Econ 506A (2008)

Topics in Advanced Theory I
GAME THEORY

Haluk Ergin & David Levine

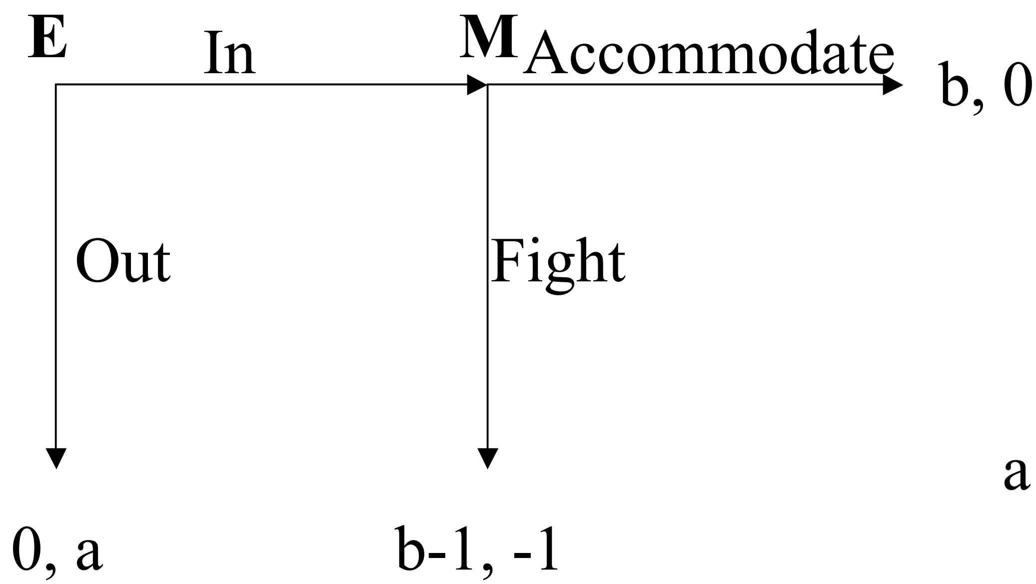
Reputation Formation

Reputation & Imperfect Information

Kreps & Wilson (1982)

Selten's Chain Store Game

Two players: Entrant (E), Monopolist (M).



a > 1 > b > 0

In the unique SPE, M accommodates and E goes in.

Finitely Repeated CS Game under Perfect Information

Time horizon: $1, \ldots, K$ periods.

A single monopolist (M) faces a different potential entrant (E_k) in each period $k = 1, \ldots, K$, where they play the CS game. All players observe past history of actions.

Payoff of the kth entrant is his payoff at stage k. (Entrants are "short-lived" players.)

Payoff of the monopolist is the sum of his payoffs in all stages. (The monopolist is a "long-lived" player.)

In the unique SPE, M always accommodates entry and E_k goes In in each period $k = 1, \ldots, K$.

Finitely Repeated CS Game
a Perturbation

Before the game starts, nature determines a type for the monopolist: $\{R(egular), T(ough)\}$. $Prob(R) = 1-\epsilon$, $Prob(T) = \epsilon$.

The Tough type monopolist *always fights entry*.

The monopolist knows his type but the entrants don't. Payoffs and strategies available to the regular monopolist (RM) and the entrants are same as before.

For small $\epsilon > 0$, is the behavior associated with the SPE of the perfect information game (E_k always goes In, RM always accommodates entry) part of some sequential equilibrium?

Strategies and Beliefs

h^k: history of play in the first k stages.
($h_t^k \in \{Out; (In, Acc); (In, F)\}$ for $1 \le t \le k$.)

$p_k(h^{k-1})$: Probability that the regular monopolist *fights* if there is entry in stage k.

$\omega_k(h^{k-1})$: Probability that the kth entrant stays *out*.

$\mu_k(h^{k-1})$: Probability that the monopolist is *Tough* at the beginning of stage k. (Belief of the entrants.)

$q_k = \mu_k + (1 - \mu_k)p_k$: Probability with which Entrant k is fought if he goes In. (He prefers to stay out $\iff q_k \ge b$.)

Consistency and Bayesian Updating

$$\mu_{k+1} = \begin{cases} \mu_k & \text{if there is no entry at stage } k \\ 0 & \text{if entry is accommodated at stage } k \\ \frac{\mu_k}{\mu_k + (1-\mu_k)p_k} & \text{if entry was fought at stage } k \end{cases}$$

- $\mu_1 = \epsilon > 0$.
- $\mu_t = 0 \Rightarrow \mu_{t+1} = 0 \Rightarrow \mu_{t+2} = 0 \Rightarrow \ldots$
- $\mu_k > 0 \Longleftrightarrow$ Entry is never accommodated in the past.
- The equilibrium rate at which the regular monopolist accommodates entry, determines the rate at which he builds a reputation for being a Tough type conditional on fighting entry.

Facts

- The R-monopolist accommodates entry at the last stage: $p_K = 0$. Hence $q_K = \mu_K$.

- If entry was accommodated sometime in the past, then the rest of the game is played like the perfect information counterpart:

$$\mu_k = 0 \implies \forall t \geq k : p_t = \omega_t = 0.$$

- At $k \leq K-1$, the R-monopolist fights entry with stricty positive probability unless he has accommodated before:

$$\mu_k > 0 \text{ at } k \leq K-1 \implies p_k > 0.$$

Stage $K-1$ (given history h^{K-2})

$\underline{b > \mu_{K-1} > 0:}$

Note $\frac{\mu_{K-1}}{\mu_{K-1}+(1-\mu_{K-1})p_{K-1}} \geq b$. ($\Rightarrow p_{K-1} \leq \frac{1-b}{b}\frac{\mu_{K-1}}{1-\mu_{K-1}} < 1$).

$0 < p_{K-1} < 1 \Rightarrow$ RM is indifferent between fighting and accommodating entry at $K-1 \Rightarrow \omega_K(h^{K-2}; In, F) = 1/a$.

Since E_K completely mixes after $(h^{K-2}; In, F)$, he is indifferent between In and Out $\Rightarrow \mu_K(h^{K-2}; In, F) = b$:

$$p_{K-1} = \frac{1-b}{b}\frac{\mu_{K-1}}{1-\mu_{K-1}} \quad \text{and} \quad q_{K-1} = \frac{\mu_{K-1}}{b}.$$

Equilibrium continuation value of RM: $V(h^{K-2}; In) = 0$.

$\underline{\mu_{K-1} \geq b:}$

$\omega_{K-1} = 1$ and $V(h^{K-2}) \geq a$. (If $\mu_{K-1} > b$ then $p_{K-1} = 1$.)

Stage $K-2$ (given history h^{K-3})

$\underline{b^2 > \mu_{K-2} > 0:}$

Note $\frac{\mu_{K-2}}{\mu_{K-2}+(1-\mu_{K-2})p_{K-2}} \geq b^2$.($\Rightarrow p_{K-2} \leq \frac{1-b^2}{b^2}\frac{\mu_{K-2}}{1-\mu_{K-2}} < 1$)

$0 < p_{K-2} < 1 \Rightarrow$ RM is indifferent between fighting and accommodating entry at $K-2 \Rightarrow \omega_{K-1}(h^{K-3}; In, F) = 1/a$.

Since E_{K-1} completely mixes after $(h^{K-3}; In, F)$, he is indifferent between In and Out $\Rightarrow \mu_{K-1}(h^{K-3}; In, F) = b^2$:

$$p_{K-2} = \frac{1-b^2}{b^2}\frac{\mu_{K-2}}{1-\mu_{K-2}} \quad \text{and} \quad q_{K-2} = \frac{\mu_{K-2}}{b^2}.$$

Equilibrium continuation value of RM: $V(h^{K-3}; In) = 0$.

$\underline{\mu_{K-2} \geq b^2:}$

$\omega_{K-2} = 1$ and $V(h^{K-3}) \geq a$. (If $\mu_{K-2} > b^2$ then $p_{K-2} = 1$.)

...by induction...

Stage $K-l$ (given history h^{K-l-1})

<u>$b^l > \mu_{K-l} > 0$:</u>

Note $\frac{\mu_{K-l}}{\mu_{K-l}+(1-\mu_{K-l})p_{K-l}} \geq b^l$.($\Rightarrow p_{K-l} \leq \frac{1-b^l}{b^l}\frac{\mu_{K-l}}{1-\mu_{K-l}} < 1$)

$0 < p_{K-l} < 1 \Rightarrow$ RM is indifferent between F and Acc. entry at $K-l \Rightarrow \omega_{K-l+1}(h^{K-l-1}; In, F) = 1/a$.

Then E_{K-l+1} is indifferent between In and Out after $(h^{K-l-1}; In, F)$ implying $\mu_{K-l+1}(h^{K-l-1}; In, F) = b^l$:

$$p_{K-l} = \frac{1-b^l}{b^l}\frac{\mu_{K-l}}{1-\mu_{K-l}} \quad \text{and} \quad q_{K-l} = \frac{\mu_{K-l}}{b^l}.$$

Eqm continuation value of RM: $V(h^{K-l-1}; In) = 0$.

<u>If $\mu_{K-l} \geq b^l$:</u>

$\omega_{K-l} = 1$ and $V(h^{K-l-1}) \geq a$. (If $\mu_{K-l} > b^l$ then $p_{K-l} = 1$.)

In all Sequential Equilibria

Beliefs

$$\mu_{K-l+1} = \begin{cases} \mu_{K-l} & \text{if there is no entry at } K-l \\ 0 & \text{if entry is accommodated at } K-l \\ \max\{b^l, \mu_{K-l}\} & \text{if } (In, F) \text{ at } K-l \text{ and } b^l \neq \mu_{K-l} \end{cases}$$

Entrants' strategies:

$$\omega_{K-l} = \begin{cases} 1 & \text{if } \mu_{K-l} > b^{l+1} \\ 1/a & \text{if } \mu_{K-l} = b^{l+1} \,\&\, K > l \geq 1 \\ 0 & \text{if } \mu_{K-l} < b^{l+1} \end{cases}$$

Regular monopolist's strategies:

$$p_{K-l} = \begin{cases} 0 & \text{if } l = 0 \\ \frac{1-b^l}{b^l} \frac{\mu_{K-l}}{1-\mu_{K-l}} & \text{if } \mu_{K-l} < b^l \,\&\, l \geq 1 \\ 1 & \text{if } \mu_{K-l} > b^l \,\&\, l \geq 1 \end{cases}$$

One seq. eqm: $\mu_{K-l+1} = b^l$ if (In, F) at $K-l$ & $b^l = \mu_{K-l}$; $\omega_K = 1/a$ if $\mu_K = b$; $p_{K-l} = 1$ when $\mu_{K-l} = b^l \,\&\, l \geq 1$.

On the Equilibrium Path

Let $l^* \geq 1$ be such that $b^{l^*+1} < \epsilon < b^{l^*}$, $k^* = \max\{1, K - l^*\}$.

- There is no entry until period k^*, $\mu_k = \epsilon$ for $k \leq k^*$.
- Entry starts with positive probability at $k \geq k^*$.
- If all entries upto $k \geq k^*$ have been met with a fight, then
  - If (In, F) at k, then μ_{k+1} becomes b^{K-k}.
  - If there is no entry at k, then $\mu_{k+1} = \mu_k$.
  - If (In, Acc) then $\mu_{k+1} = 0$.
- If there was accommodation in the past then $\mu_k = 0$, the entrant plays In, and the regular monopolist accommodates.

Reputation & Equilibrium Selection in Games with a Patient Player

Fudenberg & Levine (1989)

The Complete Information Model

A single long-run player 1 faces an infinite sequence of short-run player 2's.

At each stage $k = 0, 1, 2, \ldots$, the long-run player 1 plays the (finite normal form) game $G = (A_1, A_2, u_1, u_2)$ with the kth short-run player.

Payoff of the kth short-run player player 2 is his payoff at stage k. Payoff of the long-run player is the discounted value of the stage payoffs:

$$(1-\delta)\sum_{t=0}^{\infty} \delta^t u_1^t, \text{ where } 0 < \delta < 1.$$

Histories of past actions are observable to all players.
The extensive form game is denoted by $G(\delta)$.
β denotes a behavioral strategy profile in $G(\delta)$.

Remarks

Let B_2 denote player 2's best reply correspondence in G.

Folk Theorem: Any feasible payoff of player 1 [those in $u_1(A_1 \times \Delta(A_2))$] exceeding his "modified" minmax payoff:

$$v_1 \equiv \min_{\sigma_2 \in B_2(A_1)} \max_{a_1 \in A_1} u_1(a_1, \sigma_2).$$

is attainable in some SPE of $G(\delta)$, for δ is close to 1. (Fudenberg, Kreps, & Maskin, 1990)

What if $G(\delta)$ is perturbed to include behavioral types of 1?

Stackelberg payoff of 1:

$$u_1^* = \max_{a_1 \in A_1} \min_{\sigma_2 \in B_2(a_1)} u_1(a_1, \sigma_2).$$

Fix a **Stackelberg strategy** $a_1^* \in A_1$ of 1, i.e.

$$u_1^* = \min_{\sigma_2 \in B_2(a_1^*)} u_1(a_1^*, \sigma_2).$$

The Incomplete Information Model

The long-run player 1 has a countable number of types $\Omega = \{\omega_0, \omega_1, \omega_2, \ldots\}$. μ denotes the prior over Ω.

Payoffs of the short-run player 2's same as before. Stage payoffs of player 1 now depend on his type: $u_1(a_1, a_2, \omega)$.

Let ω_0 denote the **rational type**: $u_1(a_1, a_2, \omega_0) = u_1(a_1, a_2)$.

Let ω^* denote the **Stackelberg type**, for whom a_1^* is *strongly dominant*. That is, $u_1(a_1^*, a_2, \omega^*)$ is independent of a_2 and exceeds $u_1(a_1, a_2, \omega^*)$ for all $a_1 \neq a_1^*$ and a_2.

The extensive form game is denoted by $G(\delta, \mu)$.

β denotes a behavioral strategy profile in $G(\delta)$, where $\beta_1(\omega)$ is the behavioral strategy for type ω of player 1.

Histories

$H = (A_1 \times A_2)^\infty$ the set of all infinite histories of play.

Let $h^k = (h_1, \ldots, h_k)$ denote the truncation of $h \in H$ to the first k periods and $E^* \equiv (\{a_1^*\} \times A_2)^\infty \subset H$.

Fix a prior μ and a behavioral strategy profile β such that:

- $\mu^* \equiv \mu(\omega^*) > 0$.
- $Prob(E^*|\omega^*) = 1$. (I.e., $\beta_1(\omega^*) = a_1^*$ with probability 1.)

Note that we do not require β to be an equilibrium.

Let H^* denote the set of infinite histories $h \in H$ for which:

- h^k is reached with positive probability for each $k < \infty$.
- Player 1 always plays a_1^* on h.

A Useful Lemma

Lemma: Given μ and β as in above, let

$$\pi_t^*(h^{t-1}) \equiv Prob(a_1^t = a_1^* \,|\, h^{t-1}).$$

If $h \in H^*$ and $\bar{\pi} \in (0,1)$, then:

$$n(\bar{\pi}, h) \equiv |\{t \geq 0 \,|\, \pi_t^*(h^{t-1}) \leq \bar{\pi}\}| \leq \frac{\log \mu^*}{\log \bar{\pi}} + 1.$$

Proof: Let $h \in H^*$, by Bayes' rule

$$Prob(\omega^*|h^t) = \frac{Prob(\omega^*|h^{t-1})}{\pi_t^*(h^{t-1})}.$$

Hence if $n(\bar{\pi}, h) \geq k$, then for large enough t:

$$1 \geq Prob(\omega^*|h^t) \geq \frac{\mu^*}{\bar{\pi}^k} \;\Rightarrow\; k \leq \frac{\log \mu^*}{\log \bar{\pi}}.$$

□

The Equilibrium Selection Result

Theorem (Fudenberg & Levine, 1989)

Assume that $\mu(\omega_0) > 0$, $\mu^* \equiv \mu(\omega^*) > 0$. Then there is a constant $k(\mu^*)$ (just depending on μ^* and G) such that in any *Nash Equilibrium* β of $G(\delta, \mu)$, the payoff of ω_0 (the rational long-run player 1) is at least:

$$\delta^{k(\mu^*)} u_1^* + \left(1 - \delta^{k(\mu^*)}\right) \min_{a \in A} u_1(a).$$

Proof $\square$