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Topics in Advanced Theory I
GAME THEORY

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Bargaining under Incomplete Information

Strategic Delay in Bargaining

Admati & Perry (1987)

Alternating Offers with Strategic Delays

A seller (S) and a buyer (B) bargain over the price p of an indivisible object in continuous time ($t \in \mathbb{R}_+$). B's valuation is $v > 0$, S's valuation is 0.

The first offer p_1 is made by S at any time $t_1 \geq 0$. Subsequently players make alternating offers until they agree on a price p.

A response (acceptance or counteroffer) to an offer made at time t_k can be made at any time $t_{k+1} = t_k + 1 + \Delta_{k+1}$ where $\Delta_{k+1} \geq 0$. Interpretation:

The minimum physical time to react to an offer is one unit, but players can strategically delay their responses for any additional amount of time.

The SPE under Complete Information

A non-terminal history: $h^n = (p_k, \Delta_k)_{k=1}^n$.
($t_1 = 1 + \Delta_1$, $\Delta_1 \geq -1$, $\Delta_2, \Delta_3, \ldots \geq 0$.)

Here Δ_{k+1} is the time delay in round $k+1$ beyond the minimum of one, after offer p_k is made.

Total passage of time in to h^n is $t = n + \sum_{k=1}^n \Delta_k$.

If players agree on price p at time t, then the payoffs are:

$$u_S(p,t) = \delta^t p \qquad u_B(p,t) = \delta^t [v - p].$$

The unique SPE is the SPE of the standard alternating offers game without delay, where S immediately offers

$$p^* = \frac{v}{1+\delta}$$

which is accepted by B without additional delay.

Incomplete Information

Suppose that the B can have two types high valuation (B$_H$) or low valuation (B$_L$) where the valuations are $v_H > v_L > 0$. S's prior on B's type is $\pi^0 = Prob(B_H)$.

Note that B$_L$ is a "more patient" bargainer than B$_H$.

If the buyer's type were common knowledge then the corresponding complete information outcome

$$p_L \equiv \frac{v_L}{1+\delta} \quad \text{or} \quad p_H \equiv \frac{v_H}{1+\delta}$$

occurs with no delay, depending on B's type.

Could there be delay in the equilibrium of the incomplete information game?

Restrict attention to pure strategy sequential equilibria (β, π) where $\pi(h^n) = Prob(B_H|h^n)$.

Preliminary Observations

Lemma 2.1. *In any sequential equilibrium:*

(i) S never accepts an offer $p < \delta p_L$.

(ii) S always accepts an offer $p \geq \delta p_H$.

(iii) B never accepts an offer $p > p_H$.

(iv) B always accepts an offer $p \leq p_L$.

(v) B_H always accepts an offer $p \leq \bar{p}$, where $v_H - \bar{p} = \delta[v_H - \delta p_L]$. ($\bar{p} > p_L$)

(vi) If (p,t) and (p',t') are the eqm. outcomes for B_L and B_H respectively then $p \leq p'$ and $t \geq t'$.

(vii) Acceptance of an offer occurs with no additional delay.

Proof: Exercise.

Refinements

The incomplete information game has a large number of equilibria where off-equilibrium deviations of B are followed by optimistic beliefs of the buyer $Prob(B = B_H) = 1$.

Assumption 1 (A1) If a player can obtain the same payoff by making fewer offers, then she makes fewer offers.

Let h^n be a history that ends with an offer by S.

For $\theta \in \{L, H\}$, let $V^*_\theta(h^n, \Delta_{n+1}, p_{n+1})$ denote the supremum of the seq. eqm. payoffs of B$_\theta$ across *all* sequential equilibria of the subgame starting at $h^{n+1} \equiv (h^n, \Delta_{n+1}, p_{n+1})$ and *all* beliefs $Prob(B = B_H | h^{n+1}) \in [0, 1]$.

Given a sequential equilibrium let $U^*_\theta(h^n)$ be the sequential equilibrium continuation payoff of B_θ after h^n. Say that (Δ_{n+1}, p_{n+1}) is **bad** for B_θ in this equilibrium given h^n if

$$U^*_\theta(h^n) > V^*_\theta(h^n, \Delta_{n+1}, p_{n+1}).$$

Assumption 2 (A2) (The Intuitive Criterion) If a deviation (Δ_{n+1}, p_{n+1}) is bad for type B_θ and not for type $\mathrm{B}_{\theta'}$ given history h^n, then S puts zero probability on B_θ conditional on $h^{n+1} = (h^n, \Delta_{n+1}, p_{n+1})$.

Lemma 2.2. *In any sequential equilibrium that satisfies (A1) and (A2), consider a history h^n that ends with an offer of p by B:*

(i) If $\pi(h^n) = 0$, then S accepts p iff $p \geq \delta p_L$.

(ii) If $\pi(h^n) = 1$, then S accepts p iff $p \geq \delta p_H$.

Proof: Omitted, see A&P (1987).

Necessary Conditions under (A1) and (A2)

Given p let $\gamma^*(p)$ solve: $v_H - p = \delta^{\gamma^*(p)}[v_H - \delta p_L]$.

Proposition 3.1. *Suppose along an equilibrium path the first offer S makes is p at time t. Then,*

(i) $p \geq p_L$

(ii) If $p = p_L$, then both B_L and B_H immediately accept.

(iii) If $p > p_L$, then

- *(iiia) If B_H accepts, then B_L offers δp_L at time $t' = \max\{t+1, t+\gamma^*(p)\}$, which S accepts.*
- *(iiib) If B_H does not accept, then B_H offers δp_H at time $t+1$, which S accepts, and B_L offers δp_L at time $t' = t + 1 + \gamma^*(\delta p_H)$, which S accepts.*

Existence of Equilibrium and Uniqueness of the Equilibrium Path under (A1) and (A2)

There exists $\pi^* < v_L/v_H$ such that:

1. (Unique Pooling) If $\pi^0 < \pi^*$, then the unique equilibrium path is that S offers p_L at $t = 0$ and both types accept.

2. (Unique Separating) If $v_L/v_H < \pi^0$, then there exists a unique equilibrium path where S offers p_H at $t = 0$, B_H accepts and, B_L counteroffers δp_L at time $\gamma^*(p_H)$, which S accepts.

3. (Multiple Equilibria) if $\pi^* < \pi^0 < v_L/v_H$, then both of the above equilibrium paths as well as others exist.

Proofs: Omitted, see A&P (1987).

Foundations of Dynamic Monopoly and the Coase Conjecture

Gul, Sonnenschein & Wilson (1986)

Dynamic Monopoly

Time horizon: $t = 0, 1, 2 \ldots$ (infinite).

Players: A monopolist faces a continuum of consumers uniformly distributed on [0,1].

The monopolist produces a good at zero marginal cost. Each consumer may buy one unit of this good.

In each period t, the monopolist sets a price p_t.

Then the consumers who have not already bought the good in the past, (simultaneously) decide whether to buy one unit at price p_t.

History of past prices and the mass of buying consumers is observed by all players.

Payoffs

Consumer $q \in [0,1]$ has valuation $v = f(q)$ where $f : [0,1] \to \mathbb{R}_+$ left-continuous and non-increasing.

All consumers and the monopolist have the common discount factor $\delta \in (0,1)$.

If consumer $q \in [0,1]$ buys good at period t at price p, her payoff is $\delta^t[f(q) - p]$.

If she never buys her payoff is zero. The monopolist's payoff is the discounted value of profits.

Remark

Suppose that the monopolist posts price p_t at time t.

If a consumer with valuation v prefers to buy at price p_t iff $v \geq p_t$ and $\forall s \geq 0 : v - p_t \geq \mathbb{E}_t[\delta^s(v - p_{t+s})] \Leftrightarrow v \geq \frac{p_t - \delta^s \mathbb{E}_t p_{t+s}}{1-\delta^s}$.

(Since every consumer has zero mass and the mass of remaining consumers after any single consumer deviates remains the same.)

Hence q prefers to buy at $t \Rightarrow$ all consumers in $[0, q)$ prefer to buy at t.

We will only analyze subgames starting after histories in which all consumers less than a cutoff value $q \in [0, 1]$ *have already bought the good and left the market, and those in* $[q, 1]$ *still remain in the market.*

The Coase Conjecture

Definition: An equilibrium is **stationary** if the state of the market after any price that is lower than all preceding prices is independent of the earlier price history in the market.

Note: In a stationary equilibrium the sets of consumers accepting and rejecting depend only on the current price.

Theorem (Coase Conjecture): *For each $\epsilon > 0$, there exists $\overline{\delta} < 1$ such that for all $\delta \in (\overline{\delta}, 1)$ and for all stationary equilibria: $p_0 < \epsilon$.*

Corollary: *For each $\epsilon > 0$, there exists $\overline{\delta} < 1$ such that for all $\delta \in (\overline{\delta}, 1)$ and for all stationary equilibria, consumer q obtains an equilibrium payoff of at least $f(q) - \epsilon$.*

Example with Linear Demand: $f(q) = 1 - q$

Consider the subgame right before the monopolist determines the price at t, given that consumers $[q_t, 1]$ still remain in the market. By symmetry of such subgames to the original game: *Conjecture an SPE where*

$$p_t(q_t) = p_0(1 - q_t) \text{ where } \beta \equiv p_0 \in (0, 1).$$

Let q_{t+1} denote the cutoff consumer who is indifferent between buying at t or at $t + 1$. Given the monopolist's strategy and that consumers $[q_t, 1]$ are in the market at t, (q_{t+1}, p_{t+1}) are determined through:

$$1 - q_{t+1} = \frac{p_t - \delta p_{t+1}}{1 - \delta} \text{ and } p_{t+1} = \beta(1 - q_{t+1}).$$

$$\Rightarrow p_{t+1} = \frac{\beta}{\gamma} p_t \text{ and } (1 - q_{t+1}) = \frac{1}{\gamma} p_t \text{ where } \gamma \equiv 1 - \delta + \beta\delta.$$

Path of prices and quantities:

$$p_{t+s} = \left(\frac{\beta}{\gamma}\right)^s p_t,\ (1 - q_{t+s}) = \frac{1}{\beta}\left(\frac{\beta}{\gamma}\right)^s p_t,\ \text{and } p_t = \beta(1 - q_t)$$

Given the price path (induced by the monopolists' conjectured strategy), the consumer strategy: "*buy at t if your valuation exceeds p_t/γ.*" is optimal.

The discounted sum of profits in the subgame are:

$$\begin{aligned} V(q_t) &= \sum_{s=0}^{\infty} \delta^s p_{t+s}(q_{t+s+1} - q_{t+s}) \\ &= \frac{\gamma\beta(\gamma - \beta)}{\gamma^2 - \delta\beta^2}(1 - q_t)^2. \end{aligned}$$

There is no profitable single deviation by the monopolist at time t, if $p_t = \beta(1 - q_t)$ solves:

$$\max_{p_t \in [0, \gamma(1-q_t)]} \quad p_t(q_{t+1} - q_t) + \delta V(q_{t+1})$$
$$\text{s.t.} \quad 1 - q_{t+1} = p_t/\gamma.$$

SOC holds. FOC: $1 - q_t = 2p_t \frac{\gamma - \delta\beta}{\gamma^2 - \delta\beta^2}$ at $p_t = \beta(1 - q_t)$:

$$\Rightarrow 1 = 2\frac{\gamma/\beta - \delta}{(\gamma/\beta)^2 - \delta} \;\Rightarrow\; \beta = \frac{\bar{\delta}}{1 + \bar{\delta}} \text{ where } \bar{\delta} = \sqrt{1 - \delta}.$$

Properties of this Equilibrium as $\Delta \searrow 0$ ($\delta = e^{-r\Delta}$)

Coase conjecture: $p_t \to 0$ for all $t = 0, 1, \ldots$

Consumer payoffs: Payoff of $q \to 1 - q$.

Monopoly profits: $V(0) \to 0$.

(Almost) immediate agreement: Let $t(q)$ be the period in which q trades, then $t(q)\Delta \to 0$ for all $q \in [0, 1)$.

Bargaining and Reputation

Abreu & Gul (2000)

Bargaining with Irrational Types

Two players bargain over the division of a unit surplus in continuous time $t \geq 0$.

Each player has one irrational type. The irrational type of player $i \in \{1, 2\}$ always demands a fixed number $\alpha^i \in (0, 1)$ and accepts an offer iff she receives at least α^i. Assume

$$\alpha^1 + \alpha^2 > 1$$

The prior probability that i is irrational is $z^i \in (0, 1)$.

Player i's rate of time preference is $r^i > 0$. That is, the payoff of the rational type of player i from receiving β^i at time t is $e^{-r^i t}\beta^i$.

A bargaining game is given by $B = (\alpha^i, z^i, r^i)_{i=1,2}$.

Modeling Continuous-Time Bargaining as a War of Attrition

We will analyze the equilibria of the following game.

The War of Attrition:
Starting at $t = 0$, at each instant of time, each player decides to concede to the other player or to continue. If the first player to concede is i, then j receives his demand α^j and i receives $1 - \alpha^j$.

Justification/Motivation:
The equilibria of the above game are the limit of equilibria of discrete-time bargaining games in which both players make offers frequently (Proposition 4 in Section 4).

Note that unlike the standard war of attrition, the above game has irrational types.

Strategies

Players condition their concession behavior only on t. (This is wlog because there exists a *unique* nonterminal history h_t at any time t: h_t =neither player has conceded by t).

A strategy for player i is a cdf F^i over $\mathbb{R}_+ \cup \{+\infty\}$.

Let $j \neq i$, then $F^i(t)$ is the probability that player i concedes to player j some time in $[0, t]$.

$F^i(t)$ is player j's belief that player i will concede by time t, unconditional on the rationality of player i. Since irrational types never concede, the actual concession behavior of the rational player i is given by $\frac{F^i}{1-z^i}$.

As a result, a strategy F^i is required to satisfy:

$$\lim_{t \to +\infty} F^i(t) \leq 1 - z^i$$

Payoffs of Rational Types

Payoff of i from conceding at time $t < +\infty$ given F^j:

$$\begin{aligned} U^i(t, F^j) \;&=\; \alpha^i \int_{s<t} e^{-r^i s} dF^j(s) \\ &\quad + \frac{1}{2}(\alpha^i + 1 - \alpha^j) e^{-r^i t}(F^j(t) - F^j(t^-)) \\ &\quad + (1 - \alpha^j) e^{-r^i t}(1 - F^j(t)). \end{aligned}$$

Payoff of i from never conceding (equivalently from conceding at $t = +\infty$) given F^j:

$$U^i(+\infty, F^j) \;=\; \alpha^i \int_{s<\infty} e^{-r^i s} dF^j(s).$$

Payoff of i from the strategy profile (F^i, F^j)

$$U^i(F^i, F^j) = \frac{1}{1 - z^i} \int_{t \in [0,\infty]} U^i(t, F^j) dF^i(t)$$

Definition of Sequential Equilibrium

Definition: A strategy profile (F^1, F^2) is a **sequential equilibrium** if

$$U^i(F^i, F^j) \geq U^i(G^i, F^j)$$

for any strategy G^i of i, for all $i \in \{1, 2\}$ and $j \neq i$.

Note: There is a one-to-one correspondence between player i's strategy (defined in this paper as j's beliefs) and rational player i's actual concession behavior via the Bayes rule:

$$F^i \leftrightarrow \frac{F^i}{1 - z^i}$$

Hence, in this formulation, the consistency requirement of sequential equilibrium follows implicitly.

Description of the Sequential Equilibrium

- There exists a finite time T^0 until which each player i concedes at constant hazard rate $\lambda^i = \frac{r^j(1-\alpha^i)}{\alpha^j-(1-\alpha^i)}$:

  $\exists c^i \in [0,1]$ such that $\widehat{F}^i(t) = 1 - c^i e^{-\lambda^i t}$ for all $t \leq T^0$.
- At most one player concedes with positive probability at time zero: $(1-c^1)(1-c^2) = 0$.
- At T^0, posterior probability of irrationality of both players reach 1 and concessions stop: $1 - z^i = \widehat{F}^i(T^0)$.

We can pin down T^0, c^1, c^2 from above:

$$T^0 = \min\{(-\ln z^1)/\lambda^1, (-\ln z^2)/\lambda^2\} \text{ and } c^i = z^i e^{\lambda^i T^0}.$$

Note: Equilibrium exhibits delay (⇒ inefficiency).

Characterization

Proposition: *The unique sequential equilibrium is* $(\widehat{F}^1, \widehat{F}^2)$.

Proof: Let (F^1, F^2) be a seq. eqm. Let u^i_s denote the payoff of i from conceding at time s. Let $A^i := \{t : u^i_t = \max_s u^i_s\} \neq \emptyset$. Let $\tau^i = \inf\{t \geq 0 : F^i(t) = \lim_{s \nearrow +\infty} F^i(s)\}$.

(a) $\tau^1 = \tau^2$.

(b) If F^1 jumps at t, then F^2 does not jump at t.

(c) If F^i is continuous at t, then u^j_s is continuous at $s = t$.

(d) For any interval (t', t'') s.t. $0 \leq t' < t'' \leq \tau^1$:

- (d1) F^1 is constant on (t', t'') iff F^2 is constant on (t', t'').
- (d2) F^i is not constant on (t', t'').

(e) If $t' < t'' < \tau^1$, then $F^i(t'') > F^i(t')$ for $i = 1, 2$.

(f) F^i is continuous at $t > 0$.