FINAL EXAM SOLUTIONS

1. (a) In order to determine pure strategy Nash equilibria (PSNE) of this game, best responses are underlined on the payoff matrix which is given below;

	L	R
U	$\underline{20}, 40$	$0, \underline{60}$
D	$10, \underline{50}$	$\underline{30}, 10$

Hence, there is no PSNE.

(b) Let p be the probability of playing U for row player, and then obviously $1-p$ be the probability of playing D (i.e,, $\sigma_1 = (p, 1-p)$). So, column player randomizes if the expected payoff of playing L and R are the same. Thus,

$$u_2(\sigma_1, L) = u_2(\sigma_1, R)$$

$$(p)(40) + (1-p)(50) = (p)(60) + (1-p)(10)$$

$$(1-p)40 = 20p$$

$$\boxed{p = 2/3}$$

Similarly, let q be the probability of playing L for column player, then obviously $1-q$ be the probability of playing R (i.e,, $\sigma_2 = (q, 1-q)$). So, row player randomizes if the expected payoff of playing U and D are the same. Thus,

$$u_1(U, \sigma_2) = u_1(D, \sigma_2)$$

$$(q)(20) + (1-q)(0) = (q)(10) + (1-q)(30)$$

$$10q = (1-q)30$$

$$\boxed{q = 3/4}$$

Hence, $((\frac{2}{3}, \frac{1}{3}), (\frac{3}{4}, \frac{1}{4}))$ is the MSNE.

2. (a) Let's start determining the events in this problem. Let *wet* be the event that oil is truly present in the field, let *positive* be the event that geological survey finds the bed wet. Moreover, let's denote the complements of events *wet* and *positive* as *dry* and *negative*, respectively (i.e $P(wet) + P(dry) = 1$). We are given the following probabilities

$$P(wet) = 0.5$$

$$P(negative|wet) = 0.1$$

$$P(positive|dry) = 0.3$$

i. We need to determine $P(wet|positive)$. By using Baye's Law, we obtain

$$\begin{aligned} P(wet|positive) &= \frac{P(positive|wet)P(wet)}{P(positive|wet)P(wet) + P(positive|dry)P(dry)} \\ &= \frac{(0.9)(0.5)}{(0.9)(0.5) + (0.3)(0.5)} \\ &= 0.75 \end{aligned}$$

ii. The expected payoff from choosing to drill is

$$\begin{aligned} Eu(drill) &= [P(wet|positive)(200) + P(dry|positive)(0)] - 50 \\ &= (0.75)(200) - 50 \\ &= 100 \end{aligned}$$

If GP decides to not drill, then the payoff would be 0 which is less than the expected payoff from choosing to drill. Thus, GP will choose to drill.

iii. Now, we need to determine $P(wet|negative)$. Again, by using Baye's Law, we obtain

$$\begin{aligned} P(wet|negative) &= \frac{P(negative|wet)P(wet)}{P(negative|wet)P(wet) + P(negative|dry)P(dry)} \\ &= \frac{(0.1)(0.5)}{(0.1)(0.5) + (0.7)(0.5)} \\ &= 0.125 \end{aligned}$$

Now, the expected payoff from choosing to drill is

$$\begin{aligned} Eu(drill) &= [P(wet|negative)(200) + P(dry|negative)(0)] - 50 \\ &= (0.125)(200) - 50 \\ &= -25 \end{aligned}$$

Hence, GP will choose to not drill since drilling yields a negative expected payoff.

iv. Here, we need to determine $P(positive)$. Since dry is the complement of wet and intersection of $positive$ and wet and intersection of $positive$ and dry are disjoint, then we obtain

$$\begin{aligned} P(positive) &= P(positive \cap wet) + P(positive \cap dry) \\ &= P(positive|wet)P(wet) + P(positive|dry)P(dry) \\ &= (0.9)(0.5) + (0.3)(0.5) \\ &= 0.6 \end{aligned}$$

(b) From part (ii) we know that if the signal is positive, GP will choose to drill and the expected payoff would be 100, and if the signal is negative, will not drill and the payoff would be 0. Thus, the expected payoff from using the survey is

$$\begin{aligned} Eu(survey) &= P(positive)Eu(drill) + P(negative)Eu(notdrill) \\ &= (0.6)(100) + (0.4)(0) \\ &= 60 \end{aligned}$$

(c) In this case, by drilling GP will find oil with probability 1/2 so that payoff would be (200-50) and will find nothing with probability 1/2 so that the payoff would be (0-50). Hence, the expected payoff from making the drilling decision without using the survey is

$$\begin{aligned} Eu(drill + nosurvey) &= P(wet)(200 - 50) + P(dry)(0 - 50) \\ &= \frac{1}{2}(150) + \frac{1}{2}(-50) \\ &= 50 \end{aligned}$$

(d) Since using survey increases the expected payoff by 10, then GP would be willing to pay any price less than or equal to 10 in order to use the survey.

3. (a) Incentive compatibility constraint guarantees that competent type has no incentive to report that he is incompetent. In other words, expected utility from telling the truth for the competent manager must be greater than equal to the expected utility from telling that he is incompetent. Let's denote announcing competent type by c and incompetent type by i, i.e. $t \in \{c, i\}$. So, if competent manager reveals the truth, his expected payoff would be

$$\begin{aligned} Eu(w_0, w_1; c) &= P(profit|c)u(w_1) + P(failure|c)u(w_0) \\ &= \frac{3}{4}(w_1 - \frac{w_1^2}{2}) + \frac{1}{4}(w_0 - \frac{w_0^2}{2}) \end{aligned}$$

However, if competent manager reports that he is incompetent, he will receive 1/4 as the wage regardless of output. Thus, the utility from getting 1/4 is

$$\begin{aligned} u(1/4) &= \frac{1}{4} - \frac{(1/4)^2}{2} \\ &= \frac{7}{32} \end{aligned}$$

Hence, the incentive compatibility constraint is

$$\frac{3}{4}(w_1 - \frac{w_1^2}{2}) + \frac{1}{4}(w_0 - \frac{w_0^2}{2}) \geq \frac{7}{32}$$

(b) Individual rationality constraint requires that each type must participate voluntarily. It means participation must yield an expected payoff greater than or equal to the payoff that is obtained by outside option. If competent manager participates by telling the truth, his expected payoff would be

$$Eu(w_0, w_1; c) = \frac{3}{4}(w_1 - \frac{w_1^2}{2}) + \frac{1}{4}(w_0 - \frac{w_0^2}{2})$$

Since the outside option is worth 1/4 as income (wage) and this corresponds to a utility of 7/32 (see above), then individual rationality constraint will be

$$\frac{3}{4}(w_1 - \frac{w_1^2}{2}) + \frac{1}{4}(w_0 - \frac{w_0^2}{2}) \geq \frac{7}{32}$$

(c) We assumed that the incompetent manager cannot pretend to be competent. So, if the manager is incompetent and decides to work (participates) then he reveals his type truly implying that his payoff would be 7/32 (see above). Hence, by offering this contract firm will take away all the surplus so that profit is maximized. However, if the manager is competent and participates truly (IR and IC constraints are satisfied), then the expected profit of the firm is

$$\begin{aligned} \Pi(w_0, w_1; c) &= P(profit|c)(1 - w_1) + P(failure|c)(0 - w_0) \\ &= (\frac{3}{4})(1 - w_1) + (\frac{1}{4})(-w_0) \\ &= \frac{3}{4} - \frac{3w_1}{4} - \frac{w_0}{4} \qquad (1) \end{aligned}$$

Since a contract specifies wages w_0 and w_1 for a competent manager, they are determined as a solution to an optimal contract problem given as follows;

$$\max_{w_0, w_1} \quad \tfrac{3}{4} - \tfrac{3w_1}{4} - \tfrac{w_0}{4}$$

$$\begin{aligned} \text{s.t} \quad & \tfrac{3}{4}(w_1 - \tfrac{w_1^2}{2}) + \tfrac{1}{4}(w_0 - \tfrac{w_0^2}{2}) \geq \tfrac{7}{32} \quad \text{(IR)} \\ & \tfrac{3}{4}(w_1 - \tfrac{w_1^2}{2}) + \tfrac{1}{4}(w_0 - \tfrac{w_0^2}{2}) \geq \tfrac{7}{32} \quad \text{(IC)} \\ & w_0, w_1 \in [0, 1] \end{aligned}$$

Since pretending an incompetent manager is worth the same as outside option, IR and IC constraints are the same and bind. Then, the Lagrangian is

$$\mathcal{L}(w_0, w_1) = \frac{3}{4} - \frac{3w_1}{4} - \frac{w_0}{4} - \lambda[\frac{7}{32} - \frac{3}{4}(w_1 - \frac{w_1^2}{2}) - \frac{1}{4}(w_0 - \frac{w_0^2}{2})]$$

If we take the partial derivatives of $\mathcal{L}$ with respect to w_0 and w_1 respectively and set them equal to 0, we obtain first order conditions (FOC);

$$-\frac{1}{4} - \frac{\lambda}{4} + \frac{\lambda}{4}w_0 = 0 \tag{2}$$

$$-\frac{3}{4} - \frac{3\lambda}{4} + \frac{3\lambda}{4}w_1 = 0 \tag{3}$$

If we multiply equation (2) by -3 and add to equation (3) side by side, we obtain

$$\frac{3\lambda}{4}(w_1 - w_0) = 0$$

implying that

$$\boxed{w_0 = w_1}$$

The interpretation of this result is that competent manager also must be offered a flat wage regardless of output. Now, by using the constraint, we can solve for $w^* = w_0 = w_1$.

$$\frac{3}{4}(w^* - \frac{(w^*)^2}{2}) + \frac{1}{4}(w^* - \frac{(w^*)^2}{2}) = \frac{7}{32}$$

$$w^* - \frac{(w^*)^2}{2} = \frac{7}{32}$$

$$\frac{(w^*)^2}{2} - w^* + \frac{7}{32} = 0$$

$$\frac{16(w^*)^2 - 32w^* + 7}{32} = 0$$

$$16(w^*)^2 - 32w^* + 7 = 0$$

$$(4w^* - 7)(4w^* - 1) = 0$$

Since $w^* \in [0, 1]$, then the solution is

$$\boxed{w^* = \frac{1}{4}}$$

Therefore, competent manager should be offered 1/4 regardless of output.