Econ 506A (2008) Problem Set #2

1. (Lattices) Given a lattice $(X, \geq)$:

   (a) Observe that for any $A \subset X$ if a supremum of A exists, then it is unique. Note that this implies that the $x \vee y$ (and similarly $x \wedge y$) is unique.

   (b) Show that the partial order of the lattice can be derived from the meet or the join operation. More specifically, prove that

   $$x \leq y \Leftrightarrow x \wedge y = x \Leftrightarrow x \vee y = y.$$

   (c) By definition, the meet and join operations are *commutative* ($x \wedge y = y \wedge x$,...). Show that these operations are also *associative* ($(x \wedge y) \wedge z = x \wedge (y \wedge z)$,...). Hence, for a finite subset of $A \subset X$, the meet $\bigwedge A$ and join $\bigvee A$ are well-defined, since they do not depend on the order with which the operation is applied to the members of A.

2. (Product lattices)

   (a) Verify that in a product lattice $x \wedge y = (x_\alpha \wedge y_\alpha)_{\alpha \in I}$ and $x \vee y = (x_\alpha \vee y_\alpha)_{\alpha \in I}$.

   (b) Observe that the partial order of the product lattice $\mathbb{R}^n$ where each dimension is endowed with the usual order of real numbers is indeed the usual partial order on $\mathbb{R}^n$ which can be found in the notes.

   (c) Show that not all lattices satisfy the *distributive* properties: $x \wedge (y \vee z) = (x \vee y) \wedge (x \vee z)$ and $x \vee (y \wedge z) = (x \wedge y) \vee (x \wedge z)$, but products of chains (e.g. $\mathbb{R}^n$) do.

3. (Complete Lattices) Let $(X, \geq)$ be a partially ordered set.

   (a) Show that if $(X, \geq)$ is a lattice, then for any finite set $A \subset X$, $\inf_X(A) = \bigwedge A$ and $\sup_X(A) = \bigvee A$. In particular, any finite subset of a lattice has supremum and infimumum.

   (b) Let $\geq$ denote the usual order partial order on $\mathbb{R}^n$.

      i. Show that a compact sublattice $X \subset \mathbb{R}^n$ is *subcomplete*, i.e. for all $A \subset X$, $\inf_{\mathbb{R}^n}(A)$ and $\sup_{\mathbb{R}^n}(A)$ exist and belong to X.

      ii. Show that if $X \subset \mathbb{R}^n$ is subcomplete, then X is a compact sublattice.

iii. Find $X \subset \mathbb{R}$ such that $(X, \geq|_X)$ is complete, but not subcomplete.

4. (The induced set ordering) Show that the induced set ordering on a lattice $(X, \geq)$ is transitive and antisymmetric on $2^X \setminus \{\emptyset\}$.[1]

5. (Sublattices)

   (a) Observe that any subset of a chain is a sublattice.

   (b) Show that the meet and join operations of a sublattice S are the same as the meet and join operations of the original lattice $(X, \geq)$. Show that this need not be the case for $Y \subset X$ such that $(Y, \geq|_Y)$ is a lattice.

   (c) What you showed above implies that arbitrary intersections of sublattices of X is also a sublattice of X. Show that this need not be the case for subsets of X that are lattices. That is, if $S, T \subset X$ are lattices but not necessarily sublattices, then $S \cap T$ need not be a lattice.

   (d) Observe that the induced set ordering on a lattice $(X, \geq)$ is a partial order on the collection of nonempty sublattices.

6. (Tarski's Fixed Point Theorem) Let X be a complete lattice and endow $2^X \setminus \{\emptyset\}$ with the induced set ordering. A set $Y \subset X$ is *subcomplete* if for any $A \subset Y$, $\inf_X(A)$ and $\sup_X(A)$ exist and belong to Y. Let $f : X \to 2^X \setminus \{\emptyset\}$ be an isotone correspondence such that $f(x)$ is subcomplete for each $x \in X$. Use Tarski's fixed point theorem stated in class to show that the set of fixed points of f is nonempty, and that f has a largest and a smallest fixed point.

7. (Monotone Equilibria) Consider a Bayesian game (N, T, A, u, p). Suppose for simplicity that the number of players, the types and actions of each player is finite, and that the distribution p of types is independent. For each $i \in N$, let T_i be a partially ordered set and let A_i be a lattice. Suppose that $u_i(x_i, x_{-i}, t)$ is supermodular in x_i and it has increasing differences in x_i and (x_{-i}, t_i).

   Prove that there exist monotone pure strategy Bayesian Nash equilibria $\bar{\mathbf{a}}^*$ and $\underline{\mathbf{a}}^*$ (i.e. for all $i \in N$, $\bar{\mathbf{a}}_i^* : T_i \to A_i$ and $\underline{\mathbf{a}}_i^* : T_i \to A_i$ are isotone) such that any rationalizable strategy profile $\mathbf{a}$ is in $[\underline{\mathbf{a}}^*, \bar{\mathbf{a}}^*]$.

[1] Note that the induced set order over 2^X satisfies neither of these properties, because (vacuously) $A \geq \emptyset \geq A$ for all $A \in 2^X$.

8. (The Housing Market) An assignment μ is in the *weak core* of the housing market (N, X, R, μ^E) if there is no $N' \subset N$ and an assignment μ' such that $\mu'(N') = \mu^E(N')$ and $\mu'(i) P_i \mu(i)$ for all $i \in N'$. Note that an assignment in the core is in the weak core.

   (a) Show that there could exist an assignment in the weak core which is not in the core. This is in contrast with the roommates problem and the marriage market where the weak core and the core are identical.[2]

   (b) Show that the Walrasian equilibrium assignment is unique.[3]

   (c) In this part allow for indifferences in preferences.

      i. Prove that every Walrasian equilibrium assignment is in the weak core.

      ii. Adapt the top-trading cycles algorithm and prove that the set of Walrasian equilibria (hence the weak core) is nonempty.

      iii. Give an example showing that the core may be empty.

9. (The Roommates Problem) Find a Pareto efficient and strategyproof rule for the roommates model.

10. (The Marriage Market) The marriage market can be seen as a special case of the roommates problem. Let (M, W, R) be a marriage market and define a roommates problem $(N, \tilde{R})$ by: $N = M \cup W$, the preference R_m of each man m is extended to a preference $\tilde{R}_m$ over N by making all $m' \in M \setminus \{m\}$ unacceptable, and women's preferences are extended similarly. Show that μ is a stable matching for the roommates problem $(N, \tilde{R})$ if and only if μ is a well-defined stable matching for the marriage market (M, W, R).[4]

[2] When indifferences are allowed for, the weak core and the core may be different even in the roommates problem and the marriage market.

[3] The supporting prices, on the other hand, are not unique. They need not even be unique up to strictly increasing transformations.

[4] μ is a well-defined matching for the marriage market if it does not match different people from the same side.