Prepared Solution for Problem Set 5 Repeated Games

1. Long Run VS Short Run

	L	C	M	R
T	$10,3$	$3,10$	$13,7$	$-1,-1$
B	$9,9$	$2,2$	$12,7$	$-1,-1$

Suppose that in this simultaneous move game player 1 (row player) is a long-run player and player 2 (column player) is a short run player.

Solution:

(a) Find the pure and mixed precommitment payoffs, the static Nash payoff and the minmax payoff for the long-run player.

(a1) Minmax payoff:

It is easy to see that $\min\max U^1 = \min\{\max\{10,9\},\max\{3,2\},\max\{13,12\},\max\{-1,-1\}\}$

$= \min\{10,3,13,-1\} = -1$

(a2) Static Nash payoff:

We know that the Nash equilibrium survives the iterated deletion of strictly dominated strategy. Since R is strictly dominated by M for player 2, the game reduces to

	L	C	M
T	$10,3$	$3,10$	$13,7$
B	$9,9$	$2,2$	$12,7$

Then we find that B is strictly dominated by T for player 1. Thus the game reduces to

	L	C	M
T	$10,3$	$3,10$	$13,7$

Then we find that L and M are strictly dominated by C for player 2. Thus the game reduces to

	C
T	$3,10$

Therefore there is a unique static Nash equilibrium, which is (T,C), with payoff 3 for player 1.

(a3) Pure precommitment payoff:

If player 1 chooses T, player 2's best response will be C, resulting in a payoff 3 for player 1.

If player 1 chooses B, player 2's best response will be L, resulting in a payoff 9 for player 1.

Therefore player 1's pure precommitment will be B, with payoff 9.

(a4) Mixed precommitment payoff:

Suppose player 1 plays T with probablity p and plays B with probability $1-p$.

Then player 2's problem is to choose a strategy to maximize his payoff, subject to

$U^2(L) = 3p + 9(1-p)$

$U^2(C) = 10p + 2(1-p)$

$U^2(M) = 7$

$U^2(R) = -1$

Since player 1's payoff will be maximized when palyer 2 plays M, we solve the inqualities $\left\{\begin{array}{c} U^2(M) \geq U^2(L) \\ U^2(M) \geq U^2(C) \\ U^2(M) \geq U^2(R) \end{array}\right\}$ and get $\frac{1}{3} \leq p \leq \frac{5}{8}$.

Now we want to choose p to maximize player 1's payoff $13p + 12(1-p)$ subject to $\frac{1}{3} \leq p \leq \frac{5}{8}$. Obviously when $p = \frac{5}{8}$, the payoff is maximize and the maximum is $13\frac{5}{8} + 12\left(1-\frac{5}{8}\right) = \frac{101}{8}$.

Therefore player 1's mixed precommitment will be $\frac{5}{8}T + \frac{3}{8}B$, with payoff $\frac{101}{8}$.

(b) For large discount factors find the best and worst equilibrium for the long-run player.

(b1) The best dynamic equilibrium $\overline{v}^1$:

We know that $\overline{v}^1 = \max_{\alpha^1}\min_{\alpha^2\in BR^2(\alpha^1),\alpha^1(a^1)>0} u^1\left(a^1,\alpha^2\right)$, which can be found out by the following approach:

p	$BR^2\left(\alpha^1\right)$	min	max min
$p=0$	L	9	
$0<p<\frac{1}{3}$	L	9	
$p=\frac{1}{3}$	L, M	9	
$\frac{1}{3}< \mathbf{p} <\frac{5}{8}$	**M**	**12**	**12**
$p=\frac{5}{8}$	C, M	2	
$\frac{5}{8}<p<1$	C	2	
$p=1$	C	3	

Therefore, $\overline{v}^1=12$.

(b2) The worst dynamic equilibrium $\underline{v}^1$:

We know that $\underline{v}^1=\min_{\alpha^1}\max_{\alpha^2\in BR^2(\alpha^1),\alpha^1(a^1)>0} u^1\left(a^1,\alpha^2\right)$, which can be found out by the following approach:

p	$BR^2\left(\alpha^1\right)$	max	min max
$p=0$	L	9	
$0<p<\frac{1}{3}$	L	10	
$p=\frac{1}{3}$	L, M	13	
$\frac{1}{3}<p<\frac{5}{8}$	M	13	
$p=\frac{5}{8}$	C, M	13	
$\frac{5}{8}< \mathbf{p} < \mathbf{1}$	**C**	**3**	3
$\mathbf{p}=\mathbf{1}$	**C**	**3**	3

Therefore, $\underline{v}^1=3$.

(c) For what discount factors is the answer to part (b) valid?

(c1) Since $\overline{v}^1\geq(1-\delta)\,u^1\left(a^1,\alpha^2\right)+\delta w^1\left(a^1\right)$ and $w^1\left(a^1\right)\geq n^1$, we have $\overline{v}^1\geq(1-\delta)\,u^1\left(a^1,\alpha^2\right)+\delta n^1$. Since $\overline{v}^1=12, n^1=3, u^1\left(a^1,\alpha^2\right)=\{13,12\}$,

we get $\left\{\begin{array}{c} 12\geq 13\,(1-\delta)+3\delta \\ 12\geq 12\,(1-\delta)+3\delta \end{array}\right\}$, which implies that $\frac{1}{10}\leq\delta<1$.

(c2) We want to find the range of δ such that $\underline{v}^1\geq(1-\delta)\,u^1\left(a^1,\alpha^2\right)+\delta w^1\left(a^1\right)$ and $w^1\left(a^1\right)\leq n^1$. Note that $\underline{v}^1=3, n^1=3, u^1\left(a^1,\alpha^2\right)=\{2,3\}$,

$\left\{\begin{array}{c} 3\geq 2\,(1-\delta)+\delta w^1\left(a^1\right) \\ 3\geq 3\,(1-\delta)+\delta w^1\left(a^1\right) \end{array}\right\}$ always hold for $w^1\left(a^1\right)\leq n^1$. Therefore $\forall\delta\in[0,1)$, $\underline{v}^1=3$ can be supported.

In fact it makes sense since $\underline{v}^1=n^1=3$, and we know the static Nash equilibrium is independent of δ.

2. **Long Run VS Short Run with Noise**

A short-run supplier has the option of supplying a single indivisible item to a long-run firm. The firm has the option of paying for the item or not. If the firm pays, there is a

25% chance that the check gets lost in the mail. (Note: if the check is lost, the supplier does not receive the payment, and the firm is not charged for the item.) The firm values

the item at \$5.00, and the supplier values the item at \$1.00. The payment is \$4.00, and both parties are risk neutral. Find the best perfect public equilibrium for the firm (of the

infinitely repeated game with public randomization) as a function of the discount factor, first, assuming that the supplier can observe whether or not the check is lost in the mail,

and second assuming that the supplier can only observe whether or not payment is received.

Solution:

Denote the long-run firm by player 1 and the short-run supplier by player 2. $S_1=\{Pay, Don't\ Pay\}$, $S_2=\{Supply, Don't\ Supply\}$.

If the strategy profile is $(Pay, Supply)$, the payoff profile will be $\frac{3}{4}\,(5-4,4)+\frac{1}{4}\,(5,0)=(2,3)$

If the strategy profile is $(Don't\ Pay, Supply)$, the payoff profile will be $(5,0)=(5,0)$

If the strategy profile is either $(Pay, Don't\ Supply)$ or $(Don't\ Pay, Don'tSupply)$, the payoff profile will be $(0,1)$.

Therefore, the normal form of the stage game can be described as below:

	Supply	*Don't Supply*
Pay	$2, 3$	$0, 1$
Don't Pay	$5, 0$	$0, 1$

The extensive form will be discussed in the help session.

It is easy to see there is a unique static Nash equilibrium, which is (*Don't Pay*, *Don'tSupply*), with payoff 0 for player 1.

(a) First we assume that player 1 can observe whether or not the check is lost in the mail.

Denote the best dynamic equilibrium payoff by $\overline{v}^1$:

Suppose player 1 plays *Pay* with probablity p and plays *Don't Pay* with probability $1-p$.

We know that $\overline{v}^1 = \max_{\alpha^1} \min_{\alpha^2 \in BR^2(\alpha^1), \alpha^1(a^1)>0} u^1\left(a^1, \alpha^2\right)$, which can be found out by the following approach:

p	$BR^2\left(\alpha^1\right)$	min	max min
$p = 0$	*Don't Supply*	0	
$0 < p < \frac{1}{3}$	*Don't Supply*	0	
$p = \frac{1}{3}$	*Supply*, *Don't Supply*	0	
$\frac{1}{3} < \mathbf{p} < 1$	**Supply**	**2**	**2**
$\mathbf{p} = \mathbf{1}$	**Supply**	**2**	**2**

Therefore, $\overline{v}^1 = 2$.

Since $\overline{v}^1 \geq (1-\delta) u^1\left(a^1, \alpha^2\right) + \delta w^1\left(a^1\right)$ and $w^1\left(a^1\right) \geq n^1$, we have $\overline{v}^1 \geq (1-\delta) u^1\left(a^1, \alpha^2\right) + \delta n^1$. Since $\overline{v}^1 = 2, n^1 = 0, u^1\left(a^1, \alpha^2\right) = \{2, 5\}$,

we get $\left\{ \begin{array}{c} 2 \geq 2(1-\delta) + 0\delta \\ 2 \geq 5(1-\delta) + 0\delta \end{array} \right\}$, which implies that $\frac{3}{5} \leq \delta < 1$.

Therefore $\overline{v}^1 = \left\{ \begin{array}{c} 2 \text{ if } \frac{3}{5} \leq \delta < 1 \\ 0 \text{ if } 0 \leq \delta < \frac{3}{5} \end{array} \right\}$.

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This part is not required for solving the question, only for those who are interested in dynamic equilibrium strategies.

You may ask which strategy profile can achieve this best dynamic equilibrium payoff.

Denote by S^1 the strategy of playing *pay* with probablity $p > \frac{1}{3}$ and playing *Don't Pay* with probability $1-p$.

Denote by N^1 the strategy of playing *Don't Pay*.

Denote by T^1 the strategy of radomizing between S^1 and N^1 with playing S^1 with probablity $\frac{5\delta-3}{2\delta}$.

Denote by S^2 the strategy of playing *Supply*.

Denote by N^2 the strategy of playing *Don't Supply*.

Denote by T^2 the strategy of radomizing between S^2 and N^2 with playing S^2 with probablity $\frac{5\delta-3}{2\delta}$.

Consider the following strategy:

For player 1:

(1) At the first period, player 1 plays S^1.

(2) If the outcome is *Pay* in the previous period, player 1 continuos to play S^1; If the outcome is *Don't Pay* in the previous period, player 1 plays T^1.

(3) If the outcome is (*Don'tPay*, *Don'tSupply*) or if anyone deviates from the strategy described, then player 1 plays N^1 forever. Otherwise go back to (1).

For player 2:

(1) At the first period, player 1 plays S^2.

(2) If the outcome is *Pay* in the previous period, player 2 continuos to play S^2; If the outcome is *Don't Pay* in the previous period, player 2 plays T^2.

(3) If the outcome is (*Don'tPay*, *Don'tSupply*) or if anyone deviates from the strategy described, then player 2 plays N^2 forever. Otherwise go back to (1).

You may check that the strategy profile described above is SPE with $\frac{3}{5} \leq \delta < 1$, and the equilibrium payoff is 2 for player 1.

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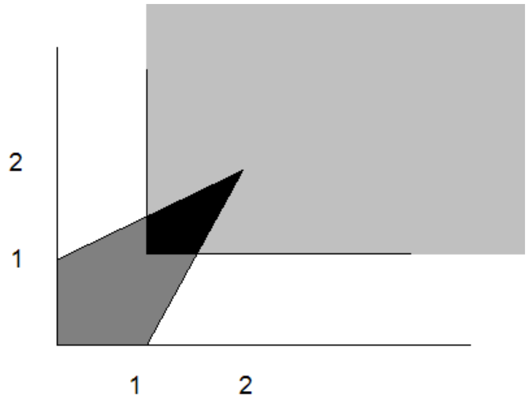

(b) Second we assume that player 2 can only observe whether or not payment is received.
Denote the best dynamic equilibrium payoff by $\overline{v}^1$.
Now we have the following conditions:
$\overline{v}^1 = (1-\delta)\,2 + \delta\left[\frac{3}{4}w^1\left(Paid\right) + \frac{1}{4}w^1\left(Not\ Paid\right)\right]$
$\overline{v}^1 \geq (1-\delta)\,5 + \delta w^1\left(Not\ Paid\right)$
$0 \leq w^1\left(Paid\right) \leq \overline{v}^1$
$0 \leq w^1\left(Not\ Paid\right) \leq \overline{v}^1$
Maximization of $\overline{v}^1$ requires that $w^1\left(Paid\right) = \overline{v}^1$ and $\overline{v}^1 = (1-\delta)\,5 + \delta w^1\left(Not\ Paid\right)$.
Thus we have
$\left\{\begin{array}{c} \overline{v}^1 = (1-\delta)\,2 + \delta\left[\frac{3}{4}\overline{v}^1 + \frac{1}{4}w^1\left(Not\ Paid\right)\right] \\ \overline{v}^1 = (1-\delta)\,5 + \delta w^1\left(Not\ Paid\right) \end{array}\right\} \Rightarrow \left\{\begin{array}{c} \overline{v}^1 = 1 \\ w^1\left(Not\ Paid\right) = 5 - \frac{4}{\delta} \end{array}\right\}$
$0 \leq w^1\left(Not\ Paid\right) \leq \overline{v}^1 \Rightarrow 0 \leq 5 - \frac{4}{\delta} \leq 1 \Rightarrow \frac{4}{5} \leq \delta < 1$
Therefore $\overline{v}^1 = \left\{\begin{array}{c} 1 \text{ if } \frac{4}{5} \leq \delta < 1 \\ 0 \text{ if } 0 \leq \delta < \frac{4}{5} \end{array}\right\}$.

3. Folk Theorem
Consider the following coordination game:

	C	D
C	$2,2$	$1,0$
D	$0,1$	$0,0$

Solution:
(a) What is the unique static Nash equilibrium?
We know that the Nash equilibrium survives the iterated deletion of strictly dominated strategy. Since D is strictly dominated by C both for player 1 and player 2, the game reduces to

	C
C	$2,2$

Therefore there is a unique static Nash equilibrium, which is (C,C), with payoff 2 for both player 1 and player 2.

(b) Sketch the socially feasible, individually rational set.
See the graph attached above, where the gray area refers to the individually rational set, the deep gray area refers to the socially feasible set, and the black area is the intersection of them, refering the socially feasible, individually rational set.

(c) Find a discount factor and subgame perfect strategies such that each player receives 1.5.
Since $(1.5, 1.5)$ is within the ocially feasible, individually rational set, it can be achieved by some SPE with certain discount factor δ.

Consider the following strategy for each player:

(1) Play D at first period.

(2) Play C if the previous outcome is (D, D); Play D otherwise.

If player i sticks to the strategy described above (the outcome will be $(D, D), (C, C), (C, C), \cdots)$, he will get $(1-\delta)\,0 + \delta 2 = 2\delta$

If player i deviates (by the one-shot deviation, the outcome will be $(C, D), (D, D), (C, C), (C, C), \cdots)$, he will get $(1-\delta)\,1 + \delta 2\delta = 2\delta^2 - \delta + 1$

We require that $2\delta \geq 2\delta^2 - \delta + 1$, which implies that $2\delta^2 - 3\delta + 1 \leq 0$. Hence we have $\frac{1}{2} \leq \delta \leq 1$.

Since the equilibrium payoff described by the above strategy profile is 2δ, let $2\delta = 1.5$ and we get $\delta = \frac{3}{4}$. Obviously $\frac{1}{2} \leq \frac{3}{4} \leq 1$.

Therefore with $\delta = \frac{3}{4}$ and by palying the strategy described above, each player will recieve 1.5 in equilibrium.

(d) Can you find an information system for which this is an equilibrium in a matching protocol?

Fix $\delta = \frac{3}{4}$.

Suppose everyone is marked with a flag, which can be either red or green. Consider the following matching protocol and strategy:

(1) Initially, everyone is marked with a red flag.

(2) If both players are marked with green flags, both play C. If at least one player is marked with a read flag, both play D.

(3) If both players are marked with green flags, the player will be marked with a green flag by playing C, and will be marked with a red flag by playing D. If at least one player is marked with a read flag, the player will be marked with a green flag by playing D, and will be marked with a red flag by playing C.

To see it will be an equilibrium,

if player i follows the strategy, he will be marked with a green flag from the second period on, and get $\left(1-\frac{3}{4}\right)0 + \frac{3}{4}2 = 1.5$,

if player i deviates, (by one-shot deviation, he will be marked with a red flag in the second period, and a green flag from the third period on), and get $\left(1-\frac{3}{4}\right)1 + \frac{3}{4}1.5 = 1.375$.

Since $1.5 > 1.375$, player i has no incentive to deviate.