PROBLEM SET #4 SOLUTIONS

1. Since Hiccup returns a utility of 4 on odd periods and 1 on even periods, the infinite horizon average discounted utility is calculated as follows;

$$
\begin{aligned}
U_{Hiccup} &= (1-\delta)[(4 + \delta(1) + \delta^2(4) + \delta^3(1) + ...)] \\
&= (1-\delta)[4(1 + \delta^2 + \delta^4 + ...) + (\delta + \delta^3 + \delta^5...)] \\
&= (1-\delta)[4(1 + (\delta^2) + (\delta^2)^2 + ...) + \delta(1 + (\delta^2) + (\delta^2)^2 + ...)] \\
&= (1-\delta)[(4 + \delta)(1 + (\delta^2) + (\delta^2)^2 + ...)] \\
&= (1-\delta)(4 + \delta)(\frac{1}{1 - \delta^2}) \\
&= \frac{(1-\delta)(4+\delta)}{(1-\delta)(1+\delta)} \\
&= \frac{4+\delta}{1+\delta}
\end{aligned}
$$

Similarly, since TwoStep returns a utility of 10 for the first period and then always a utility of 2, the infinite horizon average discounted utility is given as follows;

$$
\begin{aligned}
U_{TwoStep} &= (1-\delta)(10 + 2\delta\frac{1}{1-\delta}) \\
&= (1-\delta)10 + 2\delta \\
&= 10 - 8\delta
\end{aligned}
$$

In order to determine which investment opportunity is better, first consider the following function;

$$
\begin{aligned}
h(\delta) &= \frac{4+\delta}{1+\delta} - (10 - 8\delta) \\
&= \frac{(4+\delta) - (10 - 8\delta + 10\delta - 8\delta^2)}{1+\delta} \\
&= \frac{8\delta^2 - \delta - 6}{1+\delta}
\end{aligned}
$$

There exist a $\delta^* \in (0,1)$ such that $h(\delta^*) = 0$ (it means two utilities are the same). To see this;

$$
h(\delta^*) = 0
$$

$$
8(\delta^*)^2 - \delta^* - 6 = 0
$$

$$\delta^* \approx 0.931$$

Observe that there is also another negative real solution to this quadratic equation, but we don't need it. Secondly, we need to determine for what values of δ, function h takes positive values. Thus, we need to take the derivative of h with respect to δ;

$$\frac{dh(\delta)}{d\delta} = \frac{(16\delta - 1)(1 + \delta) - (8\delta^2 - \delta - 6)(1)}{(1 + \delta)^2}$$

$$= \frac{16\delta^2 + 15\delta - 1 - 8\delta^2 + \delta + 6}{(1 + \delta)^2}$$

$$= \frac{8\delta^2 + 16\delta + 5}{(1 + \delta)^2}$$

Hence, for any value of δ, $h'(\delta) > 0$ implying that h is an increasing function. In other words, once the function h hits 0 at the solution point, it becomes positive from then on. Also for all lower values the expression must be negative.

Therefore, if $\delta > \delta^*$, then $h(\delta) > 0$ implying that $U_{Hiccup} > U_{TwoStep}$, so it is better to invest in Hiccup. If $\delta < \delta^*$, then $h(\delta) < 0$ implying that $U_{Hiccup} < U_{TwoStep}$, so it is better to invest in TwoStep. If $\delta = \delta^*$, then $h(\delta) = 0$ implying that $U_{Hiccup} = U_{TwoStep}$, so they are equally good.
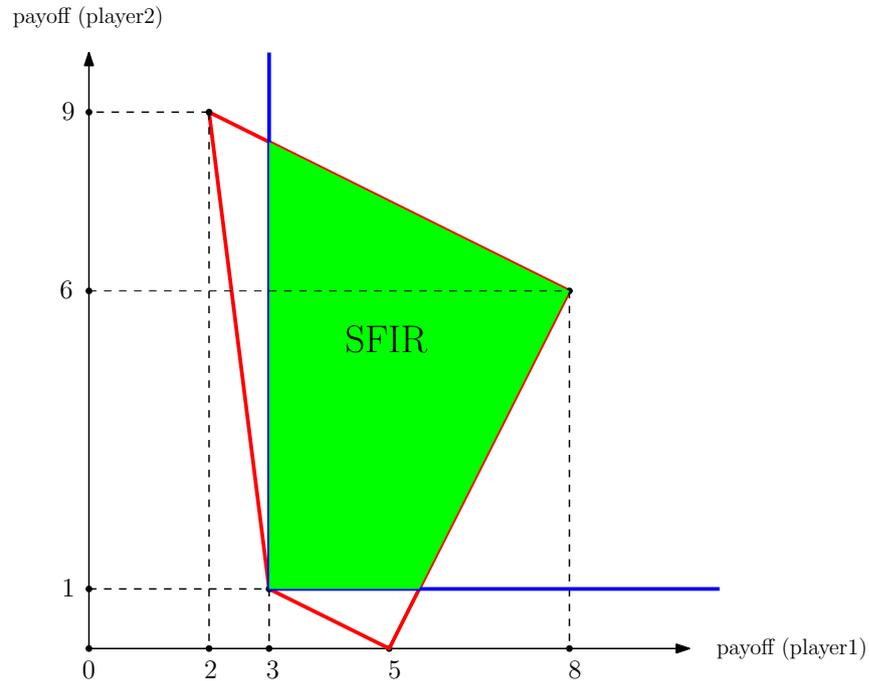
2. (a) In order to determine the static Nash equilibria (NE) of this game, best responses are underlined on the payoff matrix which is given below;

	L	R
U	$\underline{8}, 6$	$2, \underline{9}$
D	$5, 0$	$\underline{3}, \underline{1}$

Hence, (D, R) is the unique static NE.

Socially feasible region is determined by linking the payoff profiles resulted from playing pure strategies. Individually rational region is determined by finding the minmax payoff of each player. Minmax payoff of a player is the lowest payoff that the other player can force upon him. Thus, row player's minmax payoff is 3 since column player will play R to lower row player's payoff and knowing this row player plays D. Column player's minmax payoff is 1 since row player will play D to lower column player's payoff and knowing this column player plays R. Then, we obtain the individually rational region where row player's payoff must be greater than equal to 3, and column player's payoff

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must be greater than equal to 1. Hence, the socially feasible individually rational region is the intersection of socially feasible region and individually rational region which is given below;


(b) In order to determine the static Nash equilibria (NE) of this game, best responses are underlined on the payoff matrix which is given below;

	E	W
N	$\underline{7}, \underline{7}$	$\underline{4}, 0$
S	$0, \underline{4}$	$0, 0$

Hence, (N, E) is the unique static NE.

Similarly, determine the socially feasible region by linking the payoff profiles resulted from playing pure strategies. Row player's minmax payoff is 4 since column player will play W to lower row player's payoff and knowing this row player chooses to play N. Column player's minmax payoff is 4 since row player will play S to lower column player's payoff and knowing this row player chooses to play E. Then, we obtain the individually rational region where both players' payoff must be greater than equal to 4. Hence, the socially feasible individually rational region is the intersection of socially feasible region and individually rational region which is given below;

payoff (player2)

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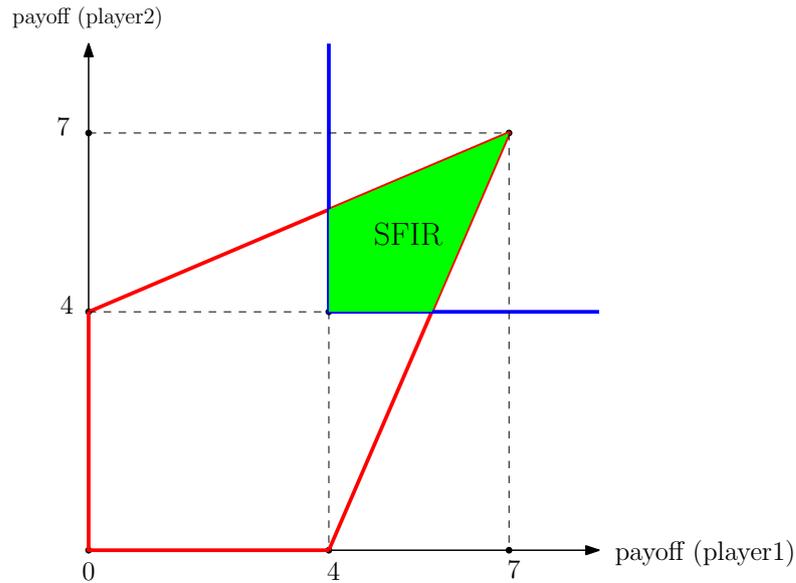
payoff (player1)

3. Grim-trigger strategies form a Subgame Perfect Nash Equilibrium (SPNE) if there is no profitable deviation in any period for any player. So, at any period t, if player1 (row player) plays H then his average discounted payoff will be 1 (player2 also follows grim-trigger strategies and plays H unless the other player deviated in the previous period). If he chooses to deviate, then he will get 100 for the current period, and 0 in the subsequent period (because player2 will punish player1 by playing L after observing the deviation). So, his average discounted payoff is $(1 - \delta)100$. Since playing H must be optimal, then

$$1 \geq (1 - \delta)100$$

$$0.01 \geq 1 - \delta$$

$$\boxed{\delta \geq 0.99}$$

However, it is not sufficient to consider optimality of player1's strategy only. Similarly, if player2 (column player) plays H in the current and subsequent periods, then her average discounted payoff is 1. If she chooses to deviate, then she receives a payoff 110 in the current period, but 0 in the subsequent periods. So, her average discounted payoff is $(1-\delta)110$. Then, by optimality condition,

$$1 \geq (1 - \delta)110$$

$$\frac{1}{110} \geq 1 - \delta$$

$$\delta \geq \frac{109}{110} \approx 0.9919$$

Hence, we need to take 0.9919 as the cutoff point (otherwise, player2 chooses to deviate). Therefore, Grim-trigger strategy forms a SPNE if $\delta \geq 0.9919$.

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