Econ 4011 04/23/2012
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PROBLEM SET #7 SOLUTIONS

Mechanism Design

1. (a) The objective function of the board of directors is the expected profit of the firm when CEO performs a high effort, and it is given as follows;

$$\begin{aligned}\Pi(w_H, e_H) &= P(success|e_H)(4v) + P(bankrupcy|e_H)(0) - w_H \\ &= (\frac{3}{4})(4v) + (\frac{1}{4})(0) - w_H \\ &= 3v - w_H \qquad (1)\end{aligned}$$

where w_H is the wage that is paid in case of high effort. Similarly, the objective function of the board of directors when the effort is low is

$$\begin{aligned}\Pi(w_L, e_L) &= P(success|e_L)(4v) + P(bankrupcy|e_L)(0) - w_L \\ &= (\frac{1}{4})(4v) + (\frac{3}{4})(0) - w_L \\ &= v - w_L \qquad (2)\end{aligned}$$

where w_L is the wage that is paid in case of low effort.

(b) Assume $w_L = 0$. Now, CEO provides high effort if the utility that he gains providing high effort exceeds the utility that he gains providing low effort. Formally,

$$u(w_H, e_H) \geq u(w_L, e_L)$$

Since $w_L = 0$, then $u(0, e_L) = log(1 + 0) = 0$. Thus, if we rearrange the above inequality, we obtain

$$log(1 + w_H) - log3 \geq 0$$

$$1 + w_H \geq 3$$

$$w_H \geq 2$$

Hence, if board wants CEO to provide high effort, then it must pay at least 2. In fact, board will pay exactly 2 since CEO's wage is the cost in objective function and must be chosen as small as possible.

(c) Board wants CEO to provide high effort if the net expected profit under high effort is greater than the net expected profit under low effort. Formally,

$$\Pi(w_H, e_H) \geq \Pi(w_L, e_L)$$

Since w_H is chosen 2, then by rearranging the above inequality we obtain

$$3v - 2 \geq v$$

$$v \geq 1$$

Hence, if v is greater than or equal to 1, board prefer to induce the CEO to provide high effort.

(d) In this case, board cannot observe the CEO's effort. However, two different objection functions can be written depending on the effort of CEO; when the effort is low, with probability 1/4 the firm will be successful and the profit of the firm is $4v - w_v$, but with probability 3/4 the firm will go bankrupt and the profit of the firm is $0 - w_0$. Thus, the objective function when the effort is low is

$$\begin{aligned}\Pi(w_v, w_0, e_L) &= P(success|e_L)(4v - w_v) + P(bankrupcy|e_L)(0 - w_0) \\ &= (\frac{1}{4})(4v - w_v) + (\frac{3}{4})(-w_0) \\ &= v - \frac{w_v}{4} - \frac{3w_0}{4} \qquad (3)\end{aligned}$$

Similarly, the objection function under the high effort is

$$\begin{aligned}\Pi(w_v, w_0, e_H) &= P(success|e_H)(4v - w_v) + P(bankrupcy|e_H)(0 - w_0) \\ &= (\frac{3}{4})(4v - w_v) + (\frac{1}{4})(-w_0) \\ &= 3v - \frac{3w_v}{4} - \frac{w_0}{4} \qquad (4)\end{aligned}$$

(e) Assume $w_0 = 0$. CEO only provides high effort if the expected utility of providing high effort exceeds the expected utility of providing low effort. Formally,

$$\begin{aligned}Eu(w, e_H) &\geq Eu(w, e_L) \\ P(success|e_H)u(w_v, e_H) + P(bankrupcy|e_H)u(w_0, e_H) &\geq P(success|e_L)u(w_v, e_L) \\ &\quad + P(bankrupcy|e_L)u(w_0, e_L) \\ \frac{3}{4}[log(1 + w_v) - log3] + \frac{1}{4}[log(1 + w_0) - log3] &\geq \frac{1}{4}[log(1 + w_v)] + \frac{3}{4}[log(1 + w_0)] \\ \frac{1}{2}log(1 + w_v) - log3 &\geq 0 \\ log(1 + w_v) &\geq 2log3 \\ log(1 + w_v) &\geq log9 \\ w_v &\geq 8\end{aligned}$$

Hence, if board wants CEO to provide high effort, then it must pay at least 8. In fact, board will pay exactly 8 since CEO's wage is the cost in objective function and must be chosen as small as possible.

(f) Board wants CEO to provide high effort if the net expected profit under high effort (equation 4) is greater than the net expected profit under low effort (equation 3). Formally,

$$\Pi(w_v, w_0, e_H) \geq \Pi(w_v, w_0, e_L)$$

Since $w_v = 8$ and $w_0 = 0$, then by rearranging the above inequality we obtain

$$3v - \frac{24}{4} \geq v - \frac{8}{4}$$

$$2v \geq 4$$

$$v \geq 2$$

Hence, if v is greater than or equal to 2, board prefer to induce the CEO to provide high effort.

2. There are two possible types for a consumer. The low-type's valuation is 1 and high type's valuation is 3 per unit of the good. Since the seller is only able to sell 1 unit or 2 units of the good, then the seller's problem is organized as a mechanism design problem as follows;

$$\begin{array}{lll} \max_{x^h, x^l} & \frac{1}{2}p^h x^h + \frac{1}{2}p^l x^l & \\ & & \\ \text{s.t} & (1 - p^i)x^i \geq 0 & \text{(IR)} \\ & (v^i - p^i)x^i \geq (v^i - p^{-i})x^{-i} & \text{(IC)} \\ & x^i \in \{1, 2\} & \end{array}$$

where $i \in \{h, l\}$. Notice that there are 2 IR constraints and 2 IC constraints in the above problem. We guess that individual rationality constraint of low type and incentive constraint of high type is binding. i.e.,

$$(v^l - p^l)x^l = 0 \tag{5}$$

$$(v^h - p^h)x^h = (v^h - p^l)x^l \tag{6}$$

Since $v^l = 1$ and $x^l \in \{1, 2\}$, then from equation (5) we obtain

$$\boxed{p^l = 1} \tag{7}$$

By combining equation (6) and (7), and given that $v^h = 3$, we obtain

$$\begin{aligned} (3-p^h)x^h &= (3-1)x^l \\ 3-p^h &= \frac{2x^l}{x^h} \\ p^h &= 3-\frac{2x^l}{x^h} \end{aligned} \tag{8}$$

If we pluq equations (7) and (8) into the objective function, we obtain

$$\max_{x^h,x^l} \quad \tfrac{1}{2}(3-\tfrac{2x^l}{x^h})x^h + \tfrac{1}{2}x^l$$

$$\text{s.t.} \qquad x^l, x^h \in \{1,2\}$$

By rearranging the terms in objective function, we obtain

$$\max_{x^h,x^l} \quad \tfrac{3}{2}x^h - \tfrac{1}{2}x^l$$

$$\text{s.t.} \qquad x^l, x^h \in \{1,2\}$$

Hence, seller chooses x^l as low as possible which is equal to 1 and chooses x^h as high as possible which is equal to 2. Therefore, seller offers two options;
(i) one unit of good for a price 1
(ii) two units of good for a price 2 each.