# e201b: homework iii—suggested answers

## the chain store paradox paradox

(a) All the game trees are annexed to the end of this document.

(b) To find a sequential equilibrium, consider what $E$ ought to play given his beliefs AND the rational actions of both incumbent types, $I_T$ and $I_W$. The tough incumbent, $I_T$, will always find it optimal to fight, whereas the weak incumbent will acquiesce. If the E plays IN then he gets an expected payoff of $Eu_E(\text{IN}) = \pi_0(b-1) + (1-\pi_0)b$, whereas if he plays OUT he gets zero for sure. A sequential equilibrium for this game is, if $\pi_0(b-1) + (1-\pi_0)b \geq 0$, the profile $((\text{IN}, (\pi_0, 1-\pi_0)), (F, A))$, where for the entrant we specify an action and beliefs, and for the incumbent(s) we specify an action at each possible node ($F$ if $I_T$ and $A$ if $I_W$). If, on the other hand, $\pi_0(b-1) + (1-\pi_0)b < 0$, a sequential equilibrium profile is $((\text{OUT}, (\pi_0, 1-\pi_0)), (F, A))$.

(c) When $\gamma \neq 1$, there is a unique equilibrium. Why? Well, in this case, then, since $\gamma = (\pi_0/(1-\pi_0))\,(1-b)/b$, it follows, after simple arithmetic, that

$$\gamma \gtrless 1 \Leftrightarrow \pi_0(1-b) \gtrless (1-\pi_0)b \Leftrightarrow 0 \gtrless (1-\pi_0)b + \pi_0(b-1),$$

that is, if it's optimal for the entrant to either play OUT or IN.

(d) If $\gamma = 1$, then $E$ is indifferent between playing OUT or IN, so any mixing between them will be a sequential equilibrium. Formally, $\forall \varepsilon \in [0,1]$, $((\varepsilon\text{IN}+(1-\varepsilon)\text{OUT}, (\pi_0, 1-\pi_0)), (F, A))$ is a sequential equilibrium.

(e) Suppose that $\gamma > 1$. Then, in the one-shot game, an entrant wouldn't enter. We want to show that there is an equilibrium where both incumbents play $A$ in the first period. This is a pooling equilibrium. If both incumbents decide to acquiesce in the first period, then the first entrant should enter the market. The second entrant's posteriors (after conditioning on the the observation that the first entrant entered and the incumbent acquiesced) are just his priors, since both incumbents are assumed to adopt the same first-period strategy. Hence, by (a), the second entrant decides to stay out. Now it remains to show that the incumbents' first-period strategy is rational. We need to show that there are no profitable deviations available to either incumbent. Deviation here means that the incumbents fight in the first period. Notice that this event is OFF THE EQUILIBRIUM PATH, so we can assign any (consistent) beliefs to the second entrant after the probability-zero event. We'll say that the entrant thinks that only weak incumbents fight (we'll show that this is indeed consistent). If $I_T$ fights, then he'll get a payoff of zero in the first period, but then the second entrant will choose to enter (that's what his beliefs tell him to do) and so the TOTAL payoff to $I_T$ as a result of the deviation will be $0 + 0 = 0 < -1 + a$, where $a - 1$ is the total payoff from playing the prescribed strategy. Hence, $I_T$ prefers to acquiesce in the first period. That $I_W$ doesn't want to deviate either is easy, so I will omit the proof.

Are such beliefs consistent? If we get the incumbents to mix in the first period between $F$ and $A$ with $\Pr(F|I_W) = \varepsilon$ and $\Pr(F|I_T) = \varepsilon^2$, then as $\varepsilon \to 0$, beliefs for the second entrant, together with his best response, converge to the proposed equilibrium and we have consistency. Thus, the equilibrium is sequential, as required.

## moral hazard

(a) The principal wishes to OPTIMALLY induce action $a^*$. This means that the wage schedule he offers the agent makes the agent prefer $a^*$ to any other action at lowest cost to the principal. We write the principal's problem below.

$$\min_{(w_s)_{s=1}^S} \sum_{s=1}^S w_s p_s(a^*) \quad \text{s.t.} \tag{*}$$

$$\sum_{s=1}^S (v(w_s) - c(a^*))p_s(a^*) \geq \sum_{s=1}^S (v(w_s) - c(a))p_s(a), \quad \forall a < a^* \tag{IC$_a$}$$

$$\sum_{s=1}^S (v(w_s) - c(a^*))p_s(a^*) \geq U. \tag{IR}$$

Let $\lambda_a$ denote the Lagrange multiplier associated with (IC$_a$) and $\mu$ that for (IR). Then the first-order conditions for this problem are

$$-p_s(a^*) + v'(w_s) \sum_{a<a^*} \lambda_a(p_s(a^*) - p_s(a)) + \mu v'(w_s)p_s(a^*) = 0,$$

which, after rearranging, yields

$$\frac{1}{v'(w_s)} = \mu + \sum_{a<a^*} \lambda_a \left(1 - \frac{p_s(a)}{p_s(a^*)}\right). \tag{FOC}$$

We're asked to find sufficient conditions under which $w_{s+1} - w_s \geq 0$ for every $s$. Noting that $w_{s+1} - w_s \geq 0 \Leftrightarrow (1/v'(w_{s+1})) - (1/v'(w_s)) \geq 0$, and using the FOCs, we see that

$$\frac{1}{v'(w_{s+1})} - \frac{1}{v'(w_s)} = \sum_{a<a^*} \lambda_a \left(\frac{p_s(a)}{p_s(a^*)} - \frac{p_{s+1}(a)}{p_{s+1}(a^*)}\right) \geq 0$$

if the following condition on probabilities holds for every state $s$ and every action $a < a^*$:

$$\frac{p_s(a)}{p_s(a^*)} \geq \frac{p_{s+1}(a)}{p_{s+1}(a^*)}.$$

This condition is known as the MONOTONE LIKELIHOOD RATIO PROPERTY.

(b) If $v(w) = 1 - \exp(-\gamma w)$, then $v'(w) = \gamma \exp(-\gamma w)$, and we get that

$$\text{“}MRS_{s+1,s}\text{”} = \frac{v'(w_{s+1})}{v'(w_s)} = \exp(-\gamma(w_{s+1} - w_s)) = \frac{\mu + \sum_{a<a^*} \lambda_a \left(1 - \frac{p_s(a)}{p_s(a^*)}\right)}{\mu + \sum_{a<a^*} \lambda_a \left(1 - \frac{p_{s+1}(a)}{p_{s+1}(a^*)}\right)}.$$

Taking logs and dividing by $-\gamma$ we finally obtain that

$$w_{s+1} - w_s = \frac{1}{\gamma}\left[\ln\left(\mu + \sum_{a<a^*} \lambda_a \left(1 - \frac{p_{s+1}(a)}{p_{s+1}(a^*)}\right)\right) - \ln\left(\mu + \sum_{a<a^*} \lambda_a \left(1 - \frac{p_s(a)}{p_s(a^*)}\right)\right)\right]. \tag{$\Delta w_s$}$$

I hope this answers the question!

(c) Well, the only things that could potentially change in ($\Delta w_s$) as a result of a change in $U$ are the Lagrange multipliers. If you look at ($\Delta w_s$) closely you can see that the wage-increment from one state to the next is only affected by the relationship between the probabilities in that state and the next. At the optimum the individual rationality constraint ought to bind, so the expected wage should rise if $U$ rises. But that does not mean that the wage in every state should rise. However, I claim that the new wage-schedule is just a shifted version of the old one: $\forall \varepsilon > 0$,

$$\text{“}MRS_{s+1,s}\text{”} = \frac{v'(w_{s+1} + \varepsilon)}{v'(w_s + \varepsilon)} = \exp(-\gamma(w_{s+1} - w_s)) = \frac{\mu + \sum_{a<a^*} \lambda_a \left(1 - \frac{p_s(a)}{p_s(a^*)}\right)}{\mu + \sum_{a<a^*} \lambda_a \left(1 - \frac{p_{s+1}(a)}{p_{s+1}(a^*)}\right)},$$

so two wage-schedules that differ only by a constant provide the agent with the same incentives. Assuming monotonicity, etc., so that the principal's problem leads to a unique solution, we can then infer that with a higher $U$ the principal picks the $\varepsilon$ that makes the IR constraint bind. (**If you disagree with this reasoning, please let me know!**)

(d) My argument here is the following. As $U$ increases, the wage schedule shifts up to satisfy the agent's IR constraint until, at some point, it's better for the principal to induce a different level of effort. It cannot be a higher effort-level because then, in order to induce such higher level, the principal needs to create more risk (or less insurance) on the agent. But if this is the case, then the agent needs a higher expected wage as compensation for the increased risk to satisfy his IR constraint. On the other hand, inducing a lower effort means that the agent faces less risk, so in order for the IR constraint to bind the principal can offer the agent a lower expected wage. The same caveat applies here, too: **if you disagree with this reasoning, please let me know!**

## adverse selection

Here, the states are private, rather than common. That is, each agent faces an individual risk of being endowed with either 2 (with probability $p$) or 0 (with probability $1-p$). Let agents be indexed by $a \in [0.1]$. Interpret $a$ as an agent's student ID, or name. The first-best allocation is found by solving the planner's problem. We need to put some structure to the uncertainty. I'll be loose and simply say that individual risks are *i.i.d.* without specifying how this works in the continuum, and I'll "make up" a law of large numbers saying that in the aggregate there is no risk: the aggregate endowment is simply $2p$. This says that $p$ percent of the population receive an endowment of 2 and $1-p$ percent receive nothing. The first-best allocation ought to equalize marginal rates of substitution, which in this case translates to equalizing consumption: $c_a = 2p$, $\forall a \in [0.1]$, so that aggregate consumption $c = \int_0^1 c_a da = 2p$ and the resource constraint is not violated.

To show that there is no incentive-compatible mechanism that improves upon autarky, consider the IC constraints from a mechanism that delivers a random transfer contingent on the reported type of an agent and resources reported are the resources reallocated. Let $(\widetilde{c}_P, \widetilde{c}_R)$ be lotteries offered as part of an incentive-compatible mechanism to the agents. Then

$$\begin{aligned} -E\exp(-\widetilde{c}_R - 2) &\geq -E\exp(-\widetilde{c}_P - 2) \\ -E\exp(-\widetilde{c}_P) &\geq -E\exp(-\widetilde{c}_R), \end{aligned}$$

which implies that $E\exp(-\widetilde{c}_R) = E\exp(-\widetilde{c}_P)$, and the result follows.

## education and employment

Pooling Equilibria—There is always a pooling equilibrium: both types of worker accept any wage and neither gets educated, and the firm offers the low wage.

Separating Equilibria—There is also always a separating equilibrium if $\overline{w} - c(\underline{\theta}) < \underline{w} < \overline{w} - c(\overline{\theta})$: the firm offers a high wage to educated workers and a low wage to uneducated workers, the highly productive worker gets educated and accepts only the high wage, and the low-productivity worker doesn't get educated and accepts any wage.

Semi-separating Equilibria—Let $\widehat{\theta} = p\overline{\theta} + (1-p)\underline{\theta}$. When is there a possible posterior belief, $q \in [0,1]$ such that $\widehat{\theta} - \underline{w} \leq q\overline{\theta} + (1-q)\underline{\theta} - \overline{w}$? That is, when is it worth the firm to offer wages that depend on education? Suppose that the firm offers a high wage if the worker is educated and a low wage if not. If $\overline{w} - c(\underline{\theta}) = \underline{w}$, then the low-productivity worker will be indifferent between getting an education and not getting one. Hence we can in principle get him to mix to generate a $q$ that will make it rational for the firm to offer the wage-schedule contingent on education. Playing with the condition that $\widehat{\theta} - \underline{w} \leq q\overline{\theta} + (1-q)\underline{\theta} - \overline{w}$ we obtain that $q \geq p - \Delta w/\Delta\theta$, where $\Delta w = \overline{w} - \underline{w}$, and similarly for $\Delta\theta$. Now, this will yield a semi-separating equilibrium when $p - \Delta w/\Delta\theta \in [0,1]$. In this case, any mixing that generates a posterior $q = \Pr(\overline{\theta}|\textit{education})$ for the firm such that $q \in [p - \Delta w/\Delta\theta, 1]$ will characterize a semi-separating equilibrium.

## courtroom drama

(a) Tree is annexed.

(b) First the **pooling equilibria**, where the defendant's posterior's are his priors. If $P_W$ and $P_L$ (the winning plaintiff and the losing plaintiff) both play $H$, then $D$ will find it optimal to play $R$, since $Eu_D(R|H) = -4/3 > -2 = Eu_D(A|H)$. Basically, if $D$ rejects everything, then both $P_W$ and $P_L$ find that everything is a best response. However, if both $P_W$ and $P_L$ play $L$, then $D$ ought to accept, since $Eu_D(R|L) = -4/3 < -1 = Eu_D(A|L)$. So the pooling equilibrium will be one where both plaintiffs play $H$, $D$ rejects $H$ and $D$ rejects $L$. The (consistent) beliefs that support $D$'s decisions are the priors at the information set where $H$ is offered and $\Pr(P_W|L) \leq 1/4$ in the other information set. (You can check that these beliefs are consistent—try the trembles where $\Pr(L|P_W) = \varepsilon^2 + \gamma\varepsilon$ and $\Pr(L|P_L) = \varepsilon$ where $\gamma \leq 2/3$.) In these equilibria, we have $D$ playing $(R, R)$. Given this strategy, both plaintiffs are indifferent between playing $L$ or $H$. So if they mix without causing $D$'s posteriors to change his optimum decisions $(R, R)$ then that would be an equilibrium.

There are no **separating equilibria** because if there were, then $D$ would reject $P_L$'s offer and accept $P_W$'s offer. But in that case both players would find it profitable to deviate, that is, pretend they were the other type.

Finally, let's look for **semi-separating equilibria** ($S^2E$), apart from those mentioned just now. Getting $D$ to mix in one information set and not in the other won't yield a $S^2E$, since both plaintiffs will want to play pure and break the incentives for $D$ to mix. But having $D$ mixing on both information sets will also break down, as a quick check shows.

The point here is that BOTH plaintiffs want to hide amongst each other; they have no reason to separate each other—not even partially.