Mechanism Design

An "auction" problem

- Single seller has a single item
- Seller does not value item
- Two buyers with independent valuations

$0 \leq v^l < v^h$ low and high valuations

$\pi^l + \pi^h = 1$ probabilities of low and high valuations

In the Auction Environment

q^l, q^h probability of getting item when low and high

p^h, p^l expected payment when low and high *individual rationality constraint*

(IR) $q^i v^i - p^i \geq 0$

- if you announce truthfully, you get at least the utility from not playing the game

incentive compatibility constraint

(IC) $q^i v^i - p^i \geq q^{-i} v^i - p^{-i}$

- you gain no benefit from lying about your type

probability constraints

(1) $0 \leq q^i \leq \pi^{-i} + \pi^i / 2$

(2) $\pi^l q^l + \pi^h q^h \leq 1/2$

Seller Problem

Maximize seller utility $U = \pi^l p^l + \pi^h p^h$

Subject to IC and IR

Guess that:

IR binds for low value

$q^l v^l - p^l = 0$

IC binds for high value

$q^h v^h - p^h = q^l v^h - p^l$

The solution

$p^l = q^l v^l$ from low IR

substitute into high IC

$p^h = (q^h - q^l)v^h + q^l v^l$

plug into utility of seller

$$U = \pi^l q^l v^l + \pi^h \left((q^h - q^l)v^h + q^l v^l\right)$$

$$U = q^l(\pi^l v^l - \pi^h v^h + \pi^h v^l) + \pi^h q^h v^h$$

$\pi^l + \pi^h = 1$ so

$$U = q^l(v^l - \pi^h v^h) + \pi^h q^h v^h$$

Case 1: $v^l > \pi^h v^h$

$U = q^l (v^l - \pi^h v^h) + \pi^h q^h v^h$

(1) $0 \le q^i \le \pi^{-i} + \pi^i / 2$

(2) $\pi^l q^l + \pi^h q^h \le 1/2$

Make q^l, q^h large as possible so

$\pi^l q^l + \pi^h q^h = 1/2$

$$U = \frac{1/2 - \pi^h q^h}{\pi^l}(v^l - \pi^h v^h) + \pi^h q^h v^h$$

$$U = \frac{1}{2\pi^l}(v^l - \pi^h v^h) + q^h \frac{\pi^h}{\pi^l}(v^h - v^l)$$

so q^h should be as large as possible

$$q^h = \pi^l + \pi^h / 2$$

plug back into (2) to find

$$q^l = \pi^l / 2$$

expected payments

$$p^l = q^l v^l,\ p^h = (q^h - q^l)v^h + q^l v^l$$

$$p^l = v^l \pi^l / 2$$

$$p^h = v^h / 2 + \pi^l v^l / 2$$

Implementation of Case 1

modified auction: each player announces their value

the highest announced value wins

if there is a tie, flip a coin; if the low value wins, he pays his value

if the high value wins he pays

$$\frac{p^h}{q^h} = \frac{v^h/2 + \pi^l v^l/2}{\pi^l + \pi^h/2}$$

under these rules

probability that high type wins is $q^h = \pi^l + \pi^h/2$

probability that low type wins is $q^l = \pi^l/2$

just as in the optimal mechanism, this means the expected payments are the same too

Case 2: $v^l < \pi^h v^h$

$U = q^l (v^l - \pi^h v^h) + \pi^h q^h v^h$

(1) $0 \le q^i \le \pi^{-i} + \pi^i / 2$

(2) $\pi^l q^l + \pi^h q^h \le 1/2$

Make q^h large as possible, q^l as small as possible

$q^h = \pi^l + \pi^h / 2$

$q^l = 0$

expected payments

$$p^l = q^l v^l,\ p^h = (q^h - q^l)v^h + q^l v^l$$

$$p^l = 0$$

$$p^h = (\pi^l + \pi^h / 2)v^h$$

Implementation of Case 2

set a fixed price equal to the highest valuation

$$v^h = \frac{p^h}{q^h} = \frac{(\pi^l + \pi^h / 2)v^h}{\pi^l + \pi^h / 2}$$

Other Applications of Mechanism Design

- general equilibrium theory
- public goods
- taxation
- price discrimination