Econ 504
Spring 2007

Homework 4
Part A

DUE: April 27

1. The following bargaining game was first proposed by Nash. Players 1 and 2 are bargaining over how to split one dollar. Both players simultaneously name shares they would like to have, s_1 and s_2, where $0 \leq s_1, s_2 \leq 1$. If $s_1 + s_2 \leq 1$ then the players receive the shares they named; if $s_1 + s_2 > 1$ then both players receive zero.

   (a) What are the pure strategy equilibria of this game?

   (b) Instead of looking at pure equilibria, look at mixed equilibria of the following form. With probability p_i player i asks for share $\underline{s}_i$ and with probability $1 - p_i$ she asks for share $\bar{s}_i > \underline{s}_i$.

      i. Show that, in any equilibrium, $\underline{s}_1 + \bar{s}_2 = 1$ and $\bar{s}_1 + \underline{s}_2 = 1$.

      ii. Find $\underline{s}_i$ and $\bar{s}_i$ and associated probabilities such that the expected share for either player is 1/4.

      iii. Show that, for any (x, y) with $x, y \geq 0$ and $x + y \leq 1$, there is a mixed strategy equilibrium in which player 1's expected share is x and player 2's expected share is y.

      iv. In class, I computed a mixed strategy equilibrium for the Bertrand-Edgeworth game in which the mixtures were continuous distributions. Here, the mixtures are discrete. Discuss: is there a mixed equilibrium with continuous distributions here? Why or why not?

2. A player, U, possesses a commodity which may be one of two qualities a or b with equal probability. Player U, however, does not learn the quality until after the game is over and he receives his payoff (hence the U for "uninformed"). The other player in the game is a potential buyer, I (for "informed"); I knows the quality of the commodity. A commodity of type a is worth \$120 to U and \$100 to I. In this case, U cannot profitably sell the commodity to I. A type b commodity, on the other hand, is worth \$60 to U and \$200 to I. Hence in this case it is possible in principle for U to sell to I. Consider a one period game in which I makes an offer to U which U may accept or reject, after which the game ends and the quality is revealed. Assume only integer bids are allowed and confine attention to pure strategy Weak Perfect Bayesian equilibria (WPBE). Let x_k denote I's bid when the commodity is of type k, $k = a, b$.

(a) Characterize all WPBE in which $x_a = x_b$. These are the pooling equilibria. Specify out of equilibrium beliefs (i.e., U's belief when he sees the "wrong" bid).

(b) Characterize all WPBE in which x_a is rejected and x_b accepted. This, of course, entails $x_a \neq x_b$. These are the separating equilibria. Are there any equilibria in which the separation works the other way, with x_a accepted and x_b rejected?

(c) There are separating equilibria in which out of equilibrium beliefs are "reasonable" in the following sense: they are consistent (as far as possible) with player I not playing a weakly dominated or iteratively weakly dominated strategy. Specify these WPBE.

3. Consider a Cournot/Stackelberg game with two firms in which firm A selects an output, immediately after which B, observing A's choice, chooses its output. Both firms operate with zero cost. Market demand can be of one of two types: high (H) or low (L). If demand is high, demand is given by:

$$Q = 4 - P$$

while if demand is low, demand is given by:

$$Q = 3 - P$$

Here Q is market demand and P is market price. High demand occurs with probability 0.2 while low demand occurs with probability 0.8. Firm A knows the true level of demand prior to making its choice; firm B does not.

(a) Express firm B's Weak Perfect Bayesian equilibrium (WPBE) output choice, q_B, as a function of q_A and $\pi(q_A)$, where $\pi(q_A)$ denotes the posterior probability assigned by B to high demand, given q_A. Is q_B increasing or decreasing in $\pi(q_A)$?

(b) Denoting by $q_A(H)$ the amount produced by A when demand is high and $q_A(L)$ when demand is low. Prove the following monotonicity property. In any Nash equilibrium,

$$q_A(H) \geq q_A(L)$$

(c) Suppose there were no information asymmetry: both firms know whether demand is high or low. What are the values of $q_A(H)$ and $q_A(L)$?

(d) Returning to the asymmetric game, argue that in any separating equilibrium:

$$q_A(H) = 2$$

and that there are a continuum of separating equilibria with

$$q_A(L) \in \left[\frac{1}{2}(3-\sqrt{5}), 1\right] \approx [0.38, 1]$$

What are the corresponding values of q_B? Comment explicitly on why the values found in c) are not consistent with separating equilibrium.

(e) To support the equilibria in d), what must π be? Are all of these equilibria reasonable?

4. Consider the bargaining with asymmetric information setting of the Myerson-Satterthwaite theorem, but now consider the following type distribution. The seller's type c is uniformly distributed on $[0,1]$. The buyer's type v is always exactly twice the seller's type. This violates the independence assumption in MS. Construct a bargaining game with a NE that is ex-post efficient, budget balanced, and individually rational.

5. There are two consumers, Sarah and Jeff, and two goods. Sarah has an endowment $\omega_1^S > 0$ of good 1 and zero endowment of good 2. Similarly, Jeff has an endowment $\omega_1^J > 0$ of good 1 and zero endowment of good 2. Sarah can produce good 2 from good 1. It costs $c(x_2)$ units of good 1 to produce x_2 units of good 2. Assume that c is as differentiable as necessary and that $c'(x_2) \geq 0$ and $c''(x_2) \geq 0$ for any x_2.

   Given the consumption bundle (x_1^S, x_2^S), Sarah has utility

$$u^S(x_1^S, x_2^S) = x_1^S.$$

   Given the consumption bundle (x_1^J, x_2^J), Jeff has utility

$$u^J(x_1^J, x_2^J) = x_1^J + v(x_2^J).$$

   Since only Jeff will consume good 2 in equilibrium, I will drop the superscript J on x_2 and write simply x_2. Assume that $v(0) = 0$, that v is as differentiable as necessary, that $v'(x_2) > 0$ and $v''(x_2) < 0$ for any x_2.

   Normalize the price of good 1 to be 1. Assume wherever necessary that solutions are interior.

   (a) Show that maximizing Sarah's profit from the manufacture and sale of good 2 is equivalent to maximizing Sarah's utility.

   (b) Find the competitive price p_2^c, quantity x_2^c of good 2, and Jeff's consumption of good 1, x_1^{Jc}. (Here and below, since you don't have an explicit functional form for v or c, the solution will necessarily be abstract.)

(c) Suppose that Sarah can act like a standard monopolist: she can set the price of good 2 but cannot dictate directly how much Jeff can purchase at that price. Write out Sarah's profit maximization problem explicitly. Find the monopoly price p_2^m, quantity x_2^m, and Jeff's consumption of good 1, x_1^{Jm}. Show that $p_2^m > p_2^c$ and that $x_2^m < x_2^c$.

(d) Suppose that Sarah can make a "take it or leave it" offer to Jeff. She can offer Jeff the consumption bundle (x_1^J, x_2) and Jeff can either accept or reject. In the former case, Sarah gets the consumption bundle $(\omega_1^S + \omega_1^J - x_1^J - c(x_2), 0)$ and Jeff gets (x_1^J, x_2). In the latter case case there is no trade. Write out Sarah's utility maximization problem explicitly. Find the x_2^d that solves this problem (d for 'price discrimination"). Show that $x_2^d = x_2^c$. Find Jeff's consumption of good 1, x_1^{Jd}.

(e) Show that Sarah can get the same utility by making a "take it or leave it offer" of the form: either buy all of x_2^d at a price of p_2^d per unit or there is no trade. Find the required unit price p_2^d. Show that $p_2^d > p_2^c$.

(f) How does p_2^d compare to p_2^m? (*Hint:* this is something of a trick question.)

(g) Suppose that instead of making a "take it or leave it offer" to Jeff, Sarah offers a "two part tariff." If Jeff buys any of good 2 at all, Jeff must pay a fixed price T plus an additional price of p_2 per unit of good 2. Thus, if Jeff buys x_2 units he pays a total of $T + p_2 x_2$. As in the standard monopoly case, Sarah does not dictate directly how much Jeff purchases. Find the T and p_2^T that maximize Sarah's profit and compute the resulting x_2^T (T for "two part tariff"). (*Hint:* Can Sarah can do any better than she does in (5d)?) State in words briefly what is going on.

6. Consider a monopolist facing inverse demand $P = 1 - Q$ (for $Q \in [0, 1]$, $P = 0$ for $Q > 1$). The monopolist has cost $C(Q) = \frac{1}{2}Q^2$.

(a) What value of Q maximizes profit?

(b) What value of Q maximizes the sum of profit and consumer surplus?

(c) Suppose that there are now two, identical firms, A and B. The cost of firm j is $C_j(q_j) = \frac{1}{2}q_j^2$, where q_j is the output of firm j. Inverse demand is again $P = 1 - Q$, with $Q = q_A + q_B$.

What values of q_A and q_B maximize the sum of firm profits? Compare to (a) and discuss briefly.

(d) The setup is the same as in (c). What values of q_A and q_B maximize the sum of proft and consumer surplus? Compare to (b) and discuss briefly.

(e) The setup is the same as in (c). Find the equilibrium of the Cournot quantity game.

(f) Suppose that the firms in (c) can write an enforceable revenue sharing agreement in which firm j gets the share s_j of the total revenue, where $s_A, s_B > 0$ and $s_A + s_B = 1$. Thus, firm j's payoff/profit in the new Cournot quantity game is $s_j PQ - \frac{1}{2}q_j^2$.

Find the equilibrium of this game.

(g) Continuing (f), for $s_A = s_B = 1/2$, compare the equilibrium of the Cournot quantity game with the answers to (c), (d), and (e), and discuss briefly.

7. (a) There are two firms, 1 and 2, producing a single homogeneous good for which demand is given by $Q^D = 1 - P$. Firm i has cost $C_i(q_i) = 0$, where q_i is the output of firm i. Firm i is located in country i. Ignore any exchange rate issues.

   The countries and firms play the following two-period game. In period one, each country chooses a per unit tax, t_i, for its own firm. If $t_i < 0$ then the tax is effectively a subsidy. In period 2, the firms observe the t_i and play a Cournot/quantity game.

   Firm i's payoff is $Pq_i - t_i q_i$. Country i's payoff is the sum of the firm's payoffs and the country's tax receipts: $(Pq_i - t_i q_i) + t_i q_i = Pq_i$, which is simply firm i's revenue.

   By assumption, the countries ignore consumer surplus for this good (perhaps because the consumers are in some other country or perhaps because consumers are not represented in the governments).

   i. Find the SPE value of t_i.
   ii. Compare SPE revenues with revenues if $t_1 = t_2 = 0$.

   (b) Same question as (a) but with the following change. The two firms, instead of producing a single homogeneous good, now produce differentiated goods. Demand for firm i is $Q_i^D = 1 - P_i + \frac{1}{2}P_j$. In period two of the two-period game, the firms now set prices rather than quantities.

   i. Find the SPE value of t_i.
   ii. Compare SPE revenues with revenues if $t_1 = t_2 = 0$.

8. Consider the following two stage Hotelling-type model. There are two firms, each with zero cost. In stage 1, each firm chooses a location on $[0, 1]$. In stage 2, the firms engage in price competition. A consumer at location λ buying from firm i at location x_i has net utility

$$\bar{v} - p_i - \alpha(x_i - \lambda)^2,$$

where $\alpha > 0$. A consumer buys only if she gets positive net utility, and she buys from the firm that gives her the highest net utility.

Find an SPE in which firm A locates at $x < 1/2$, firm B locates at $1 - x$ (assume $\bar{v}$ is sufficiently large that in equilibrium, all consumers buy), and the firms charge $p_A - p_B = p$. How does the equilibrium vary with α? Discuss briefly.