

## Econ 504 (2008) Problem Set #1

### Warm up:

1. Do the following exercises from the first homework of John Nachbar's Econ 504 (Spring 2007): 2(a), 4, and 7.
2. Do the following exercises from the Osborne-Rubinstein book. In the exercises followed by a \*, restrict attention to pure strategies.
  - (a) (Rationalizability=IESDS) 56.5.
  - (b) (Pure Strategy NE) 19.1\*. Assume that citizens are uniformly distributed, i.e.  $f(x) = 1$ .
  - (c) (Mixed Strategy NE) 36.2. Assume that  $v_1 = 3$ ,  $v_2 = 2$ , and  $v_3 = 1$ .
  - (d) (Pure Strategy BNE) 28.1\*.
  - (e) (SPE) 103.2\*.
  - (f) (Sequential Equilibrium) 226.1.
3. Consider the signaling game in the last page.
  - (a) Find a separating sequential equilibrium.
  - (b) Find a pooling sequential equilibrium. Do not forget to specify beliefs at off the equilibrium path information sets. Is there a pooling equilibrium that survives the Intuitive Criterion?
  - (c) Find a sequential equilibrium where a type of player 1 plays both L and R with positive probability.

### Medium difficulty:

1. (Existence of Symmetric NE) A two-player finite normal form game  $G = (\{1, 2\}, A, u)$  is **symmetric** if (i)  $A_1 = A_2$  and (ii)  $u_1(a, b) = u_2(b, a)$  for any  $a, b \in A_1 = A_2$ . A mixed strategy NE  $(\sigma_1, \sigma_2)$  of this game is symmetric if  $\sigma_1 = \sigma_2$ . Show that every symmetric two-player finite normal form game has a symmetric NE.
2. (Perfect Information Bargaining) Consider a modification of the alternating offers bargaining game studied in class where Player 1 makes offers in periods  $3k + 1$  and  $3k + 2$  and Player 2 makes offers in periods  $3k + 3$ , for  $k = 0, 1, 2, \dots$

- (a) Conjecture a SPE strategy profile of the players. (Hint: Suppose that the game has a unique SPE and let  $V_i(l)$  be Player  $i$ 's SPE payoff in the subgame starting in period  $3k + l$ , where  $i = 1, 2$ ,  $l = 1, 2, 3$ , and  $k = 0, 1, 2, \dots$ . One can express  $(V_1(1), V_2(1))$  in terms of  $(V_1(2), V_2(2))$ ,  $(V_1(2), V_2(2))$  in terms of  $(V_1(3), V_2(3))$ , and  $(V_1(3), V_2(3))$  in terms of  $(V_1(1), V_2(1))$ , solve for these six values, and identify the corresponding strategies.) Verify that the strategies you have found indeed form an SPE by using the single deviation property.
- (b) Is there immediate agreement in equilibrium? Show that it is an advantage for player 1 to be able to make more frequent offers than in the original alternating offers model. Note in particular that as  $\delta \rightarrow 1$ , the cake is divided in proportion to number of offers made by each player.
- (c) Show that the strategy profile you constructed in part (a) is the unique SPE.
3. (Perfect Information Bargaining) Consider a discrete version of the two player infinite horizon bargaining game studied in class where the set of feasible offers is a nonempty finite subset

$$A \subset \{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0, x + y = 1\}.$$

Except for the change that  $A$  is now finite, the rest of the game remains the same.<sup>1</sup> Find  $\bar{\delta} \in (0, 1)$  such that for any  $\delta_1, \delta_2 \in (\bar{\delta}, 1)$ , and  $(x^*, y^*) \in A$ , there is a SPE where player 1 offers the division  $(x^*, y^*)$  in period 1 and player 2 immediately accepts. How do you contrast this with the uniqueness result from the class?

4. (Perfect Information Bargaining) Consider an infinite horizon bargaining game with three players  $N = \{1, 2, 3\}$ . In each period  $t$ , one of the players is randomly selected to make an offer: Player 1 is selected with probability  $\frac{1}{2}$ , each one of Players 2 and 3 is selected with probability  $\frac{1}{4}$ . The selected Player  $i$  offers a division of the cake  $(x_t, y_t, z_t)$  where  $x_t, y_t, z_t \geq 0$  and  $x_t + y_t + z_t = 1$  ( $x_t$  denotes player 1's share  $y_t$  denotes player 2's share and  $z_t$  denotes player 3's share). The two other Players  $j$  and  $k$  observe  $i$ 's offer  $(x_t, y_t, z_t)$ , then  $j$  and  $k$  *simultaneously* accept or reject this offer. If both  $j$  and  $k$  accept then the division is carried out, if at least one of them rejects then the offer is rejected and they proceed to period  $t + 1$ .

---

<sup>1</sup>That is, the payoffs, the timeline, and the disagreement payoff of  $(0, 0)$  remains the same.

Players maximize discounted expected payoffs and have the common discount factor  $\delta \in (0, 1)$ . If no offer is ever accepted, then each player receives a payoff of zero. The selection of who makes an offer is i.i.d. across periods.

- (a) Conjecture an SPE where (as usual) players accept any division where they receive at least  $\delta$  times their continuation payoff. Write down formally the strategy profile and verify that it is indeed an SPE by using the single deviation property.
- (b) For any division  $(x, y, z)$ , construct an SPE where the cake is divided according to  $(x, y, z)$  in the first period, no matter who makes the offer.

**Difficult:**

1. (Reputation) Consider the following two player extensive form game with perfect information. The game starts by player 1's (the principal) decision to hire (H) or to not hire (N) player 2 (the agent). If player 1 does not hire, then the game ends with zero payoffs to both parties. If she chooses to hire, then player 2 decides to work (W) or to shirk (S) where:

$$u_1(H, W) = 1, u_1(H, S) = -1, u_2(H, W) = 1, u_2(H, S) = 2.$$

Consider a finite repetition of the above game, where a single long-lived agent faces a sequence  $1, \dots, K$  of short-run principals. All players observe past history of actions, payoff of the  $k$ th principal is her payoff at stage  $k$ , payoff of the agent is the sum of his payoffs in all stages.

- (a) What is the subgame perfect equilibrium of the repeated game?
- (b) Suppose now that before the game starts, nature determines a type for the agent in  $\{R(egular), D(iligent)\}$ :  $Prob(R) = 1 - \epsilon, Prob(D) = \epsilon > 0$ . The diligent type agent always works if hired.<sup>2</sup> Payoffs and strategies available to the regular agent and the principals are same as before. The agent knows his type but the entrants don't. Let  $\mu_k$  denote the belief of the principals that the agent is the diligent type.

---

<sup>2</sup>You may assume that it is physically impossible for the diligent type to shirk.

i. Find a sequential equilibrium where the belief process satisfies:

$$\mu_{K-l+1} = \begin{cases} \mu_{K-l} & \text{if the agent is not hired at } K-l \\ 0 & \text{if } (H, S) \text{ at } K-l \\ \max\left\{\left(\frac{1}{2}\right)^l, \mu_{K-l}\right\} & \text{if } (H, W) \text{ at } K-l \end{cases}$$

for  $l = 1, \dots, K-1$ , given the initial condition  $\mu_1 = \epsilon$ .

- ii. What is the sequential equilibrium payoff of the regular agent?  
 iii. (Optional) Show that if  $\epsilon \neq \left(\frac{1}{2}\right)^l$  for  $l = 1, \dots, K$ , then on the equilibrium path strategies and beliefs in any sequential equilibrium is unique.

2. (Weak Domination Theorem) In a normal form game, an action  $a_i$  is **weakly dominated** to a mixed strategy  $\sigma_i$  if:

$$\forall a_{-i} \in A_{-i} : u_i(\sigma_i, a_{-i}) \geq u_i(a_i, a_{-i}), \text{ and}$$

$$\exists a_{-i} \in A_{-i} : u_i(\sigma_i, a_{-i}) > u_i(a_i, a_{-i}), \text{ and}$$

Prove that in a finite normal form game, an action  $a_i^*$  is never a best reply to any (possibly correlated) completely mixed conjecture  $\sigma_{-i}$  of  $i$ , if and only if  $a_i^*$  is weakly dominated to a mixed strategy  $\sigma_i$ .<sup>3</sup>

---

<sup>3</sup>The correlated conjecture  $\sigma_{-i} \in \Delta(A_{-i})$  is completely mixed if  $\sigma_{-i}(a_{-i}) > 0$  for all  $a_{-i} \in A_{-i}$ .

# A Signaling Game

