

**Econ 504 (2009)
Microeconomics II**

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Optimal Auctions

The Envelope Theorem

**Source:
Milgrom and Segal (2002)**

The Envelope Theorem

Let A be an arbitrary set with generic element $\alpha \in A$. Let x, y denote generic elements of $[0, 1]$.

Given a function $f : A \times [0, 1] \rightarrow \mathbb{R}$, define:

$$V(x) = \sup_{\alpha \in A} f(\alpha, x)$$
$$A^*(x) = \{\alpha \in A : f(\alpha, x) = V(x)\}.$$

If it exists, let $f_2(\alpha, x)$ denote the partial derivative of f with respect to its second argument.

Theorem (M&S, 2002) *Let $x \in (0, 1)$ and $\alpha^*(x) \in A^*(x)$. If the derivatives $f_2(\alpha^*(x), x)$ and $V'(x)$ exist, then*

$$V'(x) = f_2(\alpha^*(x), x).$$

Absolute Continuity of V

If a function $g : [0, 1] \rightarrow \mathbb{R}$ is *absolutely continuous*, then it is differentiable almost everywhere and it is equal to the integral of its derivative (i.e. the Fundamental Theorem of Calculus is applicable to g).

Examples of Absolutely continuous functions:

- Lipschitz continuous functions, e.g. differentiable functions with bounded derivative.
- Convex [or concave] functions which are continuous at the endpoints 0 and 1.

Corollary *Suppose that V is absolutely continuous and $f(\alpha, \cdot)$ is differentiable for every $\alpha \in A$. Suppose that $\alpha^*(y) \in A^*(y)$ for almost all $y \in [0, 1]$. Then:*

$$V(x) = V(0) + \int_0^x f_2(\alpha^*(y), y) dy.$$

An Application to Auctions

The agent's possible private valuations for an object are given by $x \in [0, 1]$. Each action $\alpha \in A$ available to the agent results in a probability $q(\alpha) \in [0, 1]$ of receiving the object, and an expected payment of $t(\alpha) \in \mathbb{R}$.

Interpretations of the set of actions A :

1. $A = \mathbb{R}$ denotes possible bids in a sealed bid auction (e.g. first or second price). Given an equilibrium, $q(\alpha)$ and $t(\alpha)$ denote the agent's probability of receiving the object and expected payment conditional on bidding α .
2. $A = [0, 1]$ denotes agent's possible announcements of her valuation in a direct auction mechanism; $q(\alpha)$ and $t(\alpha)$ denote the agent's probability of receiving the object and expected payment conditional on announcing valuation α .

When her valuation is x , the agent's payoff is:

$$V(x) = \max_{\alpha \in A} f(\alpha, x)$$

where $f(\alpha, x) = q(\alpha)x - t(\alpha)$.

Note that $V : [0, 1] \rightarrow \mathbb{R}$ is absolutely continuous (since it is convex, and continuous at 0 and 1) and $f_2(\alpha, x) = q(\alpha)$. So

$$V(x) = V(0) + \int_0^x q(\alpha^*(y)) dy.$$

where $\alpha^*(y)$ denotes an optimal action for the agent when her valuation is y .

Under interpretation 1, we can set $\alpha^*(y)$ to be the agent's bid function under the given equilibrium. Under interpretation 2, if truthful revelation is an equilibrium of the direct auction mechanism, then we can set $\alpha^*(y) = y$.

Optimal Auction Design
Myerson (1981)

The Model

A single indivisible object is to be allocated to n bidders:
 $N = \{1, 2, \dots, n\}$.

Bidder i 's valuation is denoted by X_i or x_i . It is distributed with density f_i and c.d.f. F_i , in a closed interval $\mathcal{X}_i = [\underline{x}_i, \bar{x}_i] \subset \mathbb{R}$. f_i is continuous and $f_i(x_i) > 0$ for any $x_i \in \mathcal{X}_i$. Valuations are independently distributed across bidders.

A profile of valuations: $x = (x_1, \dots, x_n) \in \mathcal{X} = \mathcal{X}_1 \times \dots \times \mathcal{X}_n$.
The joint density is: $f(x) = f_1(x_1) \times \dots \times f_n(x_n)$.

Similarly, $x_{-i} \in \mathcal{X}_{-i}$ is a profile of valuations excluding x_i and f_{-i} is the joint density of valuations other than x_i .

Note: We still have independence, private values, and risk neutrality, but symmetry is dropped.

Direct Auction Mechanisms

A direct auction mechanism determines each agent's probability of receiving the object and her expected payment for each profile of valuations. That is,

Definition A **direct (auction) mechanism** is a pair (p, t) where $p : \mathcal{X} \rightarrow \mathbb{R}^n$ and $t : \mathcal{X} \rightarrow \mathbb{R}^n$ are functions, and p satisfies $p_j(x) \geq 0$ and $\sum_{i=1}^n p_i(x) \leq 1$ for all $j \in N$ and $x \in \mathcal{X}$.

$p_i(x)$ denotes the probability with which agent i receives the object and $t_i(x)$ denotes the expected payment of i , when the profile of valuations is x .

Examples of Direct Mechanisms

The direct mechanism induced by the Bayesian Nash Equilibrium of the first-price auction:

$$p_i(x) = \begin{cases} \frac{1}{|\{j \in N : x_j = x^1\}|} & \text{if } x_i = x^1 \\ 0 & \text{otherwise.} \end{cases}$$

$$t_i(x) = \begin{cases} \frac{1}{|\{j \in N : x_j = x^1\}|} b^I(x_i) & \text{if } x_i = x^1 \\ 0 & \text{otherwise.} \end{cases}$$

where $b^I(x_i) = x_i - \frac{1}{F^{n-1}(x_i)} \int_{\underline{x}}^{x_i} F^{n-1}(y_i) dy_i$ denotes the equilibrium bid function of the first-price auction.

The direct mechanism induced by the dominant strategy equilibrium of the second-price auction: $p_i(x)$ is the same,

$$t_i(x) = \begin{cases} \frac{1}{|\{j \in N : x_j = x^1\}|} x^2 & \text{if } x_i = x^1 \\ 0 & \text{otherwise.} \end{cases}$$

Feasible Mechanisms

Given a direct mechanism (p, t) , the expected utility of agent i with valuation x_i from truthfully revealing her type when everybody else truthfully reveals their type is:

$$U_i(x_i) = \int_{\mathcal{X}_{-i}} [p_i(x_i, x_{-i})x_i - t_i(x_i, x_{-i})] f_{-i}(x_{-i}) dx_{-i}.$$

Definition A direct mechanism (p, t) is **feasible** if it is

1. **Incentive Compatible (IC):** For all $i \in N$, $x_i, x'_i \in \mathcal{X}_i$

$$U_i(x_i) \geq \int_{\mathcal{X}_{-i}} [p_i(x'_i, x_{-i})x_i - t_i(x'_i, x_{-i})] f_{-i}(x_{-i}) dx_{-i}.$$

2. **Individually Rational (IR):** For all $i \in N$, $x_i \in \mathcal{X}_i$

$$U_i(x_i) \geq 0.$$

Characterization of Feasibility

Given a direct mechanism (p, t) , the *expected* probability with which agent i receives the object if she announces x_i and everybody else truthfully reveals their type is:

$$Q_i(x_i) = \int_{\mathcal{X}_{-i}} p_i(x_i, x_{-i}) f_{-i}(x_{-i}) dx_{-i}.$$

Lemma 1 *The direct mechanism (p, t) is feasible if and only if the following are satisfied for all $i \in N$:*

1. *For all $x_i, y_i \in \mathcal{X}_i$: $x_i \leq y_i \implies Q_i(x_i) \leq Q_i(y_i)$.*
2. *For all $x_i \in \mathcal{X}_i$:*

$$U_i(x_i) = U_i(\underline{x}_i) + \int_{\underline{x}_i}^{x_i} Q_i(y_i) dy_i$$

3. *$U_i(\underline{x}_i) \geq 0$.*

The Seller's Expected Revenues from a Feasible Mechanism

Lemma 2 *If the direct mechanism (p, t) is feasible, then the seller's expected revenues is given by:*

$$\int_{\mathcal{X}} \left[\sum_{i=1}^n \left(x_i - \frac{1 - F_i(x_i)}{f_i(x_i)} \right) p_i(x) \right] f(x) dx - \sum_{i=1}^n U_i(\underline{x}_i).$$

The Revenue Equivalence Theorem

Corollary (*The Revenue/Payoff Equivalence Theorem*)

The seller's expected revenues and the agents' expected payoffs from a feasible auction mechanism is completely determined by the probability function p and the numbers $U_i(\underline{x}_i)$ for all i .

That is, once we know who gets the object in each possible situation (as specified by p) and the expected payoffs of the lowest valuation types, then the seller's expected revenues and the agents' expected payoffs do not depend on the payment function t .

Example: In the first-price and second-price auction equilibria we studied, the object is always allocated to the highest valuation bidder and the payoffs of the lowest type bidders is zero. Therefore, the seller's expected revenues and the bidders' expected payoffs are the same in both cases.

Optimal Auctions

A direct auction mechanism (p, t) is **optimal** if it maximizes the seller's expected revenues subject to feasibility.

Corollary *Suppose that*

1. $p: \mathcal{X} \rightarrow \mathbb{R}^n$ maximizes

$$\int_{\mathcal{X}} \left[\sum_{i=1}^n \left(x_i - \frac{1 - F_i(x_i)}{f_i(x_i)} \right) p_i(x) \right] f(x) dx$$

subject to:

(a) For all $i \in N$, $x_i, y_i \in \mathcal{X}_i$: $x_i \leq y_i \implies Q_i(x_i) \leq Q_i(y_i)$.

(b) $p_j(x) \geq 0$ and $\sum_{i=1}^n p_i(x) \leq 1$ for all $j \in N$ and $x \in \mathcal{X}$.

2. $t: \mathcal{X} \rightarrow \mathbb{R}^n$ is given by:

$$t_i(x) = p_i(x)x_i - \int_{\underline{x}_i}^{x_i} p_i(y_i, x_{-i}) dy_i.$$

Then, (p, t) is an optimal direct auction mechanism.

The Regular Case

Regularity assumption: The function

$$c_i(x_i) = x_i - \frac{1 - F_i(x_i)}{f_i(x_i)}$$

is strictly increasing for each $i \in N$.

Corollary *Suppose that regularity is satisfied.*

Define $p : \mathcal{X} \rightarrow \mathbb{R}^n$ as follows: If $\max\{c_i(x_i) : i \in N\} < 0$, then the seller keeps the object. Otherwise, the seller gives the object to the bidder with highest $c_i(x_i)$ value (breaking ties arbitrarily).

Define $t : \mathcal{X} \rightarrow \mathbb{R}^n$ by:

$$t_i(x) = p_i(x)x_i - \int_{\underline{x}_i}^{x_i} p_i(y_i, x_{-i}) dy_i.$$

Then, (p, t) is an optimal direct auction mechanism.

An Example

Suppose the X_i is distributed uniformly on $\mathcal{X}_i = [0, 1]$, for each $i \in N$. That is, $f_i(x_i) = 1$, $F_i(x_i) = x_i$, and

$$c_i(x_i) = 2x_i - 1.$$

Regularity is satisfied.

By symmetry of this example, the bidder with the highest $c_i(x_i)$ is the bidder with the highest valuation.

Therefore, the optimal auction allocates the object to the bidder with the highest valuation x_i iff $c_i(x_i) \geq 0$ iff $x_i \geq 1/2$. If the highest valuation is less than $1/2$, then the seller keeps the object.

Notes:

- The optimal auction entails a reserve price of $1/2$.
- The optimal auction is ex-post inefficient.