

**Econ 504 (2008)  
Microeconomics II**

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**Auctions II**

# **Interdependent Values**

**Milgrom & Weber (1982)**

## Aside: Affiliation

For any  $z = (z_1, \dots, z_n), z' = (z'_1, \dots, z'_n) \in \mathbb{R}^n$ , define

$$z \vee z' = (\max\{z_1, z'_1\}, \dots, \max\{z_n, z'_n\})$$

$$z \wedge z' = (\min\{z_1, z'_1\}, \dots, \min\{z_n, z'_n\})$$

Let  $X_1, \dots, X_n$  be random variables with joint density  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ .  $X_1, \dots, X_n$  are **affiliated** if:

$$f(z \vee z')f(z \wedge z') \geq f(z)f(z')$$

for all  $z, z' \in \mathbb{R}^n$ .

**Theorem:** Assume that  $X_1, \dots, X_n$  have a joint density  $f$  and  $X_1, \dots, X_n$  are affiliated. Then:

1.  $X_1, \dots, X_{n-1}$  are affiliated.
2. If  $g_i : \mathbb{R} \rightarrow \mathbb{R}$  is nondecreasing for  $i = 1, \dots, n$ , then  $g_1(X_1), \dots, g_n(X_n)$  are affiliated.
3. If  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  is nondecreasing and  $1 \leq k \leq n$ , then:

$$\mathbb{E}[h(X_1, \dots, X_n) | X_1 = x_1, \dots, X_k = x_k]$$

is nondecreasing in  $(x_1, \dots, x_k)$ .

4. If additionally  $f$  is symmetric in its  $n$  arguments, then  $X_1, X_{-1}^1, \dots, X_{-1}^{n-1}$  are affiliated.

# The symmetric affiliated values model

$N = \{1, 2, \dots, n\}$  bidders, bidding for a single indivisible object.

$X_i$  is a random variable denoting bidder  $i$ 's private information / signal. A particular realization is denoted by  $x_i$ .  $X_1, \dots, X_n$  have joint density  $f$ .

**Notation:**  $Y_1 = X_{-1}^1 = \max_{i \neq 1} X_i$ ,  $g = f_{X_{-1}^1}$ ,  $G = F_{X_{-1}^1}$ .

## Valuations:

There is a function  $u : \mathbb{R}^n \rightarrow \mathbb{R}$  such that the valuation of  $i$  is given by the random variable  $V_i = u(X_i, X_{-i})$ .

Bidders are risk neutral.

# Assumptions:

1.  $X_1, \dots, X_n$  are affiliated
2. Density  $f$  is symmetric in its  $n$  arguments.
3.  $\underline{x} > -\infty$  where  $\underline{x} =$  the infimum of the support of  $X_i$ .
4.  $u$  is nonnegative, nondecreasing, strictly increasing in its first argument, symmetric in its last  $n-1$  arguments, and twice continuously differentiable.
5.  $\mathbb{E}[V_i] < \infty$ .

**Definition:**  $v(x, y) = \mathbb{E}[V_1 | X_1 = x, Y_1 = y]$ .

Note: By affiliation  $v$  is nondecreasing in  $x$  and  $y$ .

**Assumption 5.**  $v(x, y)$  is strictly increasing in  $x$ .

# The Second-Price Auction equilibrium

**Theorem:** *The bid function given by:*

$$b(x) = v(x, x) \equiv \mathbb{E}[V_1 | X_1 = x, Y_1 = x]$$

*yields a symmetric Bayesian Nash equilibrium of the second price auction.*

Notes:

This generalizes the earlier, independent private values equilibrium of the second-price auction.

However in the current model the above bid function need not give a dominant strategy equilibrium.

# The (Ascending) English Auction

1. The auctioneer sets a price of  $u(\underline{x}, \dots, \underline{x})$  and gradually raises it.
2. Bidders may drop out at any time. A bidder who drops out can not reenter the auction at a higher price.
3. At any point during the auction the history of bidders who have dropped out and the prices at which they dropped out is common knowledge.
4. Strategies specify at what price to drop out as a function of the history.
5. The auction is over after  $n - 1$  bidders drop out. The remaining bidder receives the object and pays the price at which the last bidder has dropped out.

## Remark

Since the model is symmetric, in a symmetric equilibrium the identities of the bidders who have dropped out is not informative.

In this case, it is sufficient to keep track of the number of bidders who have dropped out and at which prices.

# The (Ascending) English Auction equilibrium

A strategy is a vector of functions  $b = (b^0, \dots, b^{n-2})$  where  $b^k(x; p_1, \dots, p_k) \geq p_k$  denotes the price at which bidder with signal  $x$  will drop out given that exactly  $k$  people already dropped out at prices  $p_1 \leq p_2 \leq \dots \leq p_k$ .

Iteratively define the strategy  $b = (b^0, \dots, b^{n-2})$  by:

$$b^0(x) = u(x, x, \dots, x)$$

$$b^k(x; p_1, \dots, p_k) = u(x, x, \dots, x, x^{n-k+1}, \dots, x^n)$$

where  $x^l$  solves  $b^{n-l}(x^l)(p_1, \dots, p_{n-l}) = p_{n-l+1}$  for  $l = n - k + 1, \dots, n$ .

**Theorem:** *The strategy  $b = (b^0, \dots, b^{n-2})$  above yields a symmetric equilibrium of the English auction.*

# The First-Price Auction equilibrium

Heuristic derivation.

**Theorem:** *The bid function given by*

$$b(x) = \int_{\underline{x}}^x v(y, y) dL(y|x)$$

where

$$L(y|x) = \exp\left(-\int_y^x \frac{g(\alpha|\alpha)}{G(\alpha|\alpha)} d\alpha\right)$$

*yields a symmetric Bayesian Nash equilibrium of the first price auction.*

**Lemma 1:**  $\frac{g(y|x)}{G(y|x)}$  is nondecreasing in  $x$ .

**Lemma 2:** If  $x > \underline{x}$  then  $L(\cdot|x)$  above is a cumulative distribution function on  $[\underline{x}, x]$ .

# Revenue Comparisons

**Theorem:** *The expected revenue from the English auction is at least as much as the expected revenue from the second-price auction.*

**Theorem:** *The expected revenue from the second-price auction is at least as much as the expected revenue from the first-price auction.*