Econ 506A (2008)

Topics in Advanced Theory I
GAME THEORY

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Supermodular Games

Milgrom and Roberts (1990)

Supermodular Games

Players: $N = \{1, \ldots, n\}$.

Pure strategies of i: A_i is a compact sublattice of $\mathbb{R}^{k_i}$. Then, $\underline{x}_i := \inf_{\mathbb{R}^{k_i}} A_i$ and $\bar{x}_i := \sup_{\mathbb{R}^{k_i}} A_i$ exist & are in A_i.

Payoffs: $u_i : A \to \mathbb{R}$:

1. Upper semi-continuous in x_i, continuous in x_{-i}.
2. Supermodular in x_i, increasing differences in x_i and x_{-i}.

Strategic complementarities: *When other players increase their choice variables, it becomes more profitable for player i to increase hers as well.*

Notes: The following results apply (w/out any continuity cond.) when pure strategy sets are *finite* lattices. They also apply to general complete lattices if the continuity conditions are appropriately reformulated (see M&R).

Example: Linear Bertrand Oligopoly with Differentiated Products

$x_i \in A_i = [0, M]$ denotes the price set by firm i.

Demand for firm i:

$$d_i(x) = a_i - b_i^i x_i + \sum_{j \neq i} b_i^j x_j$$

where $a_i, b_i^i, b_i^j \geq 0$ are constants. Profit of firm i:

$$u_i(x) = (x_i - c_i) d_i(x)$$

Increasing differences:

$$\frac{d^2 u_i}{dx_i dx_j} = b_i^j \geq 0 \text{ for } i \neq j$$

Example: Partnership Game

$N = \{1, 2\}$. Player 1 is the supplier of capital and player 2 is the supplier of labor: $A_1 = [0, \bar{K}]$, $A_2 = [0, \bar{L}]$.

Payoffs:

$$u_1(K, L) = t\frac{K^\alpha L^\beta}{2} - K \quad \text{and} \quad u_2(K, L) = t\frac{K^\alpha L^\beta}{2} - L$$

where $t, \alpha, \beta > 0$ are constants such that $\alpha + \beta < 1$ (decreasing returns to scale).

Increasing differences:

$$\frac{d^2 u_i}{dLdK} = \frac{\alpha\beta}{2K^{1-\alpha}L^{1-\beta}} > 0 \quad \text{for } i = 1, 2.$$

Also note that u_1 has increasing differences in t and K; u_2 has increasing differences in t and L.

Example: Cournot Duopoly

$x_i \in A_i = [0, M]$ denotes the quantity supplied by firm i.

Inverse demand (market clearing price):

$$P(x) = a - x_1 - x_2.$$

where $a \geq 0$ is a constant. Profit of firm i:

$$u_i(x) = x_i[P(x) - c_i]$$

Decreasing differences!

$$\frac{d^2 u_i}{dx_i dx_j} = -1 \text{ for } i \neq j.$$

If we reverse the order on one of the players' pure strategy sets, then Cournot Duopoly becomes supermodular and we can apply the folowing results. (What about Cournot oligopoly with $n \geq 3$?)

Example: Diamond Search Model

$x_i \in A_i = [0, M]$ denotes the effort exerted by player i to find a trading partner.

Probability of trade of player i is $t \times g(avg(x_{-i})) \times x_i$, where $avg(x_{-i}) = \frac{1}{n-1}\sum_{j \neq i} x_j$, $g : [0, M] \to \mathbb{R}_+$ is nondecreasing, and $t > 0$ is constant. Payoff of i:

$$u_i(x) = tg(avg(x_{-i}))x_i - \frac{x_i^2}{2}$$

Increasing differences:

$$\frac{du_i}{dx_i} \text{ increasing in } x_j \text{ for } i \neq j$$

Also note that u_i has increasing differences in t and x_i.

Best reply

Lemma *Let* (N, A, u) *be a supermodular game. Then,*

1. $\forall x \in A$, *the pure strategy best replies of* i *to* x_{-i}*:*

$$B_i(x) = \arg\max_{y_i \in A_i} u_i(y_i, x_{-i})$$

is a nonempty compact sublattice of $\mathbb{R}^{k_i}$*.*

2. *The correspondence* B_i *is isotone and upper semi-continuous.*

3. *The extremal best replies*

$$\bar{B}_i(x) := \sup B_i(x) \text{ and } \underline{B}_i(x) := \inf B_i(x)$$

exist and are isotone.

Rationalizability and Nash Equilibria

Theorem *Given a supermodular game* (N, A, u), *the limits:*

$$\bar{z} = \lim_{k\to\infty} \bar{B}^k(\bar{x}) \quad \textit{and} \quad \underline{z} = \lim_{k\to\infty} \underline{B}^k(\underline{x})$$

exist and are pure strategy Nash equilibria.

Furthermore, rationalizable strategy profiles are a subset of $[\underline{z}, \bar{z}] = \{x \in A : \underline{z} \le x \le \bar{z}\}$.

Lemma *Let* $\bar{x}^k := \bar{B}^k(\bar{x})$.

1. *If* $x_i \not\le \bar{B}_i(\bar{x})$, *then* x_i *is strictly dominated by* $x_i \wedge \bar{B}_i(\bar{x})$.
2. *If* $x_i \not\le \bar{x}_i^{k+1}$, *then* $u_i(x_i \wedge \bar{x}_i^{k+1}, y_{-i}) > u_i(x_i, y_{-i})\ \forall y \le \bar{x}^k$.

Corollaries

In the following fix, a supermodular game (N, A, u).

Corollary *Each player i has a minimum and a maximum rationalizable strategy given by $\underline{z}_i$ and $\bar{z}_i$.*

Corollary *There exists a pure strategy Nash equilibrium.*

Corollary *If there is a unique Nash equilibrium, then there is a unique rationalizable strategy profile.*

Corollary *Suppose that the game is symmetric (unchanged by permutations of player indices). Then, it has a symmetric pure strategy Nash equilibrium. Furthermore, if it has unique symmetric Nash equilibrium, then there is a unique rationalizable strategy profile.*

Comparative statics

Theorem *Let T be a partially ordered set. Suppose that $(N, A, u(\cdot, t))_{t \in T}$ is a parametrized family of supermodular games such that:*

u_i has increasing differences in x_i and t.

Then, the largest and smallest Nash equilibria $\overline{z}(t)$ and $\underline{z}(t)$ are isotone in t.