17 - 2x_F - 7 + x_F - 1 = 0$$

$$9 - x_F = 0$$

$$\boxed{x_F = 9}$$

Since $x_F = 9$ and $x_S = 7 - \frac{x_F}{2}$, then we obtain

$$x_S = 7 - \frac{9}{2}$$

$$\boxed{x_S = 2.5}$$

Hence, $(x_S, x_F) = (2.5, 9)$ is the Stackelberg equilibrium.

3. We are going to solve this game by backward induction. Let's label the pirates as P1, P2, P3, P4 and P5. Suppose first three pirates are dead, and P4 makes an offer. No matter what he offers, P5 will reject it, and according to the procedure (remember if at least half of the pirates vote against the division, then it is not accepted) P4 will be dead implying that P5 gets 50 gold coins. Knowing this, P3 will offer 1 coin to P4 and nothing to P5 and keeps the rest for himself. This will be accepted because P3 and P4 will say yes even though P5 will vote no. Knowing this, P2 will offer 2 coin to P4 and 1 to P5 and nothing to P3. Then, this division will be accepted since P2, P4 and P5 will vote yes. Knowing this, P1 will offer 1 coin to P3, 2 coin to P5 and nothing to P2 and P4. Then, this offer will be accepted since P1, P3 and P5 will vote yes.

Hence, P1 offers $(47, 0, 1, 0, 2)$ and P1, P3, P5 votes yes and P2, P4 votes no implying that division is accepted and game ends.