Econ 504 (2008)
Microeconomics II

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Game Theory - Basics I

Normal Form Games

Definition

A **normal form game** is a triplet $(N, A = \prod_{i \in N} A_i, u = (u_i)_{i \in N})$:

- $N = \{1, \ldots, n\}$ is a finite set of players.

- A_i is the set of actions (pure strategies) of player i.

- $u_i \colon A \to \mathbb{R}$ is player i's expected utility index over action profiles.

A normal form game is **finite** if A is finite.

The game is common knowledge among players.

Mixed Strategies

$\Delta(X)$: the set of **probability distributions** over a set X.

$\Delta(A_i)$: **mixed strategies** of player i. (deliberate randomization by i, j's belief about i's play, steady state population proportions, pure strategies in a perturbed game)

A mixed strategy profile can be:

independent: $\sigma = (\sigma_1, \ldots, \sigma_n) \in \Delta(A_1) \times \ldots \times \Delta(A_n)$,

or **correlated**: $\sigma \in \Delta(A)$

Payoffs are extended to mixed strategies by taking expectations:

$$u_i(\sigma) = \mathbb{E}_\sigma u_i = \sum_{a \in A} \sigma(a) u_i(a).$$

Best Reply

Player i is **rational** if he maximizes his expected payoff subject to a belief about others' play.

Let $\sigma_{-i} \in \Delta\left(A_{-i}\right)$. a_i^* is a **pure best reply** to σ_{-i} if:

$$\forall a_i \in A_i : \quad u_i(a_i^*, \sigma_{-i}) \geq u_i(a_i, \sigma_{-i}).$$

σ_i^* is a **mixed best reply** of i to σ_{-i} if:

$$\forall \sigma_i \in \Delta(A_i) : \quad u_i(\sigma_i^*, \sigma_{-i}) \geq u_i(\sigma_i, \sigma_{-i}).$$

$B_i^p(\sigma_{-i})$: i's pure best replies to σ_{-i}.

$B_i(\sigma_{-i})$: i's mixed best replies to σ_{-i}.

Note: $B_i(\sigma_{-i}) = \Delta\left(B_i^p(\sigma_{-i})\right)$.

Strict Domination

σ_i' **strictly dominates** σ_i if:

$$\forall \sigma_{-i} \in \Delta(A_{-i}) : \quad u_i(\sigma_i', \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i}).$$

Note: Alternative definitions where quantifiers are changed to independently mixed strategy profiles σ_{-i}, or to action profiles a_{-i} are the same.

Theorem: In a finite normal form game, an action a_i^* is never a best reply to any (possibly correlated) conjecture σ_{-i} of i iff a_i^* is strictly dominated to a mixed strategy σ_i.

Separation: Suppose C and D are nonempty, convex, disjoint sets in $\mathbb{R}^m$, and C is closed. Then, $\exists r \in \mathbb{R}^m \setminus \{0\}$:

$$\forall x \in C, y \in cl(D): \quad r \cdot x \geq r \cdot y.$$

Proof of Thm: Suppose that a_i^* is not strictly dominated.

Let $A_{-i} = \{a_{-i}^k \,|\, k = 1, ..., m\}$, $u_i(\sigma_i, \cdot) = \left(u_i(\sigma_i, a_{-i}^k)\right)_{k=1}^m$,

$$C = \{u_i(a_i^*, \cdot) - u_i(\sigma_i, \cdot) \mid \sigma_i \in \Delta(A_i)\}.$$

Assumptions above are satisfied for C and $D = (-\infty, 0)^m$. So there is $r \in \mathbb{R}^m \setminus \{0\}$ as in above.

Verify $r \geq 0$. Let $\sigma_{-i}(a_{-i}^k) = r_k / \sum_{l=1}^m r_l$. For any σ_i:

$$u_i(a_i^*, \sigma_{-i}) - u_i(\sigma_i, \sigma_{-i}) = \left(\sum_{l=1}^m r_l\right)^{-1} r \cdot [u_i(a_i^*, \cdot) - u_i(\sigma_i, \cdot)] \geq 0.$$

Common Knowledge of Rationality and Iterated Elimination of Strictly Dominated Strategies

Iterated Elimination of Strictly Dominated Strategies:

Let $A^0 = A$. Inductively, let $A_i^{t+1} = B_i^p\left(\Delta\left(A_{-i}^t\right)\right)$.

$$A^{IESDS} = \bigcap_{t=0}^{\infty} A^t.$$

Proposition: The outcome of IESDS is independent of the order of elimination of the strictly dominated strategies. (See O&R section 4.2.2).

CK Rationality & IESDS Example 1: Simplified Price Competition

	High	Medium	Low
High	6,6	0,10	0,8
Medium	10, 0	5,5	0,8
Low	8,0	8,0	4,4

CK Rationality & IESDS Example 2: Cournot Duopoly

Two firms produce the same good at constant marginal cost $c \in (0, 1)$.

They choose quantities $q_i, q_j \in [0, 1]$.

The inverse demand in the market is given by

$$p(q_1, q_2) = \max\{0, 1 - (q_1 + q_2)\}.$$

Payoffs: $u_i(q_i, q_j) = (p(q_1, q_2) - c) \times q_i$, for $i \neq j$.

Best reply of: $q_i^*(q_j) = \max\{0, (1 - q_j - c)/2\}$.

Rationality + Common Knowledge Rationality $\Rightarrow$

What about the Cournot Oligopoly with 3 or more firms?

Nash Equilibrium

A strategy profile $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*) \in \Delta(A_1) \times \ldots \times \Delta(A_n)$ is a **Nash Equilibrium** (**NE**) if for any $i \in N$:

$$\forall \sigma_i \in \Delta(A_i): \quad u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*).$$

Note:

1. σ^* is a NE if $\sigma_i^* \in B_i(\sigma_{-i}^*)$ for any $i \in N$.
2. A finite normal form game may not have a pure strategy NE.
3. If σ^* is a NE then $supp(\sigma^*) \subset A^{IESDS}$.

Theorem *(Nash, 1950) Every finite normal form game has a Nash equilibrium in mixed strategies.*

Kakutani's Fixed Point Theorem: Let C be a convex, compact subset of of $\mathbb{R}^m$ and let $f\colon C \to C$ be a correspondence such that:

- For any $x \in C$, $f(x)$ is non-empty and convex
- f has a closed graph (is usc): for all sequences (x_k) and (y_k) in C such that $y_k \in f(x_k)$ for all k, $x_k \to x$, and $y_k \to y$, we have $y \in f(x)$.

Then there exists $x^* \in C$ such that $x^* \in f(x^*)$.

Proof Let $C = \Delta(A_1) \times \ldots \times \Delta(A_n)$ & define $B\colon C \to C$ by

$$B(\sigma) = B_1(\sigma_{-1}) \times \ldots \times B_i(\sigma_{-i}) \times \ldots \times B_n(\sigma_{-n}) \quad \sigma \in C.$$

C is convex and compact. B is non-empty, convex valued, and has a closed graph. Hence there is $\sigma^* \in B(\sigma^*)$. By the definition of B, σ^* is a NE. $\square$

Bayesian Games

Simultaneous Move Games with Payoff uncertainty

A **Bayesian Game** is a tuple (N, T, p, A, u) where:

- $N = \{1, \ldots, n\}$ is a finite set of players,
- $A = A_1 \times \ldots \times A_n$, where A_i is the set i's actions,
- $T = T_1 \times \ldots \times T_n$, where T_i is the set of i's types,
- $p = (p_i)_{i \in N}$, where $p_i(\cdot | t_i)$ is a probability distribution over T_{-i} (i's prior over others' types conditional on his),
- $u = (u_i)_{i \in N}$ where u_i: $A \times T \rightarrow \mathbb{R}$ is player i's vNM payoff function.

Examples

- **Auctions** Players are bidders, actions are bids, and types are valuations. Payoff functions are determined by the auction mechanism and bidders' risk preferences.

- **Double Auctions** Two players, one buyer and one seller. The buyer owns one unit of indivisible object. Types are valuations for the good. Actions of the buyer are bids, actions of the seller are asks. Payoff functions are determined by the double auction mechanism.

- **Cournot Oligopoly** Players are firms, actions are production levels, and types are firms' marginal costs. Supply levels and the market demand determines the payoffs.

  etc.

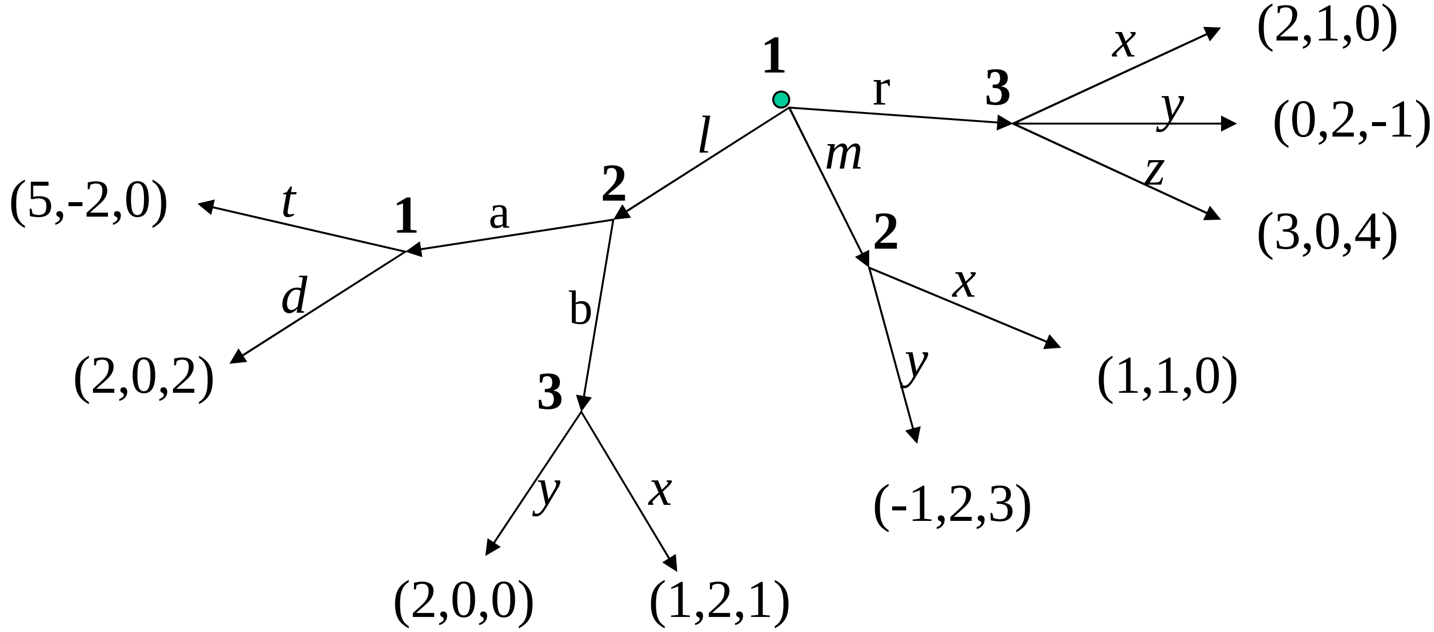
Bayesian Nash Equilibrium

Given a Bayesian Game, a pure strategy **Bayesian Nash Equilibrium** (**BNE**) is a profile $\mathbf{a}^* = (\mathbf{a}_i^*)_{i \in N}$ where $\mathbf{a}_i^* : T_i \to A_i$, s.t. for all $i \in N$, $t_i \in T_i$, and $a_i \in A_i$:

$$\int_{T_{-i}} u_i(\mathbf{a}^*(t); t) p_i(dt_{-i}|t_i) \geq \int_{T_{-i}} u_i(a_i, \mathbf{a}^*_{-i}(t_{-i}); t) p_i(dt_{-i}|t_i).$$

Extensive Form Games with Perfect Information

A perfect info. extensive form game



An **extensive form game with perfect information** is a tuple $\Gamma = (N, H, P, u)$ such that:

- *Players:* $N = \{1, \ldots, n\}$ is a finite set of players,
- *Histories:* H is a set of sequences satisfying:

  (i) the empty sequence $\emptyset$ is in H,

  (ii) if $(a^1, \ldots, a^k) \in H$ and $l < k$ then $(a^1, \ldots, a^l) \in H$,

  (iii) if $(a^1, \ldots, a^k) \in H$ for any k, then $(a^1, a^2, \ldots) \in H$.

  A history $h \in H$ is *terminal* if it is infinite or if there is no a s.t. $(h, a) \in H$. Z denotes terminal histories.

- *Player function:* a function P: $H \setminus Z \to N$.
- *Payoffs:* a vNM payoff function u_i: $Z \to \mathbb{R}$ for each i.

Extensive Form Strategies

A strategy of a player is a **complete contingent-plan**, determining which action he will take after each history he is to move (*including histories that will not be reached according to his own strategy!*)

Let $A(h) = \{a \mid (h, a) \in H\}$ for each $h \in H \setminus Z$.

A **pure strategy** of player i is a function s_i that associates an action in $A(h)$ to each history h for which $P(h) = i$.

A **behavioral strategy** β_i of player i associates a probability distribution in $\Delta(A(h))$ to each history h for which $P(h) = i$.

Nash Equilibrium

Given $\Gamma = (N, H, P, u)$, a profile of behavioral strategies β and a non-terminal history h, let

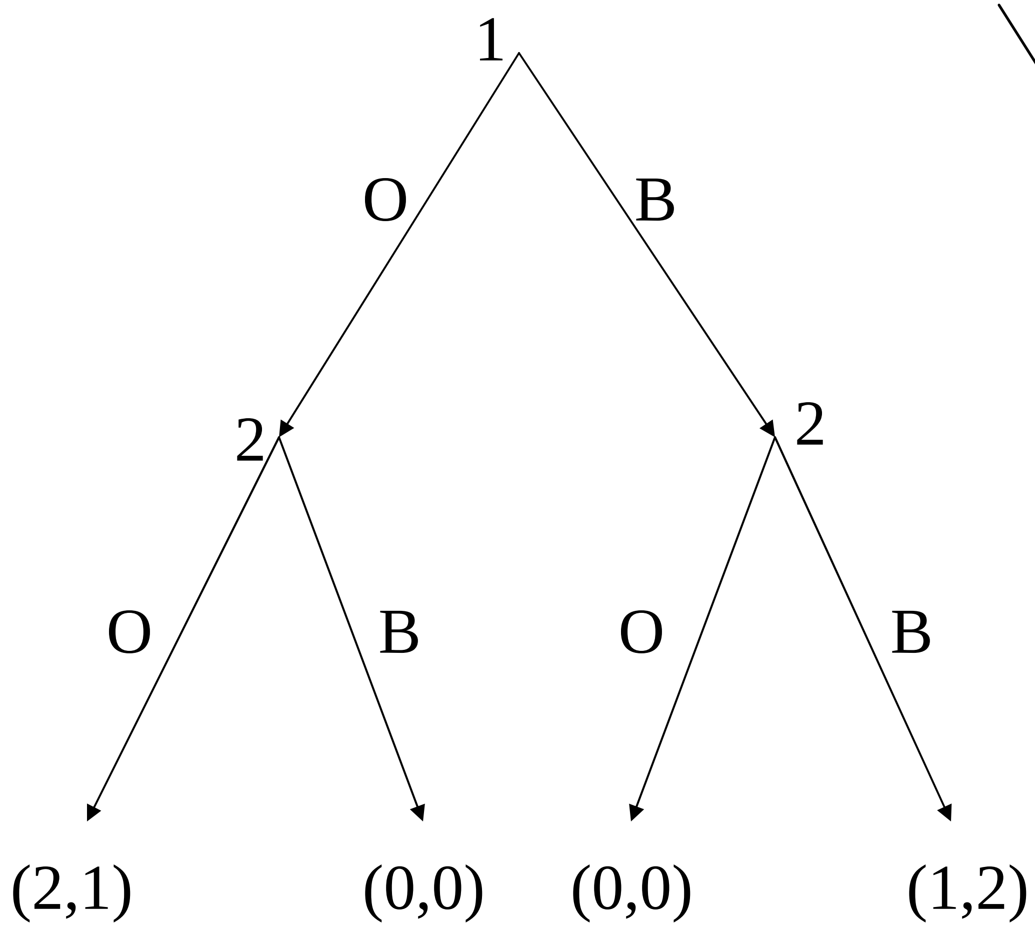
$$\mathcal{O}(\beta|h)$$

denote the distribution over Z induced by β starting at history h. Let $\mathcal{O}(\beta) := \mathcal{O}(\beta|\emptyset)$. (see O&R p213)

A profile of behavioral strategies β^* is a **Nash Equilibrium** if for any i and β_i:

$$u_i(\mathcal{O}(\beta^*)) \geq u_i(\mathcal{O}(\beta_i, \beta^*_{-i})).$$

Battle of the Sexes with perfect information

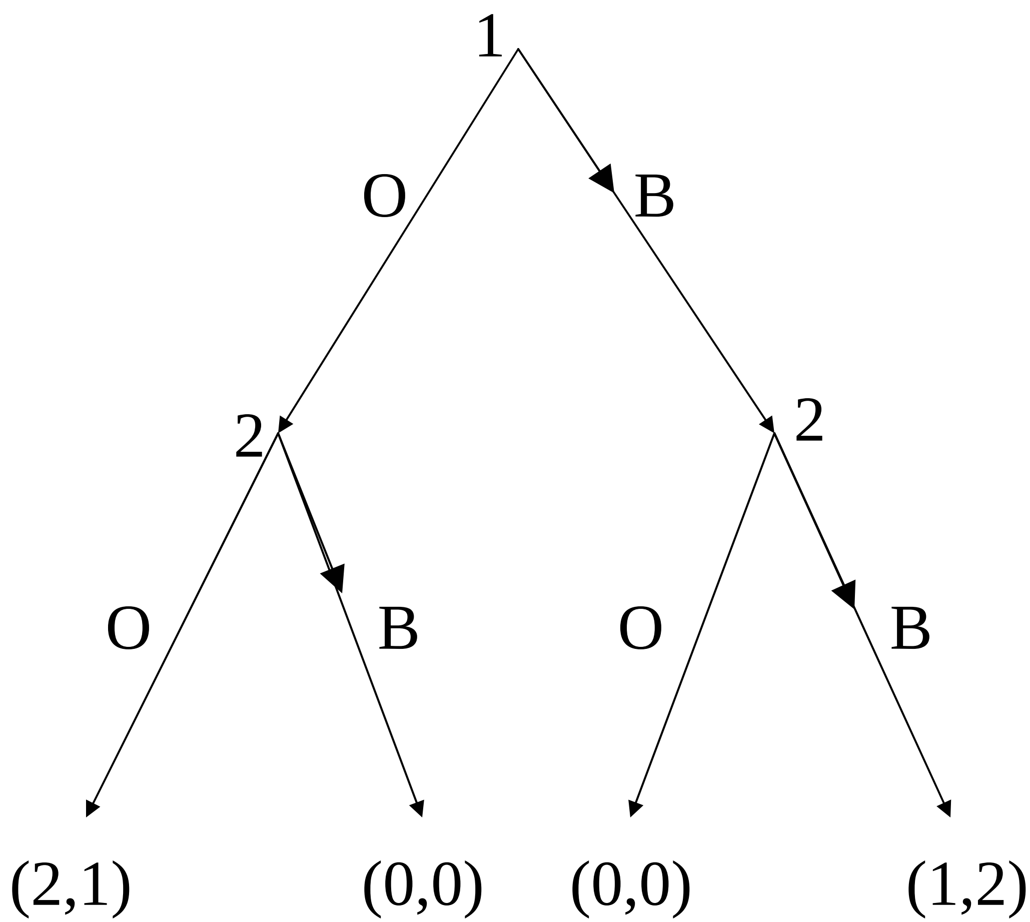


Normal Form (XY=[X|O,Y|B]) :

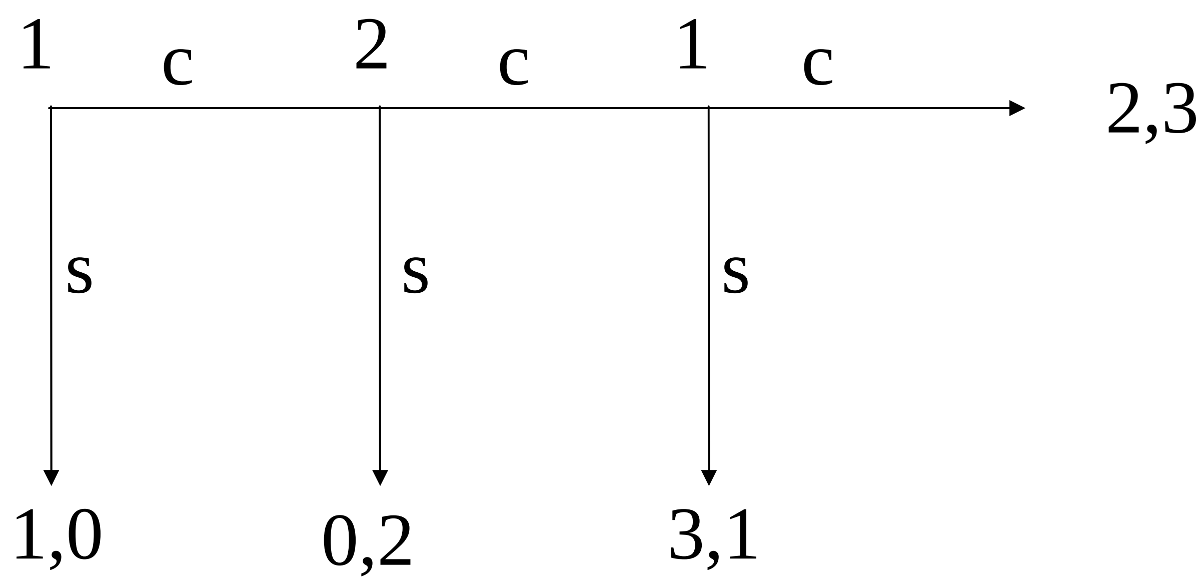
	OO	OB	BO	BB
O	2,1	2,1	0,0	0,0
B	0,0	1,2	0,0	1,2

Nash Equilibria?

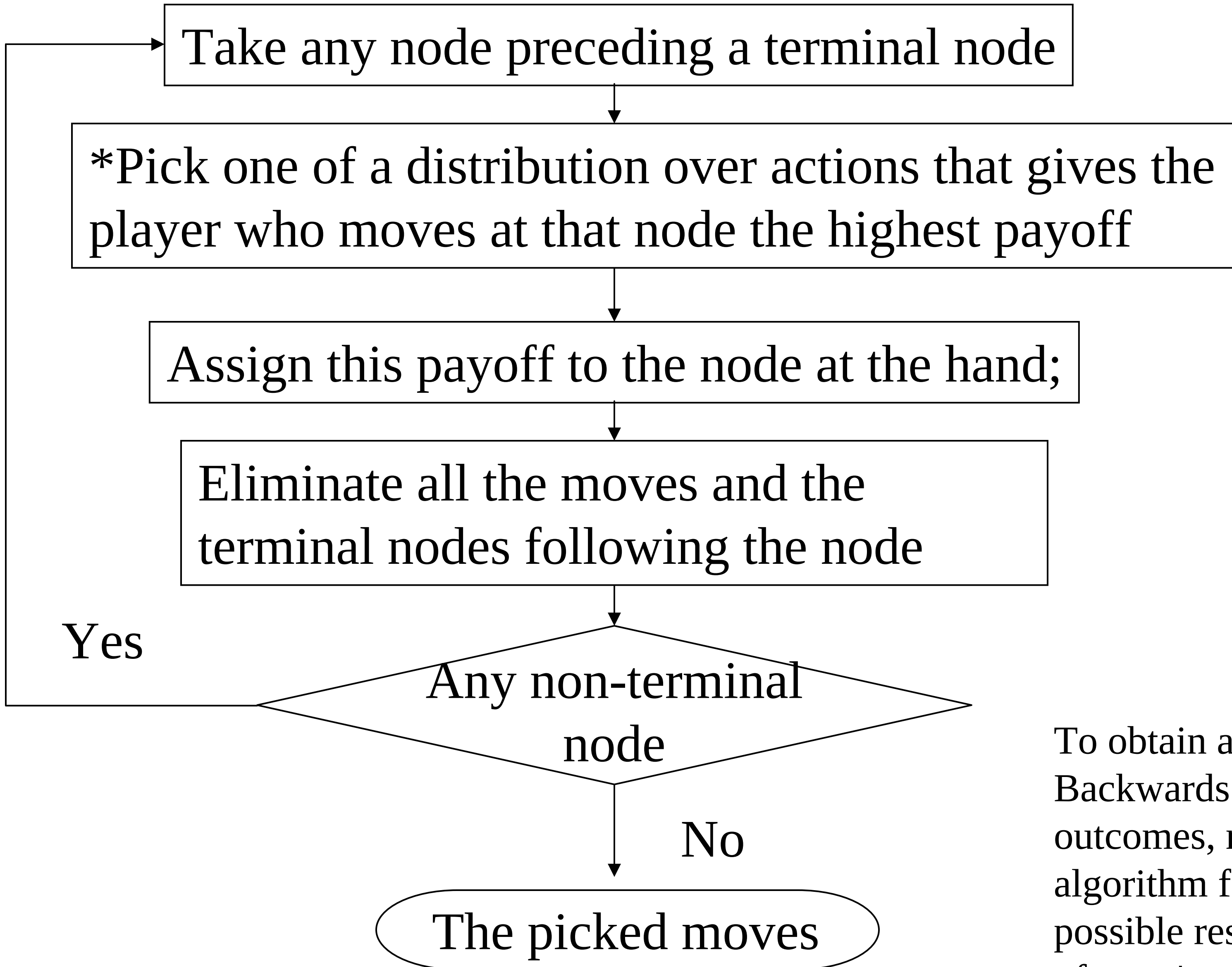
What is wrong with this NE?



Sequential Rationality & Backwards Induction



Backwards Induction (for finite & perfect info. games)



To obtain all Backwards Induction outcomes, repeat this algorithm for all possible resolutions of step *.

Subgame Perfection

A profile of behavioral strategies β^* is a **Subgame Perfect (Nash) Equilibrium (SPE)** if for any $h \in H \setminus Z$, $i \in N$, and β_i:

$$u_i(\mathcal{O}(\beta^*|h)) \geq u_i(\mathcal{O}(\beta_i, \beta^*_{-i}|h)).$$

Definition: Γ is **finite** if H is finite. Γ has **finite horizon** if the maximum length of histories in H is finite.

Theorem: In finite horizon perfect information games, backwards induction outcomes correspond exactly to subgame perfect equilibria.

Single-Deviation Principle

An perfect info. extensive-form game is **continuous at infinity** if, given any $\epsilon > 0$, there exists some t such that, for any two terminal histories that agree in the first t actions, the payoff difference of each player is less than ϵ.

Theorem: Assume that Γ is continuous at infinity. Then β is a SPE iff for any nonterminal history h, player $i = P(h)$ cannot increase his conditional payoff at h by deviating from his strategy only at h, i.e. for any β'_i that agrees with β_i except on h:

$$u_i(\mathcal{O}(\beta_i, \beta_{-i}|h)) \geq u_i(\mathcal{O}(\beta'_i, \beta_{-i}|h)).$$

Alternating Offers Bargaining Infinite Horizon (Rubinstein, 1982)

The Model

- Two players $N = \{1,2\}$ bargain over sharing a dollar over time.
- A division where player 1 receives x and player 2 receives y, is denoted by (x,y) where $x+y=1$, $x\geq 0$, $y\geq 0$.
- Both players are impatient. They have a discount factor $\delta\in(0,1)$. If they agree at time t about a division (x_t,y_t), then the utility of each player is:

$$U_1(x_t,t) = \delta^{t-1} x_t \quad \text{and} \quad U_2(y_t,t) = \delta^{t-1} y_t$$

- If they never reach an agreement, then they both receive 0.

Alternating offers bargaining model infinite time horizon

$T = \{1,2,\ldots\}$

If t is odd,

- Player 1 offers some (x_t,y_t),
- Player 2 Accepts or Rejects the offer
- If the offer is Accepted, the game ends yielding $(\delta^{t-1}x_t, \delta^{t-1}y_t)$,
- Otherwise, we proceed to date t+1.

If t is even

- Player 2 offers some (x_t,y_t),
- Player 1 Accepts or Rejects the offer
- If the offer is Accepted, the game ends yielding payoff $(\delta^{t-1}x_t, \delta^{t-1}y_t)$,
- Otherwise, we proceed to date t+1.

SPE of ∞-period bargaining

Theorem: At any t, proposer offers the other player $\delta/(1+\delta)$, keeping himself $1/(1+\delta)$, while the other player accepts an offer iff she gets at least $\delta/(1+\delta)$.

Proof: Single-deviation principle: Take any t, at which i offers, j accepts/rejects. According to the strategies in the continuation game, at t+1, j will get $1/(1+\delta)$. Hence, it is a best response for j to accept an offer iff she gets at least $\delta/(1+\delta)$. Given this, i must offer $\delta/(1+\delta)$.

Uniqueness of SPE

Uniqueness of SPE payoffs:

M=supremum SPE payoff of a player in a subgame where she starts making an offer.

m=infimum SPE payoff of a player in a subgame where she starts making an offer.