Econ 504 (2008)
Microeconomics II

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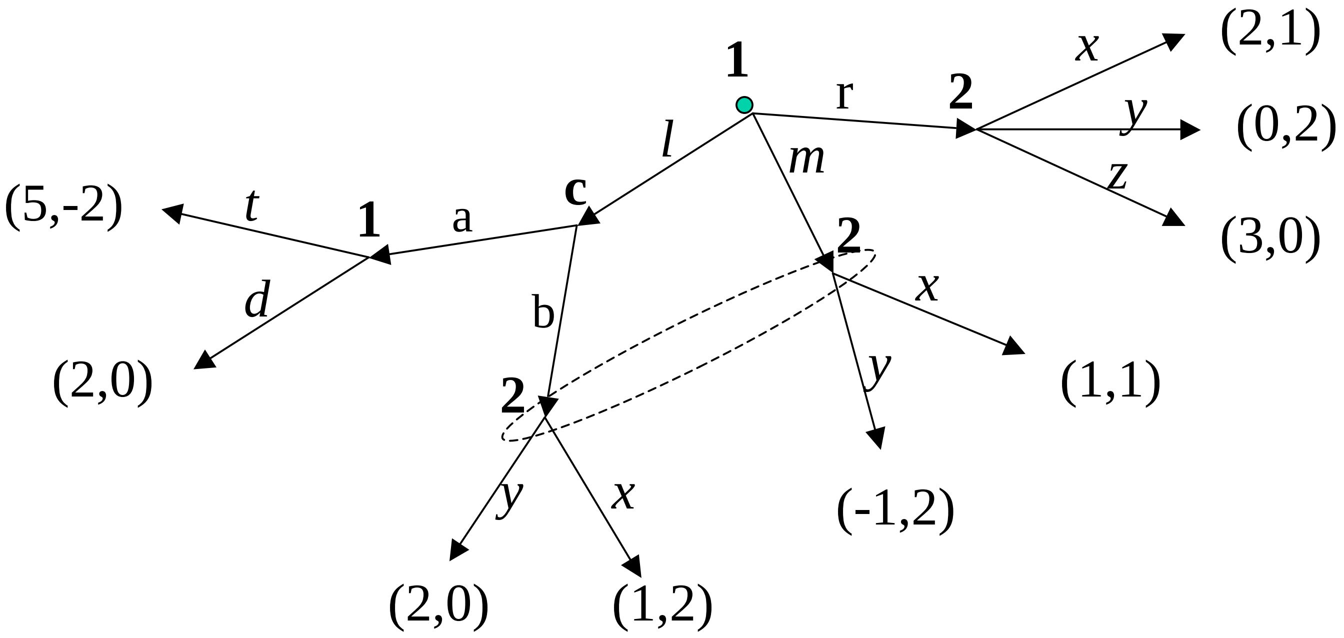
Game Theory - Basics II

General Extensive Form Games and Sequential Equilibrium

An extensive form game



$f_c(a|(l))=f_c(b|(l))=1/2,$

$I_1=\{\{\emptyset\},\{(l,a)\}\}, I_2=\{\{r\},\{(m),(l,b)\}\},\ldots$

An **extensive form game** $\Gamma = (N, H, \mathcal{I}, u, P, f_c)$ consists of

- *Players:* a finite set of players $N = \{1, \ldots, n\}$,
- *Histories:* a set of sequences H as in before. The set of terminal histories Z and $A(h)$, $h \in H \setminus Z$ defined as in before.
- *Payoffs:* a vNM payoff function $u_i \colon Z \to \mathbb{R}$ for each i,
- *Player function:* a function $P \colon H \setminus Z \to N \cup \{c\}$. If $P(h) = c$ (the chance move), a probability distribution $f_c(\cdot|h)$ over $A(h)$ is also specified.
- *Information sets:* for each i, a partition $\mathcal{I}_i$ of $\{h \in H | P(h) = i\}$ s.t. if $h, h' \in I_i \in \mathcal{I}_i$ then $A(h) = A(h')$.

Perfect Recall & Perfect Information

- *An extensive form game has* **perfect information** *if all information sets are singletons:*

- *An extensive form game has* **perfect recall** *if players never forget what they knew and the actions they took earlier:*

  Formally, let $X_i(h)$ be the sequence of the information sets i visits on h and the actions he takes at them. Then perfect recall requires that $X_i(h) = X_i(h')$ whenever h and h' are in the same information set $I_i \in \mathcal{I}_i$.

Strategies and Outcomes

A **pure strategy** of player $i \in N$ in an extensive form game is a function s_i that associates an action in $A(I_i)$ to each information set $I_i \in \mathcal{I}_i$. (where $A(I_i) := A(h)$ for $h \in I_i$.)

A **behavioral strategy** of player i is a collection $(\beta_i(I_i))_{I_i \in \mathcal{I}_i}$ of *independent* probability distributions, where $\beta_i(I_i)$ is a probability distribution over $A_i(I_i)$.

For general extensive form games we can analogously define the outcome function. Given any profile of behavioral strategies β and non-terminal history h, $\mathcal{O}(\beta|h)$ denotes the distribution over Z induced by β starting at history h.

Nash Equilibrium and Subgame Perfect Equilibrium

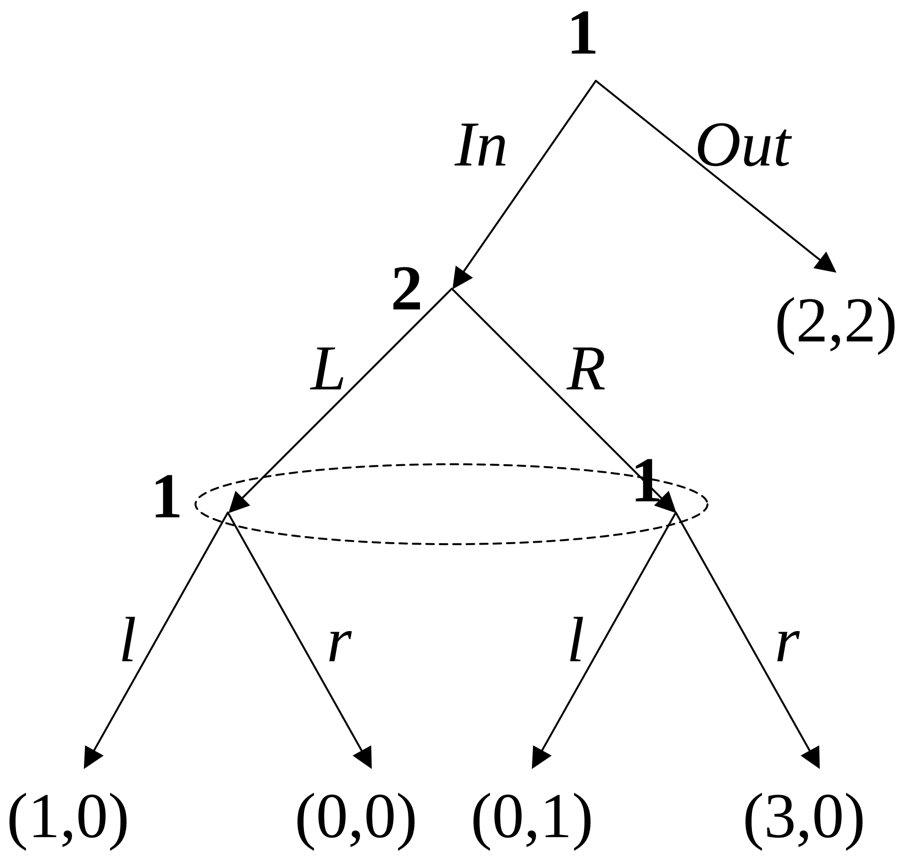
A profile of behavioral strategies β^* is a **Nash Equilibrium** if for any i and β_i: $u_i(\mathcal{O}(\beta^*)) \geq u_i(\mathcal{O}(\beta_i, \beta^*_{-i}))$.

A **subgame** is part of a game that can be considered as a game itself: a history h **defines a subgame**, if $\forall I \in \mathcal{I}$ s.t. there is $(h, h') \in I$, we have that any $\tilde{h} \in I$ can be written as $\tilde{h} = (h, h'')$.

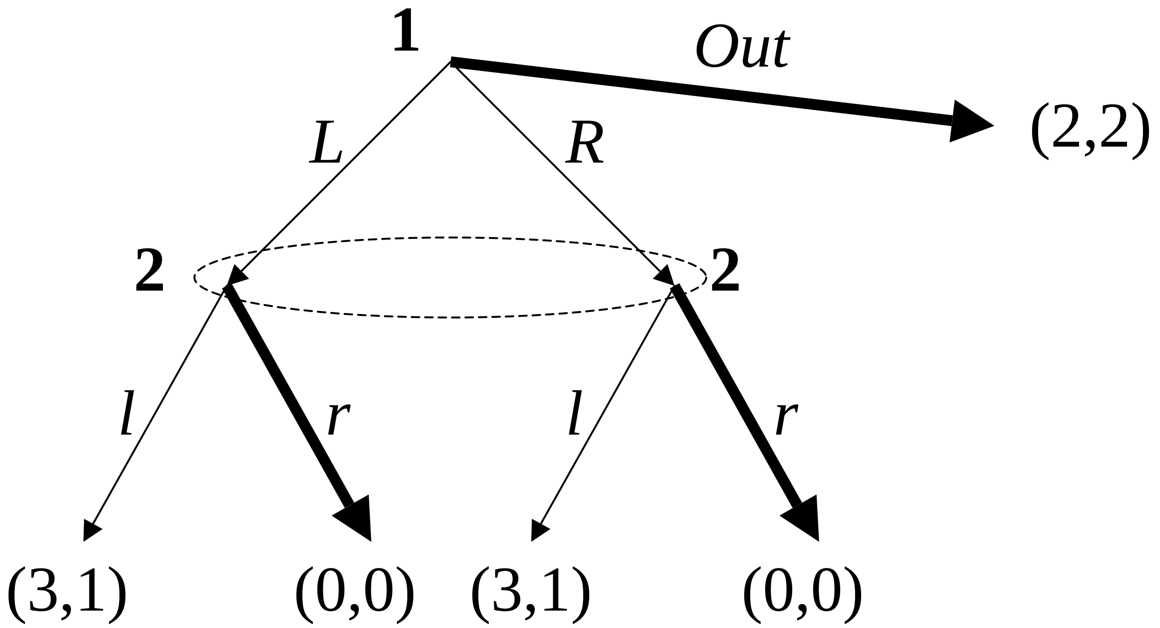
A profile of behavioral strategies β^* is a **Subgame Perfect (Nash) Equilibrium** if for any h that defines a subgame of Γ, any $i \in N$, and β_i:

$$u_i(\mathcal{O}(\beta^*|h)) \geq u_i(\mathcal{O}(\beta_i, \beta^* {-i}|h)).$$

The subgame perfect equilibrium



What is wrong with this equilibrium?

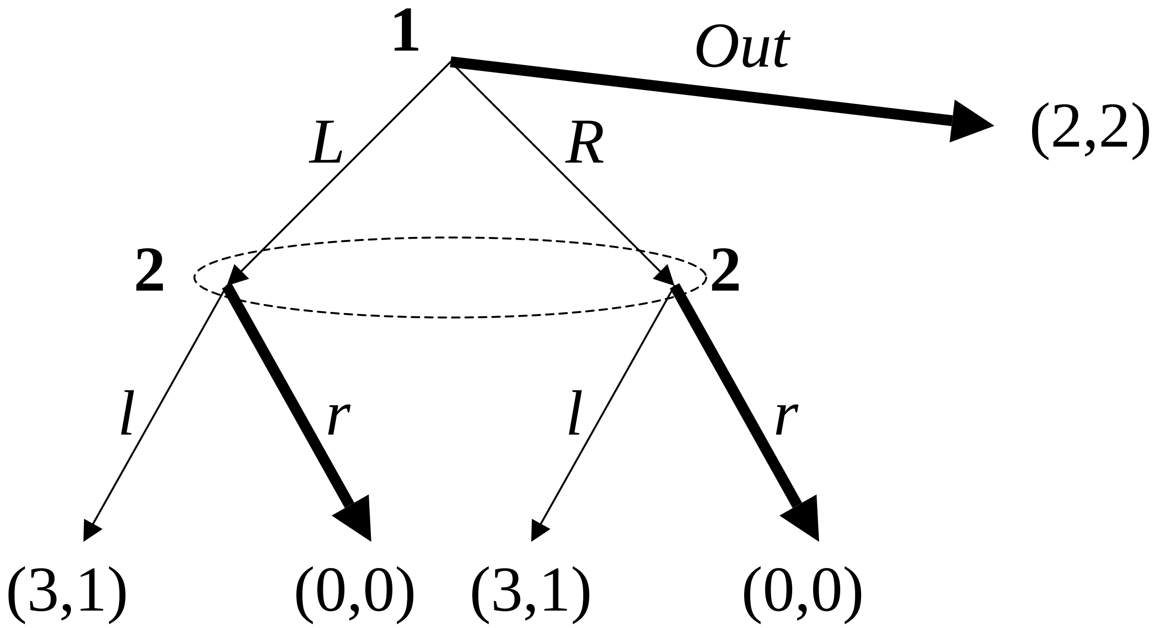


Sequential Rationality

A player i is **sequentially rational** if at each of his information sets, he maximizes his continuation expected utility given his beliefs and the other players' strategies:

- for each $I_i \in \mathcal{I}_i$, there is a probability distribution $\mu(\cdot|I_i)$ over the histories in I_i representing i's beliefs,
- given the others' strategies β_{-i}, for each $I_i \in \mathcal{I}_i$, β_i maximizes $\mathbb{E}_{\mu(\cdot|I_i)} u_i(\mathcal{O}(\beta_i, \beta_{-i}|h))$.

What if players tremble?



Assessments, Consistency, and Sequential Rationality

An **assessment** is a pair (β, μ) where β is a behavioral strategy profile and $\mu(\cdot|I)$ is a probability distribution over I for each $I \in \mathcal{I}$.

An assessment (β, μ) is **sequentially rational** if for each i, $I_i \in \mathcal{I}_i$, and β_i':

$$\mathbb{E}_{\mu(\cdot|I_i)} u_i(\mathcal{O}(\beta_i, \beta_{-i}|h)) \geq \mathbb{E}_{\mu(\cdot|I_i)} u_i(\mathcal{O}(\beta_i', \beta_{-i}|h)).$$

An assessment (β, μ) is **consistent** if there is a sequence (β^k, μ^k) of assessments s.t.

- $(\beta^k, \mu^k) \to (\beta, \mu)$,
- each β^k is completely mixed, and
- μ^k is derived from β^k using Bayes' rule.

Sequential Equilibrium

A **sequential equilibrium** is a sequentially rational and consistent assessment (β, μ).

Definition: An extensive form game Γ is **finite** if H is finite.

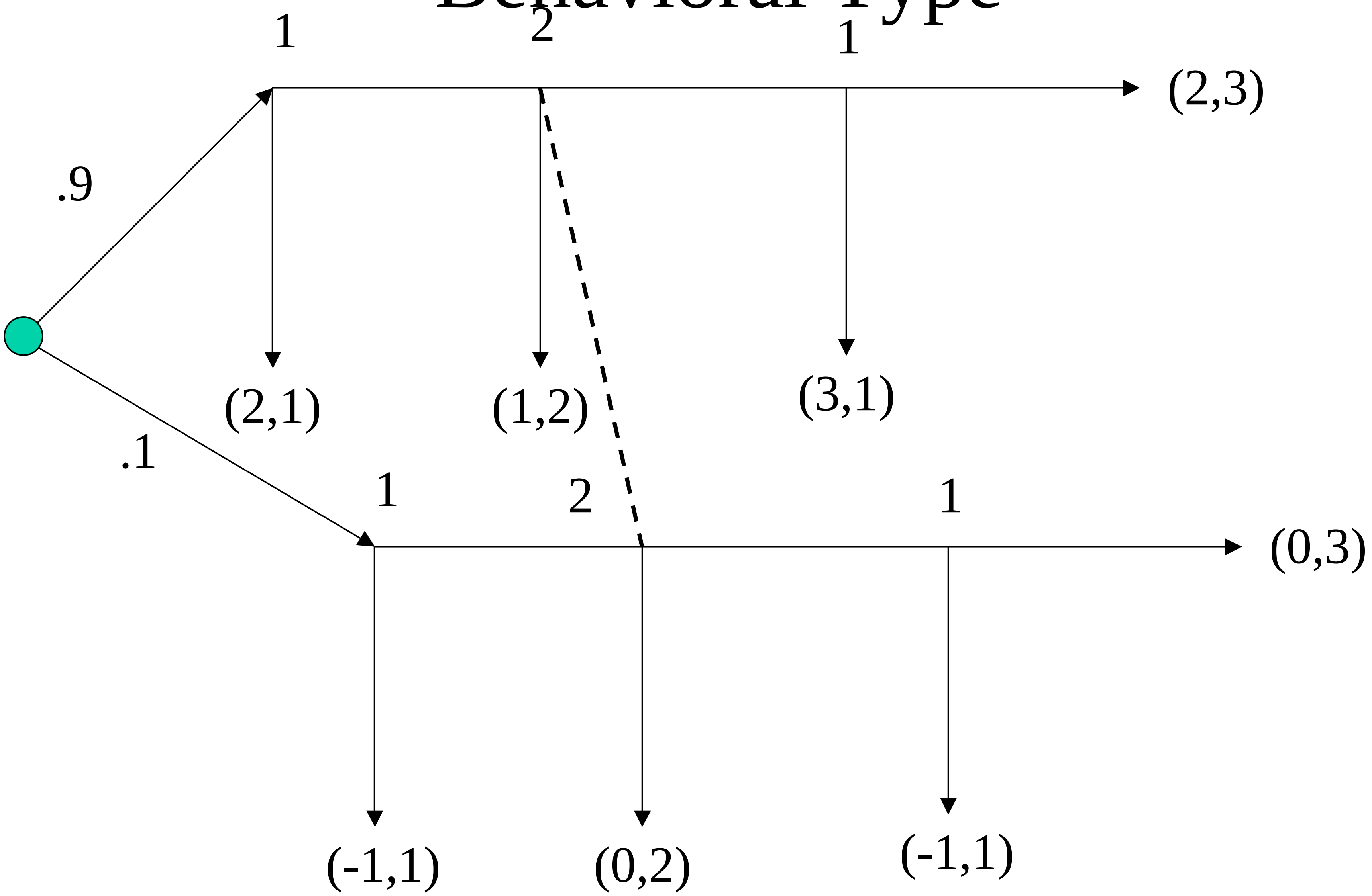
Theorem Every finite extensive form game with perfect recall Γ has a sequential equilibrium.

Single-Deviation Principle for Consistent Assessments

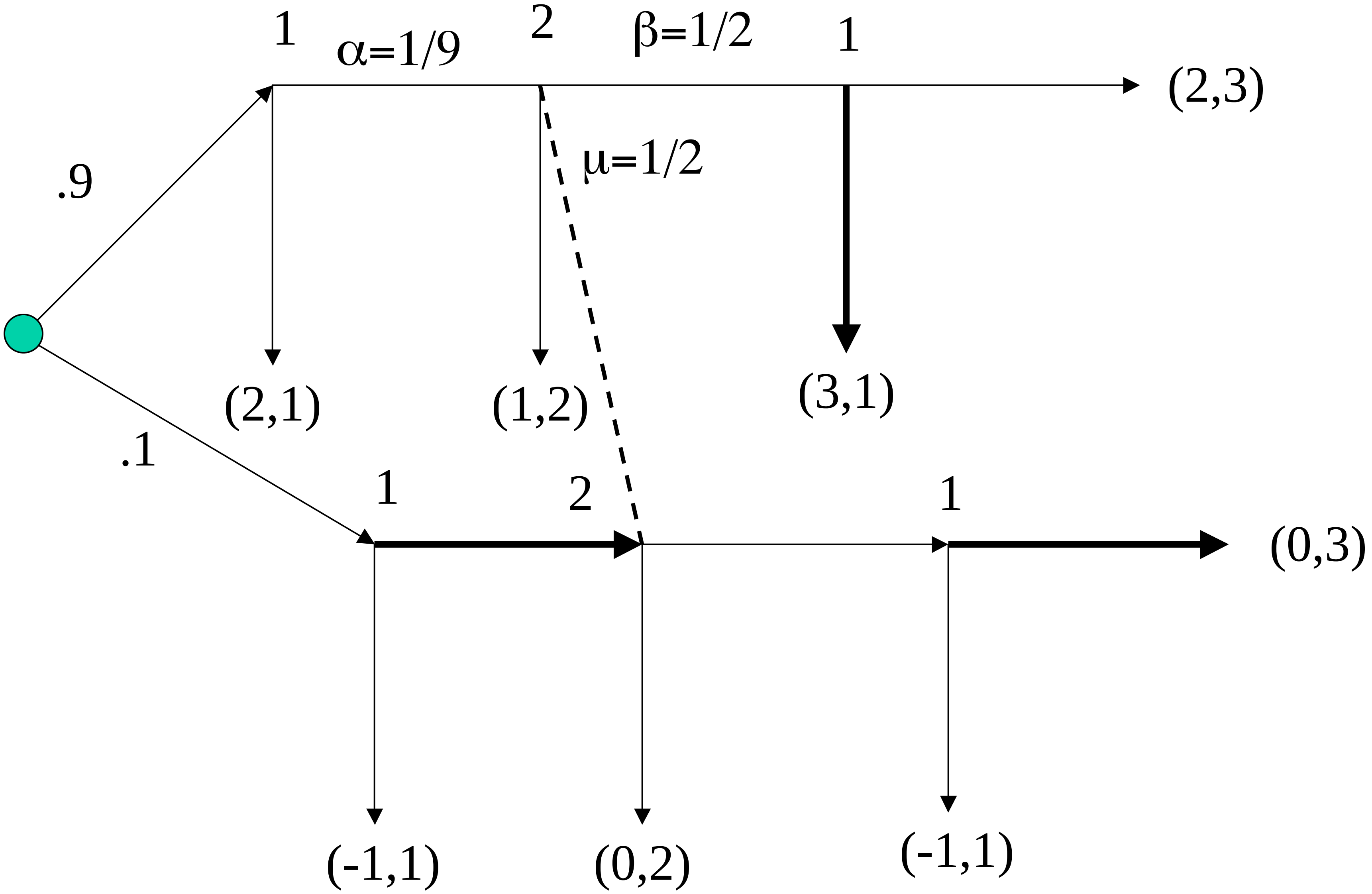
Theorem: Let Γ be an extensive form game of perfect recall that is continuous at infinity. Suppose that $A(h)$ is finite for all $h \in H$ and let (β, μ) be a consistent assessment. Then (β, μ) is sequentially rational iff for any player i and any information set $I_i \in \mathcal{I}_i$, player i cannot increase his conditional payoff at I_i by deviating from his strategy only at I_i, i.e. for any β_i' that agrees with β_i except on I_i:

$$\mathbb{E}_{\mu(\cdot|I_i)} u_i(\mathcal{O}(\beta)|h) \geq \mathbb{E}_{\mu(\cdot|I_i)} u_i(\mathcal{O}(\beta_i', \beta_{-i})|h).$$

Example: The Centipede Game with a Behavioral Type



The Sequential Equilibrium



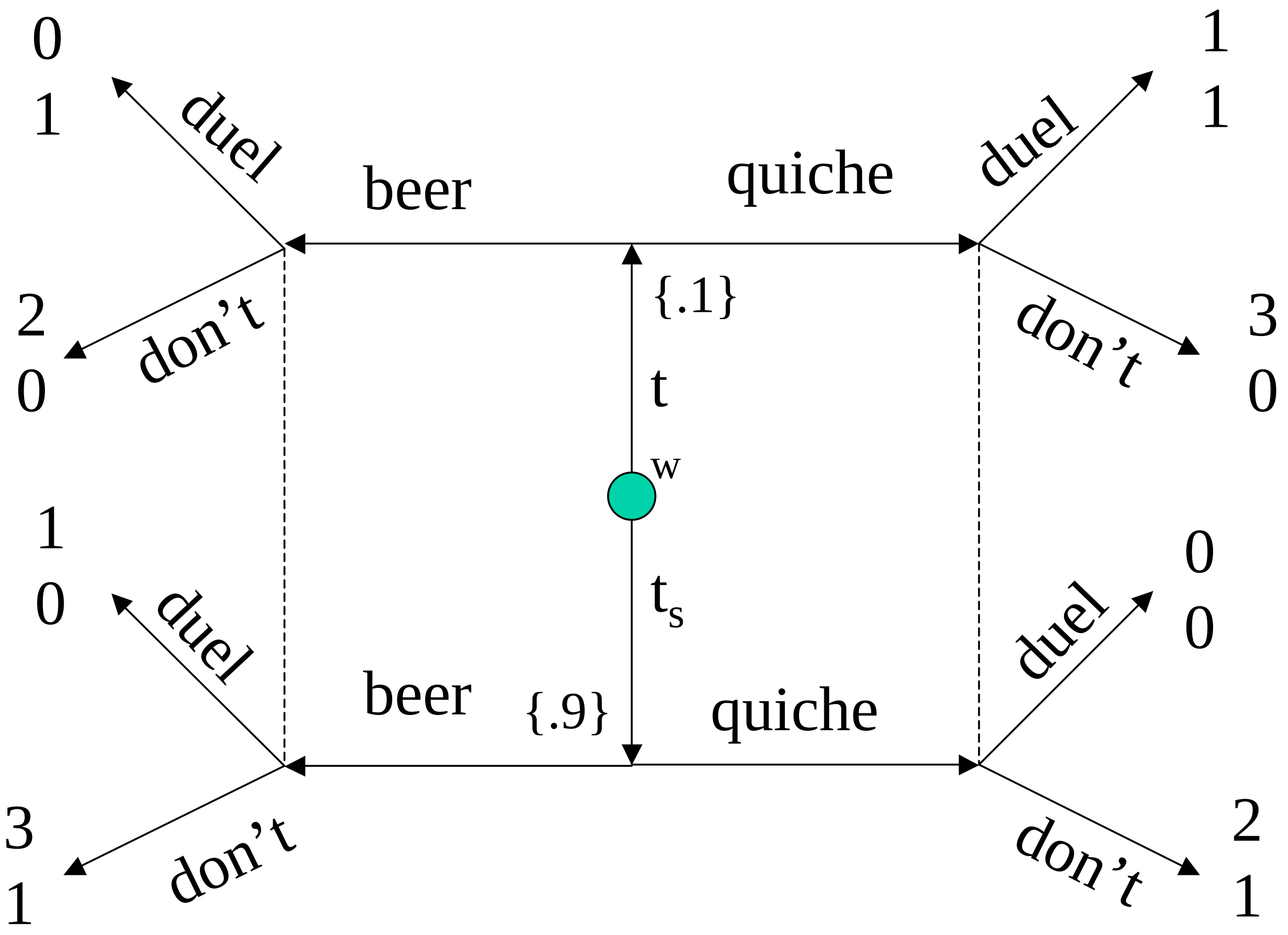
Signaling Games & the Intuitive Criterion
Cho & Kreps (1987)

Signaling Game -- Definition

- Two Players: (S)ender, (R)eceiver

1. Nature selects a type t from T with probability p(t);
2. Sender observes t, and then chooses a message m from M;
3. Receiver observes m (but not t), and then chooses an action a from A;
4. Payoffs are $U_S(t,m,a)$ and $U_R(t,m,a)$.

Beer – Quiche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don't
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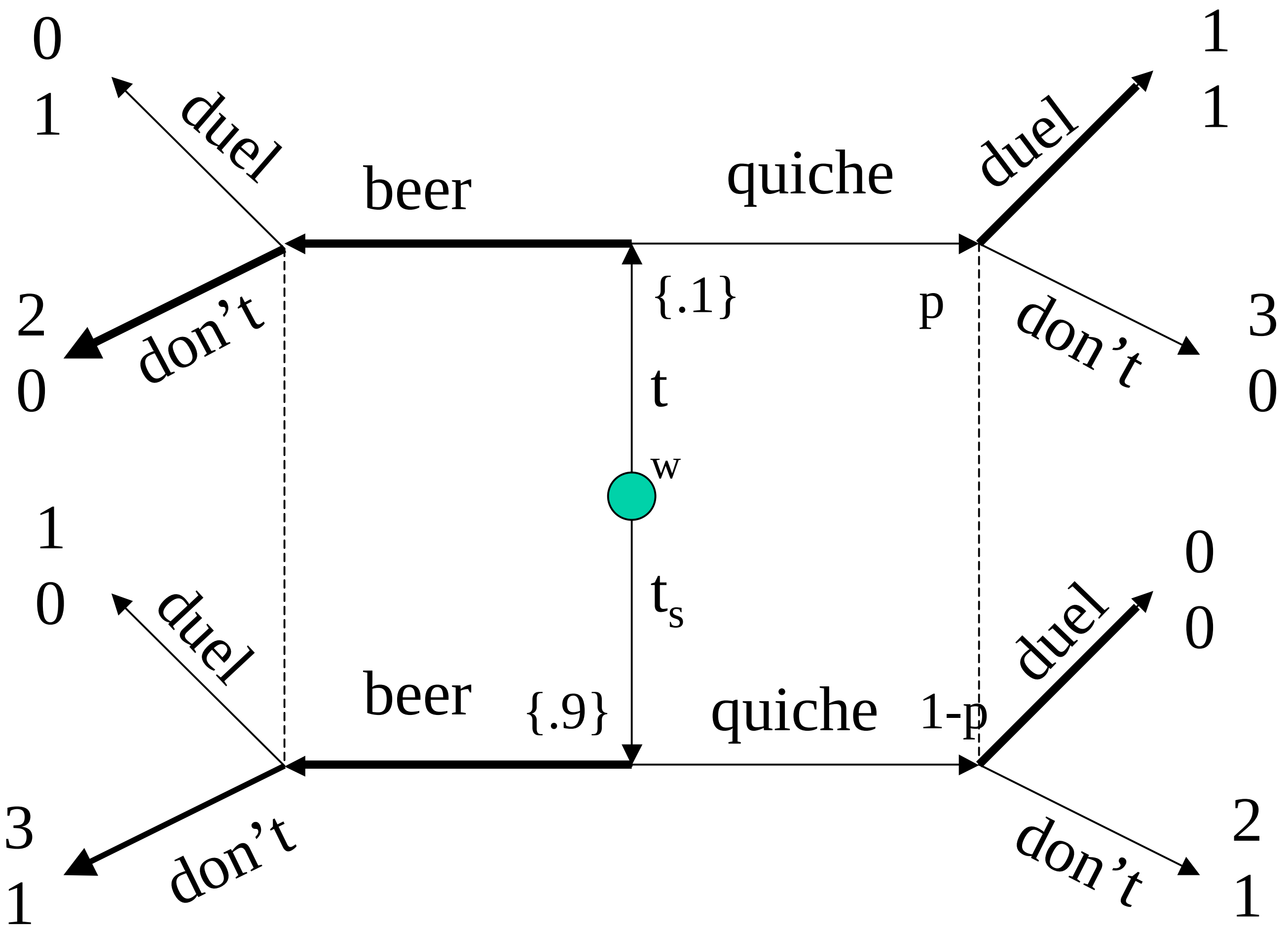
1

Types of Sequential Equilibria (SE)

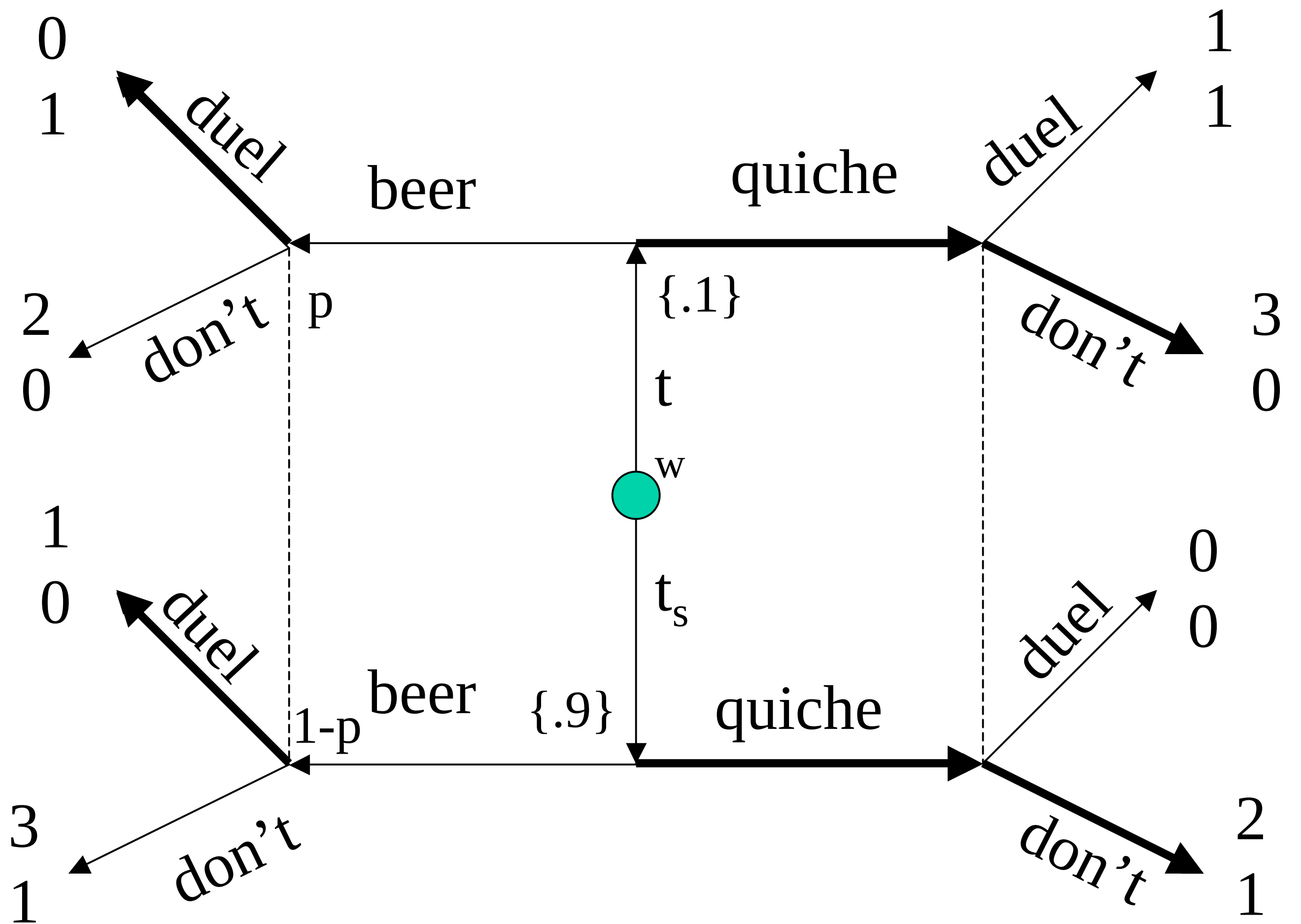
- A **pooling equilibrium** is an equilibrium in which all types of sender send the same message.
- A **separating equilibrium** is an equilibrium in which all types of sender send different messages.
- A **partially separating/pooling equilibrium** is an equilibrium in which some types of sender send the same message, while some others sends some other messages.

Pooling SE where “Beer” is Played



p=μ(t_w|quiche) > 1/2, and
p=1/2 & Prob (duel | quiche) ≥1/2

Pooling SE where “Quiche” is Played



p=μ(t_w|beer) >1/2, and

p=1/2 & Prob (duel | beer) ≥1/2

Intuitive Criterion

Given a SE, if t's equilibrium payoff denoted U*(t) is greater than t's highest possible payoff from m, i.e:

$$U^*(t) > \max_{a \in A} U_S(t, m, a) \qquad (*)$$

then after receiving message m_j, the Receiver's should place zero probability on Sender being of type t, i.e. $\mu(t|m)=0$.

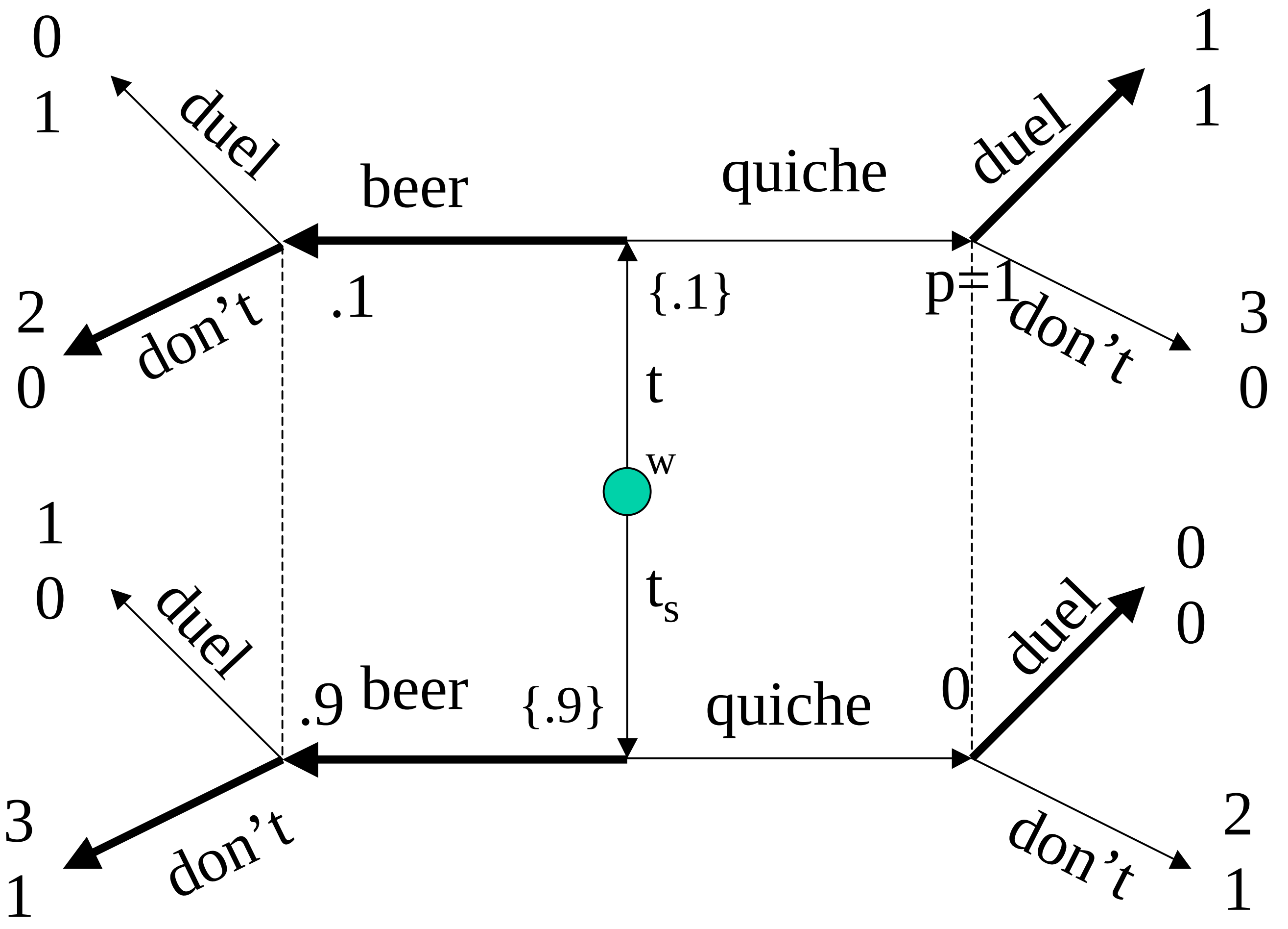
Note: This is possible if (*) does not hold for at least one type in T.

Cho & Kreps:

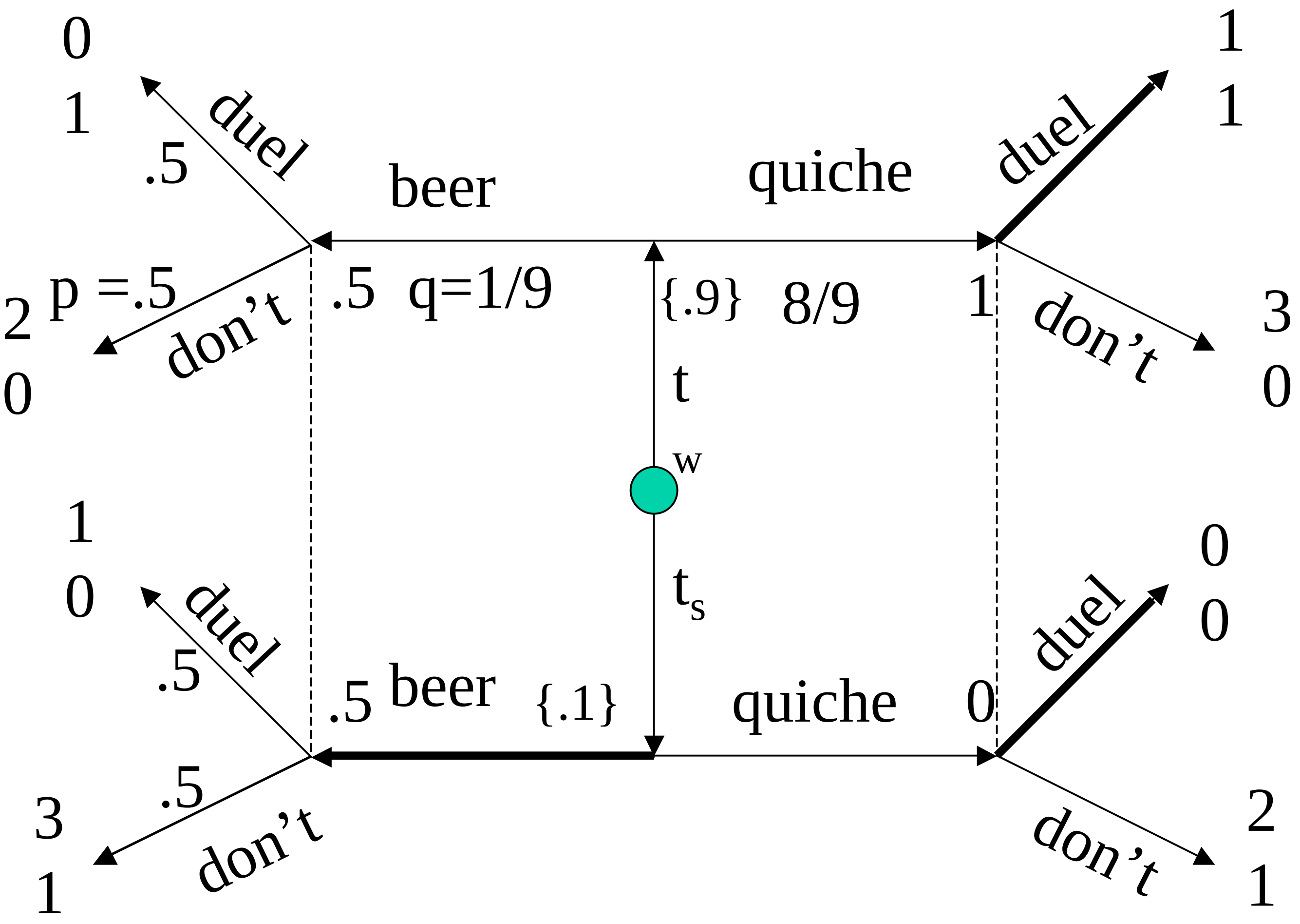
It is as if the sender, if he were of strong type, is (by drinking beer) implicitly making the speech:

> “*I am drinking beer, which ought to convince you that I am of the strong type. For (given the Quiche-pooling equilibrium) I would never wish to drink beer if I were of the weak type. While, if I am of the strong type, and if drinking beer convinces you, then as you see it is in my interest to drink beer.*”

The only pooling SE that survives the Intuitive Criterion



A Mixed SE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quiche
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don't
.5
q=1/9
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8/9
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duel
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duel
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beer
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don't
don't
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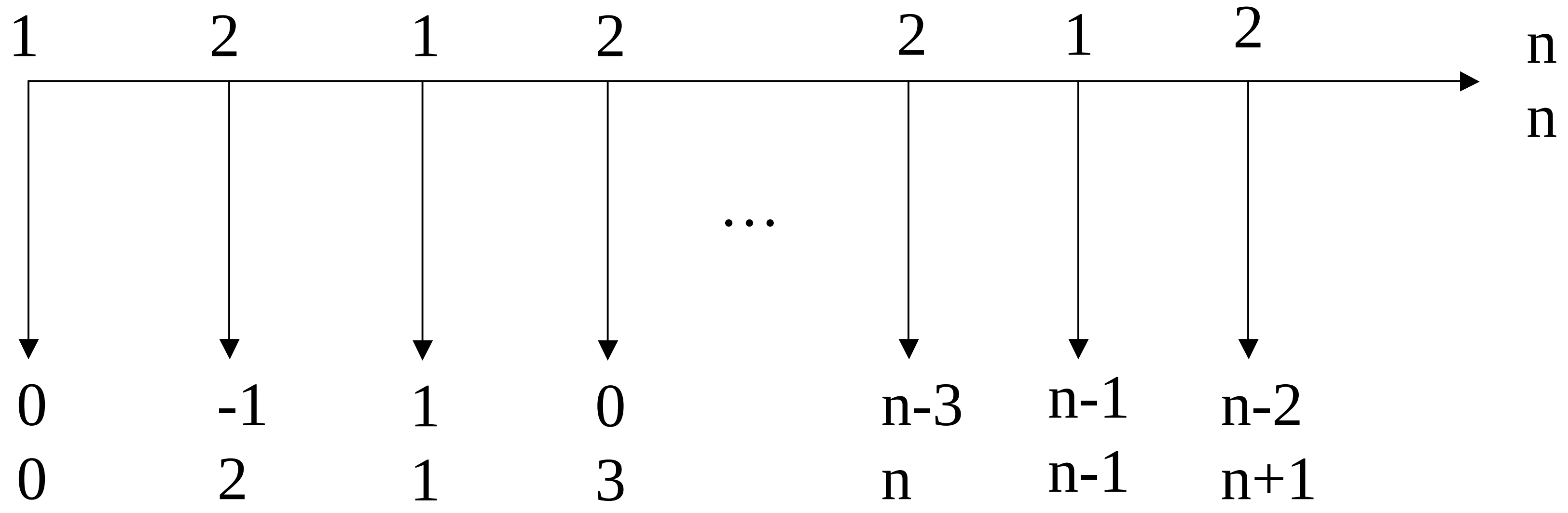
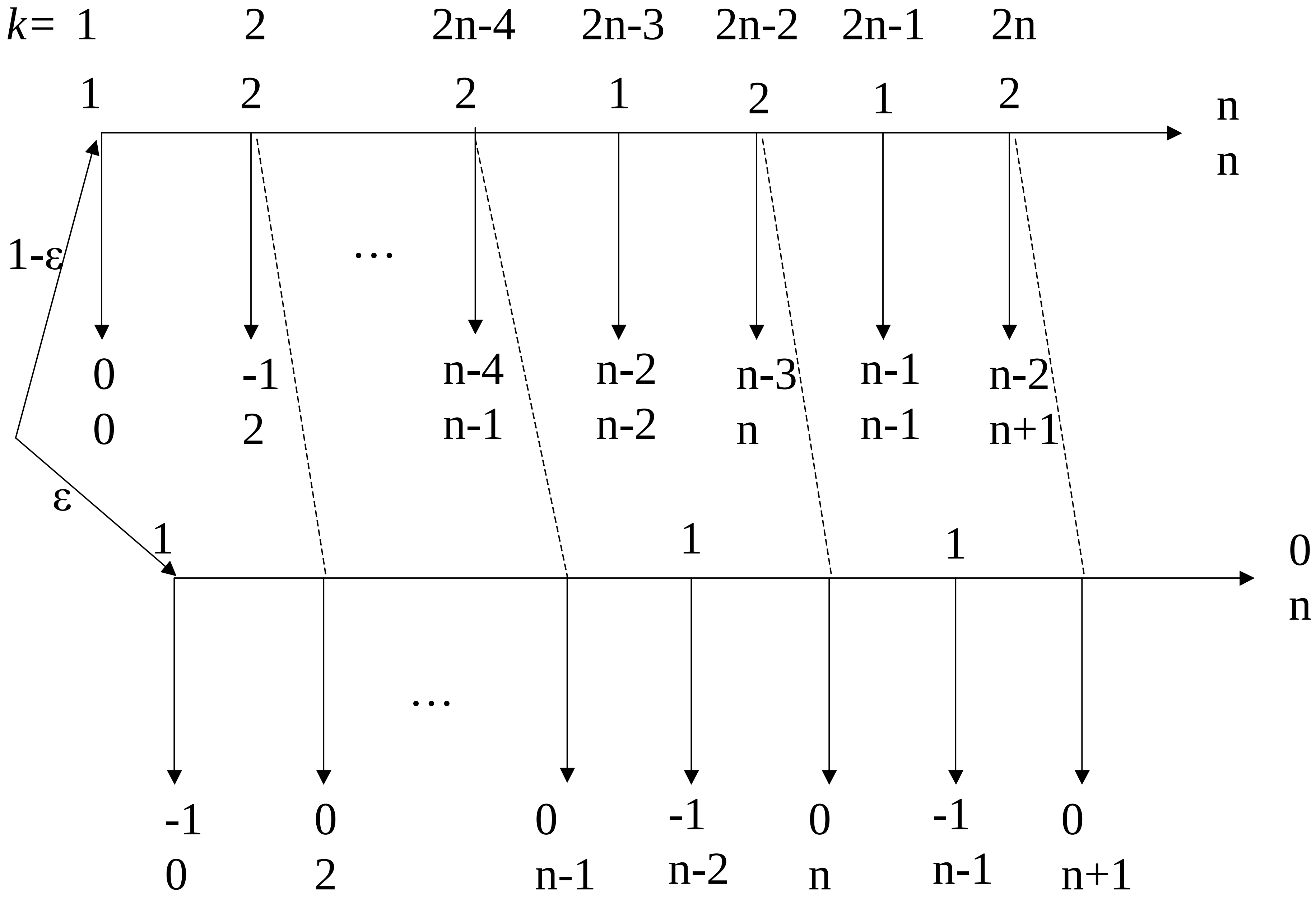
Reputation Formation

Centipede Game - Complete Information (horizon *2n*)

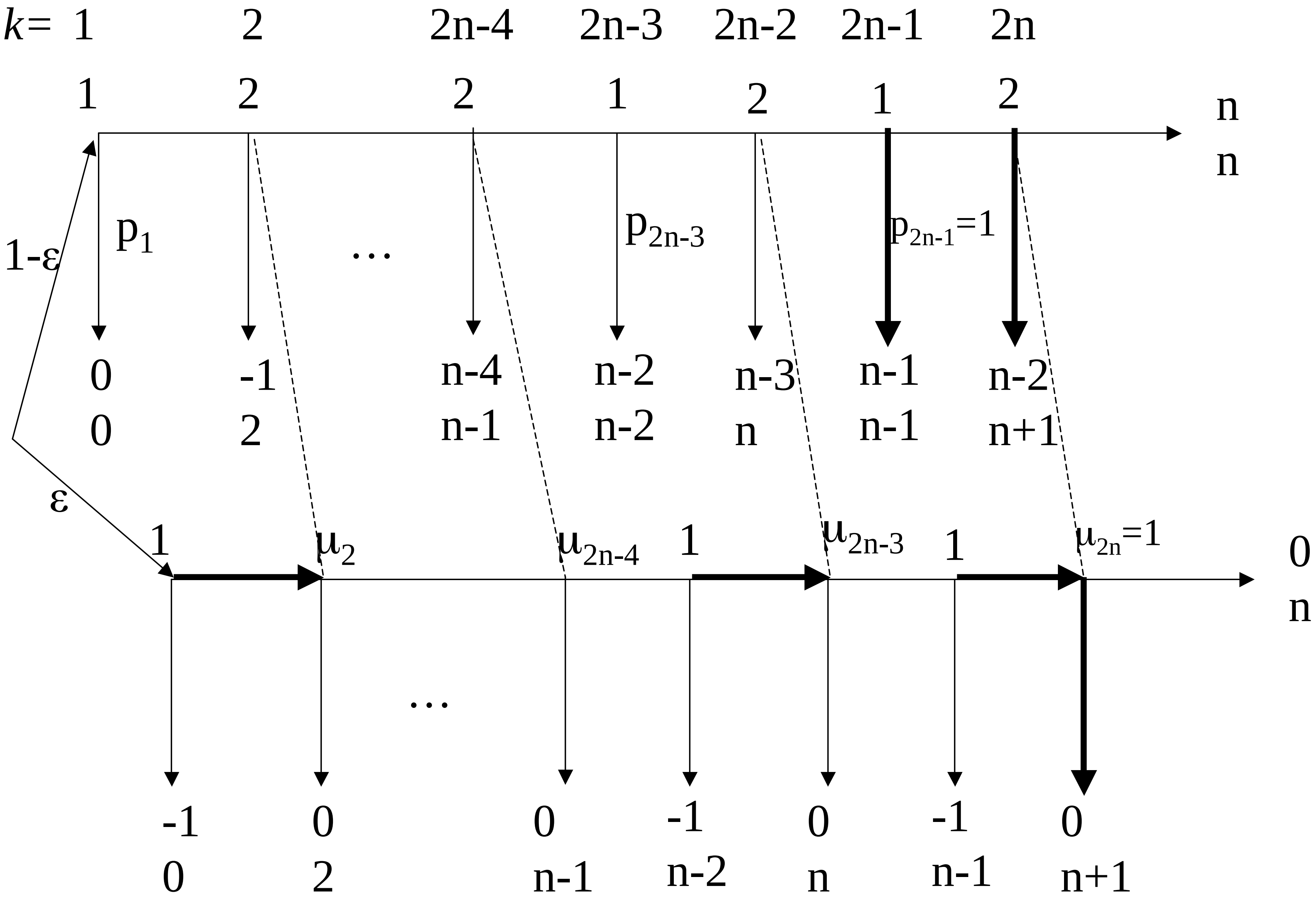


Step k:	1	2	3	4	…	2n-2	2n-1	2n	
	1	2	1	2		2	1	2	
	0 0	-1 2	1 1	0 3		n-3 n	n-1 n-1	n-2 n+1	n n

Centipede Game – with doubt (Small ε 2 (0,1/2))



Centipede Game – with doubt



Facts:

- At any even $k \leq 2n-2$, 2 goes across with positive probability.
- At any odd $k \leq 2n-3$, (rational) 1 goes across with positive probability.
- **Cutoff Property:**
  - If 2 goes across w/p 1 at k, then 1 goes across w/p 1 at k-1,
  - If 1 goes across w/p 1 at k, then 2 goes across w/p 1 at k-1,

So there is k* such that

  - At $k < k^*$ players go across with probability 1
  - At $k = k^*,\ldots,2n-2$, players completely mix. ($\varepsilon < 1/2$)

- If 1 goes across w/p 1 at k, then 2's posterior at k+1 is her prior at k-1.

Sequential Equilibrium Beliefs

At an even $k < k^*$: $\mu_k = \varepsilon$.

At an even $k \geq k^*$, let x be 2's payoff from stopping. Since 2 mixes at k:

$x = \mu_k(x+1) + (1- \mu_k)[(x-1)p_{k+1} +(1-p_{k+1})(x+1)]$

$\Leftrightarrow (1- \mu_k)\, p_{k+1} = 1/2$

$$\mu_{k+2} = \frac{\mu_k}{\mu_k + (1-\mu_k)(1-p_{k+1})} = \frac{\mu_k}{\mu_k + (1-\mu_k) - p_{k+1}(1-\mu_k)} = 2\mu_k$$

$$\mu_k = \frac{\mu_{k+2}}{2} \Rightarrow \forall \text{even } k \geq k^*: \quad \mu_k = \frac{1}{2^{(2n-k)/2}}$$ since $\mu_{2n}=1$.

Sequential Equilibrium Strategies

- Let l^* be defined by: $$\frac{1}{2^{l^*}} > \varepsilon \geq \frac{1}{2^{l^*+1}}$$

Set $k^*=\max\{1, 2n - 2l^* - 1\}$.

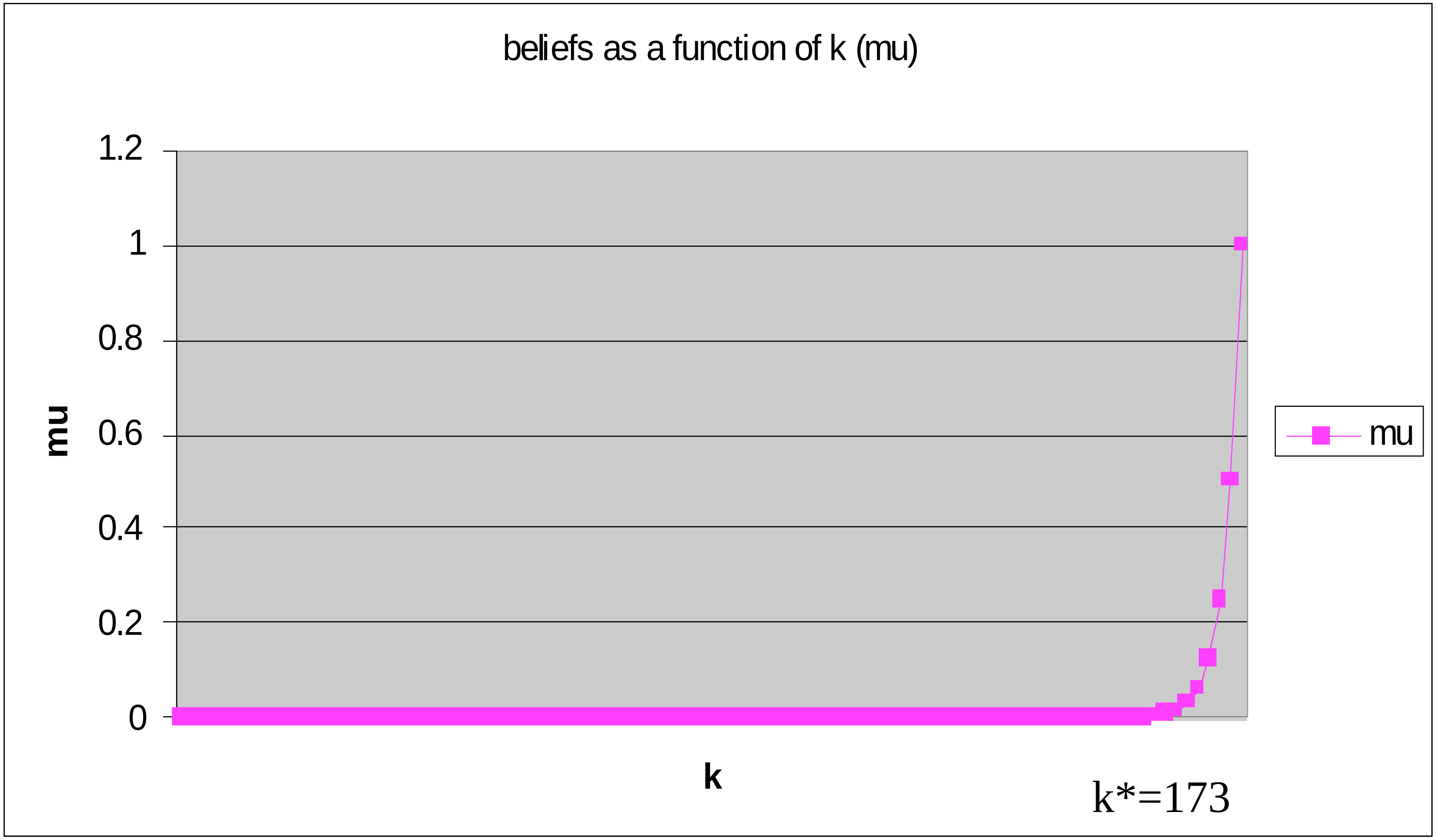
- At $k < k^*$: Both players go across w/p 1
- At $k = k^*,\dots, 2n-2$:
  - Player 2 goes across w/p 1/2
  - Player 1 goes across w/p p_k given by:

$$\mu_{k+1} = \frac{\mu_{k-1}}{\mu_{k-1} + (1 - p_k)(1 - \mu_{k-1})} \Rightarrow p_k = \frac{\mu_{k+1} - \mu_{k-1}}{(1 - \mu_{k-1})\mu_{k+1}}$$

beliefs are as in the previous slide ($\mu_0 = \varepsilon$).

- At k=2n-1,2n; both player stop w/p 1.

Example n=100, ε=0.001



k*=173