Econ 506A (2008) Problem Set #1

1. (Perfect Information Bargaining) Derive the SPE of the two player infinite horizon bargaining game where the set of feasible expected utility pairs are:

$$X = \{(x_1, x_2) \in \mathbb{R}^2 \,|\, x_1 \geq 0, x_2 \geq 0, x_1^2 + x_2^2 \leq 1\}.$$

2. (Perfect Information Bargaining) Consider a discrete version of the general two player infinite horizon bargaining game studied in class where the set of feasible expected utility pairs that could be offered is a nonempty finite subset

$$X \subset \{(x_1, x_2) \in \mathbb{R}^2 \,|\, x_1 > 0, x_2 > 0, x_2 \leq g(x_1)\}.$$

Except for the change that X is now finite, the rest of the game[1] remains the same. Define the Pareto frontier of the feasible set as:

$$P = \{x \in X \,|\, \text{There does not exist } y \in X \text{ such that } y \geq x \,\&\, y \neq x\}$$

Find $\bar{\delta} \in (0,1)$ such that for any $\delta_1, \delta_2 \in (\bar{\delta}, 1)$, and $x^* \in P$, there is a SPE where player 1 offers x^* in period 0 and player 2 immediately accepts. How do you contrast this with the uniqueness result from the class?

3. (Perfect Information Bargaining) Two players bargain over time to sign a contract $x = (x_1, x_2) \in \mathbb{R}^2$. Player i has a discount factor $\delta_i \in (0,1)$. If a contract x is signed at period $t = 0, 1, 2, \ldots$ then player i's payoff is $\delta_i^t x_i$, if a contract is never signed then each player gets a disagreement payoff of zero. The set of contracts that player i can offer to player j is given by:

$$X^i = \left\{(x_1, x_2) \in \mathbb{R}^2 | \alpha_i x_i + (1 - \alpha_i) x_j = 1, x_1, x_2 \geq 0\right\}$$

where $\alpha_1, \alpha_2 \in (0,1)$ and $\alpha_1 + \alpha_2 \geq 1$.

Player 1 makes offers in even periods, and player 2 in odd periods. Player i offers a contract $x \in X^i$, if player j accepts then the contract x is signed and the game ends; if player j rejects, then we proceed to the next period when j makes an offer. Note that the game reduces to the standard (split-the-pie) alternating offers bargaining game when $\alpha_1 + \alpha_2 = 1$.

[1] That is, the payoffs, the timeline, and the assumptions that g is continuous, concave, strictly decreasing with $g(0) > 0$, and the disagreement payoff is $D = (0, 0)$.

(a) Find a SPE and argue that it is unique.

(b) What happens to the SPE payoffs as $(\delta_1, \delta_2) \to (1, 1)$? Is there a fundamental difference between the cases $\alpha_1 + \alpha_2 > 1$ and $\alpha_1 + \alpha_2 = 1$? How do you interpret that?

4. (Perfect Information Bargaining) Two players bargain over time in order to take a joint decision $x = (x_1, x_2) \in [0, 1]^2$. Player i's "ideal point" is $\bar{x}^i \in [0, 1]^2$, his payoff from reaching the decision x at time $t = 1, 2, \ldots$ is:

$$u_i(x, t) = \delta_i^{t-1}[1 - (x_1 - \bar{x}_1^i)^2 - \gamma(x_2 - \bar{x}_2^i)^2]$$

and his payoff from disagreement is 0. Assume that $0 < \gamma < 1$, $0 < \delta_i < 1$, $\bar{x}^1 = (1, 0)$, and $\bar{x}^2 = (0, 1)$.

The bargaining process is as follows. In odd periods, player 1 makes an offer $x_1 \in [0, 1]$, player 2 may either reject the offer in which case we proceed to the next period, or he may accept and choose $x_2 \in [0, 1]$ in which case the decision is (x_1, x_2) and the game is over. In even periods, player 2 makes an offer $x_1 \in [0, 1]$, player 1 may either reject the offer in which case we proceed to the next period, or he may accept and choose $x_2 \in [0, 1]$ in which case the decision is (x_1, x_2) and the game is over.

(a) Verify that in SPE, at a subgame where player i just offered x_1, if player j accepts then he chooses $x_2 = \bar{x}_2^j$. Draw the achievable utility pairs U^i if j accepts some offer $x_1 \in [0, 1]$ of i:

$$U^i = \left\{ \left(u_i(x_1, \bar{x}_2^j, 0), u_j(x_1, \bar{x}_2^j, 0) \right) \in \mathbb{R}^2 \ : \ x_1 \in [0, 1] \right\}$$

for $i = 1, 2$ and $j \neq i$.

(b) Find an SPE and argue that it is unique. You do not need to solve for the equilibrium strategies explicitly, a graphical argument using U^1 and U^2 suffices.

(c) What happens to the SPE payoffs as $(\delta_1, \delta_2) \to (1, 1)$? Again a graphical argument suffices. How do you contrast this limiting result to the SPE payoffs of the standard alternating offers bargaining model as $(\delta_1, \delta_2) \to (1, 1)$?

5. (Admati and Perry (1987))

(a) Prove that if the buyer's valuation is common knowledge, then the unique SPE of the bargaining game is the SPE of the standard alternating offers game without additional delay.

(b) Prove part (iii) of Lemma 2.1.

(c) Construct a sequential equilibrium which does not satisfy the conclusions of Proposition 3.1. (So it also can not satisfy both (A1) and (A2).)

6. (Dynamic Monopoly) Fill in the following details from our discussion of the linear demand example:

   (a) Derive the stated explicit formula for $V(q_t)$.

   (b) Verify that the SOC for the monopolist's problem holds.

   (c) Derive the FOC for the monopolist's problem and solve it for β.

7. (Dynamic Monopoly) Consider the subgame perfect equilibrium of the dynamic monopoly model with linear demand. In this exercise allow for the monopolist's discount rate δ_M to be different from the consumers' common discount rate δ_C.

   (a) Solve for an SPE where the price set by the monopolist is a constant fraction of the highest valuation still in the market:

$$p_t(q_t) = p_0(1 - q_t) \text{ where } \beta \equiv p_0 \in (0, 1).$$

   (b) Suppose that $\delta_C = e^{-r_C\Delta}$ and $\delta_M = e^{-r_M\Delta}$ where $r_C, r_M > 0$ are fixed constants. Show that $p_0 \to 0$ as $\Delta \to 0$.

   (c) Show that for any price $\bar{p} \in (0, 1)$ and $\bar{\delta} \in (0, 1)$, there exist $\delta_C, \delta_M \in (\bar{\delta}, 1)$ for which $p_0 = \bar{p}$. Briefly interpret/contrast what you showed in (b) and (c) in terms of the Coase conjecture.

8. (Abreu and Gul (2000))

   (a) Prove that Equation (1) in p93 and $du_t^i/dt = 0\ \forall t \in (0, \tau^1)$ implies that F^j is differentiable on $(0, \tau^1)$ with continuous derivative f^j. (Hint: You need to explicitly compute f^j using the definition of a derivative.)

   (b) Prove that the solution of the differential equation following Equation (1) in p93 is $F^j(t) = 1 - c^j e^{-\lambda^j t}$ where c^j is an undetermined constant.

9. (Reputation formation) Go over the backwards induction argument used to derive the sequential equilibrium of the perturbation of the finitely repeated chain store game. Fill in the missing details.

10. (Reputation formation) Consider the following two player extensive form game with perfect information. The game starts by player 1's (the principal) decision to hire (H) or to not hire (N) player 2 (the agent). If player 1 does not hire, then the game ends with zero payoffs to both parties. If she chooses to hire, then player 2 decides to work (W) or to shirk (S) where:

$$u_1(H,W) = 1, u_1(H,S) = -1, u_2(H,W) = 1, u_2(H,S) = 2.$$

Consider a finite repetition of the above game, where a single long-lived agent faces a sequence $1, \ldots, K$ of short-run principals. All players observe past history of actions, payoff of the kth principal is her payoff at stage k, payoff of the agent is the sum of his payoffs in all stages.

(a) What is the subgame perfect equilibrium of the repeated game?

(b) Suppose now that before the game starts, nature determines a type for the agent in $\{R(egular), D(iligent)\}$: $Prob(R) = 1-\epsilon, Prob(D) = \epsilon > 0$. The diligent type agent always works if hired.[2] Payoffs and strategies available to the regular agent and the principals are same as before. The agent knows his type but the entrants don't. Let μ_k denote the belief of the principals that the agent is the diligent type.

i. Find a sequential equilibrium where the belief process satisfies:

$$\mu_{K-l+1} = \begin{cases} \mu_{K-l} & \text{if the agent is not hired at } K-l \\ 0 & \text{if } (H,S) \text{ at } K-l \\ \max\left\{\left(\frac{1}{2}\right)^l, \mu_{K-l}\right\} & \text{if } (H,W) \text{ at } K-l \end{cases}$$

for $l = 1, \ldots, K-1$, given the initial condition $\mu_1 = \epsilon$.

ii. What is the sequential equilibrium payoff of the regular agent?

iii. (Optional) Show that if $\epsilon \neq \left(\frac{1}{2}\right)^l$ for $l = 1, \ldots, K$, then on the equilibrium path strategies and beliefs in any sequential equilibrium is unique.

[2] You may assume that it is physically impossible for the diligent type to shirk.

11. (Reputation formation) Consider the following two player extensive form stage game. First, player 1 (the principal) decides to hire or not to hire player 2 (the agent). If she does not hire, then the game ends with zero payoffs to both parties. If she hires, then nature determines a type of task $T \in \{A, B\}$ each with probability $\frac{1}{2}$. Player 2 observes T and chooses a type of effort $e_\tau \in \{e_A, e_B\}$.[3] The payoffs are given by:

$$u_i(T, e_\tau) = \begin{cases} 1 & \text{if } T = \tau \\ -2 & \text{if } T \neq \tau \end{cases} \qquad i = 1, 2.$$

The above stage game is infinitely repeated, where the tasks are i.i.d. across time and a single long-lived agent faces a sequence of short-run principals. All principals observe past effort decisions in stages when there was hiring. Payoff of the principals are their stage game payoffs. Payoff of the agent is the discounted value of her stage game payoffs, her discount rate is $\delta \in (0, 1)$.

Before the game starts, nature determines a type for the agent in $\{R(egular), A\}$: $Prob(R) = 1 - \epsilon$, $Prob(A) = \epsilon > 0$. Payoffs and strategies of the regular agent and the principals are as in above. The A-type agent always performs e_A irrespective of the task. Let μ_k denote the sequential equilibrium belief of the principals that the agent is the A-type, let p_k^T denote their belief that the regular agent chooses e_T if she is hired at period k and the realized task is T.[4]

(a) Show that for large enough δ there is a sequential equilibrium with no hiring on the equilibrium path.

(b) Show that if $\epsilon > \frac{2}{3}$, then in all sequential equilibria there is no hiring on the equilibrium path.

(c) Show that in a sequential equilibrium, if the kth principal hires with positive probability then $p_k^T \geq \frac{1}{3}$ for $T = A, B$.

(d) Show that there is a function $f\colon (0, \frac{2}{3}] \longrightarrow (0, 1)$ such that:

i. For any k, if μ_k denotes the kth principal's belief after a public history when she hires with positive probability and μ_{k+1} denotes her posterior after hiring and observing e_A at stage k, then $\mu_{k+1} \geq f(\mu_k)$.

[3]Player 1 does *not* observe T. You can think of e_T as the type of effort appropriate to successfully accomplish the task T.

[4]This may be different from the actual probability with which the regular agent plays T since the latter may in principle also depend on the agent's private history of past tasks.

ii. f is continuous, strictly increasing and $f(t) > t$ for all $t \in (0, \frac{2}{3}]$.

(e) Show that for large enough δ, there is no hiring in all sequential equilibria with the following property: If $\mu_k = 0$, then from there on principals always hire and the agent always chooses the effort e_T that matches the task T.

12. (Repeated Games with Perfect Monitoring) Consider the infinite repetition of a finite two-player normal form game G. Assume that there is no public randomization device and restrict attention to pure strategies.

    (a) Suppose that $\bar{a}$ is a strictly enforceable action profile such that:

    $$(*) \quad u_i(a_i, \bar{a}_{-i}) - u_i(\bar{a}) < u_i(\bar{a}) - u_i(p_i, p_{-i}) \quad \forall i \in N, a_i \in A_i,$$

    where p_j denotes an action of j that min-maxes player $-j$. Construct a machine with two states that implements an SPE of $G(\infty)$ for δ close enough to 1, where $\bar{a}$ is played on the equilibrium path.

    (b) Suppose now that $(*)$ need not hold. Modify the machine in the previous part by adding more states such that it implements an SPE of $G(\infty)$ for δ close enough to 1, where $\bar{a}$ is played on the equilibrium path .

    (c) Show that the NEU condition is not necessary for the folk theorem if the NSM condition is not satisfied.[5]

13. (Repeated Games with Perfect Monitoring) Let $X = \{(x_1, x_2) \mid x_1, x_2 \geq 0, x_1 + x_2 = 1\}$ denote all possible divisions of a unit pie between two players. The ultimatum game is a two player extensive form game where initially player 1 offers $x \in X$, then player 2 observes x and decides to accept or reject. If she accepts then player i's payoff is x_i, if she rejects then both players have zero payoff. Consider the infinite repetition $G^\delta(\infty)$ of the ultimatum game with common discount rate $\delta \in (0, 1)$ where each player observes all past history of actions. Let $x \in X$. Show that there is $\bar{\delta} \in (0, 1)$ such that for all $\delta \in (\bar{\delta}, 1)$, there is a SPE of $G^\delta(\infty)$ where every period on the equilibrium path, player 1 offers x and player 2 accepts.

[5]The definitions of NEU and NSM are in the slides.