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Topics in Advanced Theory I
GAME THEORY

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Lattices, Sublattices, Fixed Points, and Supermodularity

Lattices

Partial orders

A binary relation $\geq$ on a set X is a **partial order** if it is:

1. *Reflexive:* $x \geq x$ for all $x \in X$.
2. *Transitive:* $x \geq y, y \geq z \Rightarrow x \geq z$.
3. *Antisymmetric:* $x \geq y, y \geq x \Rightarrow x = y$.

Two elements x, y in a partially ordered set are **ordered** if $x \geq y$ or $y \geq x$. A partial order is a **chain** (total/strict/linear order) if any two elements are ordered.

We write $x \leq y$ to denote $y \geq x$. We write $x > y$ to denote $x \geq y$ and not $x \leq y$ (similarly for $x < y$).

Lattices

A partially ordered set $(X, \geq)$ is a **lattice** if for any $x, y \in X$, there exist elements $x \vee y, x \wedge y \in X$ such that:

1. $x \vee y \geq x, y$ and $\forall z \in X$ s.t. $z \geq x, y$ we have $z \geq x \vee y$.
2. $x \wedge y \leq x, y$ and $\forall z \in X$ s.t. $z \leq x, y$ we have $z \leq x \wedge y$.

$x \vee y$ is called the **join** and $x \wedge y$ is called the **meet**.

Examples

Euclidean space: $\mathbb{R}^n$ is a lattice under the partial order:

$$x \leq y \Leftrightarrow x_i \leq y_i \text{ for all } i = 1, \ldots, n.$$

The meet and join of $x, y \in \mathbb{R}^n$ are given by:

$$x \wedge y = (\min\{x_1, y_1\}, \min\{x_2, y_2\}, \ldots, \min\{x_n, y_n\}),$$

$$x \vee y = (\max\{x_1, y_1\}, \max\{x_2, y_2\}, \ldots, \max\{x_n, y_n\}).$$

When $n = 1$, $\mathbb{R}$ is a chain.

Power set: Given a set X, the power set 2^X which denotes the set of all subsets of X is a lattice under the partial order $\supset$. The meet and join of $A, B \in 2^X$ are given by:

$$A \wedge B = A \cap B \text{ and } A \vee B = A \cup B.$$

Examples (continued)

General Product Lattices: Given a collection of lattices $(X_\alpha, \geq_\alpha)_{\alpha \in I}$, define the binary relation $\geq$ on the Cartesian product $X = \prod_{\alpha \in I} X_\alpha$ by:

$$x \geq y \iff \forall \alpha \in I : x_\alpha \geq_\alpha y_\alpha.$$

Then, $(X, \geq)$ is a lattice, $x \wedge y = (x_\alpha \wedge y_\alpha)_{\alpha \in I}$, and $x \vee y = (x_\alpha \vee y_\alpha)_{\alpha \in I}$. $(X, \geq)$ is called the **product lattice**.

Tarski's Fixed Point Theorem

Complete lattices

Given a partially ordered set $(X, \geq)$ and $A \subset X$:

1. An **upper bound** of A is $\bar{x} \in X$ s.t. $\forall x \in A : \bar{x} \geq x$.
2. A **least upper bound/supremum** of A is an upper bound $x^* \in X$ of A s.t. $\bar{x} \geq x^*$ for any upper bound $\bar{x}$ of A.

When supremum of A exists, it is unique, and denoted by $\sup_X(A)$. Lower bound and $\inf_X(A)$ are defined similarly.

Note: Any finite subset of a lattice has a supremum and an infimumum.

A lattice $(X, \geq)$ is **complete** if for all nonempty $A \subset X$, $\inf_X(A)$ and $\sup_X(A)$ exist.

Tarski's Fixed Point Theorem

Given a lattice $(X, \geq)$ and a function $f : X \to X$:

- f is **isotone** if $x \geq y \Rightarrow f(x) \geq f(y)$.
- $x^* \in X$ is a **fixed point** of f if $f(x^*) = x^*$.

Theorem *(Tarski (1955)) Let $(X, \geq)$ be a complete lattice and let $f : X \to X$ be an isotone function. Then,*

1. *The set of fixed points of f is nonempty, and*
   (a) $\sup_X(\{x \in X : f(x) \geq x\})$ *is the largest fixed point.*
   (b) $\inf_X(\{x \in X : f(x) \leq x\})$ *is the smallest fixed point.*

2. *The set of fixed points of f is a nonempty complete lattice.*

Constructive Proof of (1a) when X is Finite

Let $\bar{x} = \sup_X X$ and $x^n = f^n(\bar{x})$. Then

$$x^0 \geq x^1 \geq x^2 \geq \dots$$

By finiteness of X, there exists n^* such that $x^{n^*} = x^{n^*+1}$.

Let $\bar{x}^* := x^{n^*}$. Note that:

- $\bar{x}^*$ is a fixed point of f.
- $f(x) \geq x \Rightarrow \bar{x}^* \geq x$.

So, $\bar{x}^* = \sup_X(\{x \in X : f(x) \geq x)\})$ is the largest fixed point of f .

Sublattices

Let $(X, \geq)$ be a lattice. We extend the partial order $\geq$ to all subsets of $S, T \subset X$ using *the induced set ordering:*

$$S \geq T \iff \forall x \in S, \forall y \in T : x \vee y \in S \text{ and } x \wedge y \in T.$$

The subset $S \subset X$ is a **sublattice** of X if $S \geq S$, or more explicitly if for all $x, y \in S$: $x \wedge y, x \vee y \in S$.

Notes: The meet and join operations of a sublattice S are the same as the meet and join operations of $(X, \geq)$. That is not true for all $Y \subset X$ such that $(Y, \geq|_Y)$ is a lattice.

Arbitrary intersections of sublattices of X is a sublattice of X. This is not true for subsets of X that are lattices.

Characterization of the Sublattices of a Finite Product of Lattices

A sublattice of a product lattice can be characterized by a collection of pairwise restrictions:

Theorem *Let* $(X, \geq)$ *be the product of finitely many lattices* $(X_1, \geq_1), \ldots,$ $(X_n, \geq_n)$. *Let* S *be an arbitrary subset of* X *and for any* i, j *with* $i \neq j$, *define:*

$\tilde{S}_{ij} = \{(y_i, y_j) \in X_i \times X_j : \exists x \in S$ *such that* $y_i = x_i$ *and* $y_j = x_j\}$.

and $S_{ij} = \tilde{S}_{ij} \times X_{-ij}$.

Then, S *is a sublattice of* X *if and only if*

1. $\tilde{S}_{ij}$ *is a sublattice of* $X_i \times X_j$ *for* $i \neq j$.
2. $S = \cap_{i \neq j} S_{ij}$.

Supermodularity

Given a lattice $(X, \geq)$, a function $f : X \to \mathbb{R}$ is **supermodular** if for all $x, y \in X$:

$$f(x) + f(y) \leq f(x \wedge y) + f(x \vee y).$$

f is **submodular** if $-f$ is supermodular.

Note that if all pairs are ordered under $\geq$ (e.g. $\mathbb{R}$), then *all* functions are supermodular and submodular!

Increasing Differences

Let $f : X \times Y \to \mathbb{R}$ where X and Y are two lattices. Then, f has **increasing differences** if for all $x, x' \in X$ with $x \geq x'$, the difference $f(x, y) - f(x', y)$ is nondecreasing in y.

(Definition does not change if coordinates are reversed.)

Let $(X, \geq)$ be a product of finitely many lattices $(X_i, \geq_i)_{i=1}^n$ and $f : X \to \mathbb{R}$. Then, f has **increasing differences** if $f(\cdot, \cdot, x_{-ij}) : X_i \times X_j \to \mathbb{R}$ has increasing differences for all i, j with $i \neq j$ and $x_{-ij} \in X_{-ij}$.

Lemma *Let $(X, \geq)$ be a product of finitely many lattices $(X_i, \geq_i)_{i=1}^n$ and $f : X \to \mathbb{R}$. If f is has increasing differences, then $\forall i$:*

$$x \geq y \Rightarrow f(x_i, x_{-i}) - f(y_i, x_{-i}) \geq f(x_i, y_{-i}) - f(y_i, y_{-i}).$$

Supermodularity on Finite Products of Lattices

Let $(X, \geq)$ be a product of finitely many lattices $(X_i, \geq_i)_{i=1}^n$ and $f : X \to \mathbb{R}$. Then, f is **coordinatewise supermodular** if for all i and $x_{-i} \in X_{-i}$, $f(\cdot, x_{-i}) : X_i \to \mathbb{R}$ is supermodular.

Theorem *Let $(X, \geq)$ be a product of finitely many lattices $(X_i, \geq_i)_{i=1}^n$ and $f : X \to \mathbb{R}$. Then, f is supermodular if and only if f is coordinatewise supermodular and has increasing differences.*

If X is a product of chains (e.g. $\mathbb{R}^n$), then any function f satisfies coordinatewise supermodularity, implying:

Corollary *If X is a product of chains, then $f : X \to \mathbb{R}$ is supermodular if and only if it has increasing differences.*

Supermodularity on $\mathbb{R}^n$

Lemma *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be twice continuously differentiable. Then,* f *has increasing differences if and only if* $\frac{d^2 f}{dx_1 dx_2} \geq 0$.

Corollary *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be twice continuously differentiable. Then,* f *is supermodular if and only if it has nonnegative mixed second order derivatives, i.e.* $\frac{d^2 f}{dx_i dx_j} \geq 0$ *for all* $i \neq j$.

Topkis' Monotonicity Theorem

Theorem *Let X be a lattice, T be a partially ordered set, $f : X \times T \to \mathbb{R}$, and $S : T \to 2^X \setminus \{\emptyset\}$. Assume that $f(x,t)$ is supermodular in x for each t, and it has increasing differences in x and t in the following sense:*

$$f(x,t) - f(x',t) \geq f(x,t') - f(x',t')$$

whenever $x \geq x'$ and $t \geq t'$. Define $x^* : T \to 2^X$ by:

$$x^*(t) = \arg\max_{x \in S(t)} f(x,t).$$

If $t \geq t'$ and $S(t) \geq S(t')$, then $x^*(t) \geq x^*(t')$.

A function $f : X \to Y$, where X and Y are sets endowed with arbitrary binary relations, is **isotone** (or **monotone**) if $x \geq y \Rightarrow f(x) \geq f(y)$.

Corollary *If in the monotonicity theorem, $S : T \to 2^X \setminus \{\emptyset\}$ isotone, then $\forall t \in T$, $S(t)$ and $x^*(t)$ are sublattices of X.*