Econ 506A (2008)

Topics in Advanced Theory I
GAME THEORY

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Matching

The Housing Market: One-to-One Assignment of Indivisible Objects

The Housing Market Model

N: Set of agents

X: Set of indivisible objects (houses, offices, dorm rooms...).

$|N| = |X| = n$.

$R = (R_i)_{i \in N}$: Preference profile of the agents.

R_i is a linear order over X (complete, transitive, antisymmetric).

P_i denotes the strict part: $x P_i y \Leftrightarrow x R_i y \,\&\, \neg[y R_i x]$.

An *assignment* is a one-to-one function $\mu : N \to X$.

$\mu(i)$ denotes the indivisible object that agent i receives under assignment μ.

The Core

Let μ^E denote the initial endowment assignment.

An assignment μ is in the *core* of (N, X, R, μ^E) if there is no $N' \subset N$ and an assignment μ' such that:

1. $\mu'(N') = \mu^E(N')$.
2. $\mu'(i) R_i \mu(i)$ for all $i \in N'$ and $\mu'(i) P_i \mu(i)$ for some $i \in N'$.

Note that if μ is in the core, then it is:

1. *Pareto efficient*: $\nexists$ another assignment μ' such that $\mu'(i) R_i \mu(i)$ for all $i \in N$ and $\mu'(i) P_i \mu(i)$ for some $i \in N$.
2. *Individually Rational*: $\mu(i) R_i \mu^E(i)$ for all $i \in N$.

Note: In the more general model allowing for weak preferences, Shapley & Scarf (1974) show that the "weak core" of the housing market is nonempty (see exercises).

Gale's Top-trading Cycles Algorithm

Step 1: Let every agent point to the owner of her most preferred object (possibly to herself). There is at least one cycle (including self-cycles). Assign every agent who is part of a cycle to her most preferred object. Let N^1 be the set of remaining agents and X^1 be the set of remaining objects. Note that $X^1 = \mu^E(N^1)$.

Step $k \geq 2$: Let every agent in N^{k-1} point to the owner of her most preferred object in X^{k-1}. Assign every agent who is part of a cycle to her most preferred object in X^{k-1}. Let N^k be the set of remaining agents and X^k be the set of remaining objects. Note that $X^k = \mu^E(N^k)$.

The algorithm terminates when every agent is assigned an object. Let μ^{TTC} denote the resulting assignment.

Theorem *(Roth & Postlewaite (1977)) μ^{TTC} is the unique assignment in the core.*

An Example

$N = \{1, 2, 3, 4, 5, 6\}$, $X = \{a, b, c, d, e, f\}$,

R_1	R_2	R_3	R_4	R_5	R_6
a	a	b	c	d	c
$\vdots$	$\underline{d}$	$\vdots$	$\vdots$	f	e
$\underline{b}$	$\vdots$	$\underline{c}$	$\underline{a}$	$\underline{e}$	$\underline{f}$
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$

$$\mu^E = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ b & d & c & a & e & f \end{pmatrix}$$

Step 1: $N^1 = \{2, 5, 6\}$ $X^1 = \{d, e, f\}$.

Step 2: $N^2 = \{5, 6\}$ $X^2 = \{e, f\}$.

Step 3: $N^3 = X^3 = \emptyset$.

$$\mu^{TTC} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ a & d & b & c & f & e \end{pmatrix}$$

Prices Supporting μ^{TTC} as a Walrasian Equilibrium

Given a housing market (N, X, μ^E, R), a vector of prices $p = (p_x)_{x \in X} \subset \mathbb{R}_+^X$ and an assignment μ constitute a *Walrasian Equilibrium* if for all $i \in N$:

1. $p_{\mu(i)} \leq p_{\mu^E(i)}$, and
2. $\forall x \in X$: $p_x \leq p_{\mu^E(i)}$ implies $\mu(i) R_i x$.

Proposition *(Gale, Shapley & Scarf (1974)) For any given a housing market (N, X, μ^E, R), there is a price vector p such that (μ^{TTC}, p) is a Walrasian equilibrium.*

Proof Choose the supporting prices in the reverse order of the step in which an object is assigned in the TTC algorithm: e.g. $p_x = n - k$ if $x \in X^{k-1} \setminus X^k$. □

Note that any price vector which ranks the objects in the same way also works.

Strategic Aspects & Characterization of the Core

In the following, fix (N, X, μ^E).

A *rule* f is a map which associates an assignment with every preference profile. $f_i(R)$ denotes the object that agent i receives in the assignment $f(R)$.

Let *core* denote the rule which associates the unique core assignment of (N, X, μ^E, R) to every preference profile R.

A rule f is *strategyproof* if for all R, i, and R'_i:

$$f_i(R_i, R_{-i})\, R_i\, f_i(R'_i, R_{-i}).$$

Proposition *The core is strategyproof.*

Theorem *(Ma (1994)) A rule is individually rational, strategyproof, and Pareto efficient if and only if it is the core.*

The Roommates Problem:
One-Sided One-to-One Matching

The Roommates Model

$N = \{1, \ldots, n\}$: Set of agents.

$R = (R_i)_{i \in N}$: Preference profile of the agents.

R_i is a linear order over N, denoting agent i's preferences over potential roommates and living alone.

A *matching* is a function $\mu : N \to N$ such that

$$\forall i \in N : \mu(\mu(i)) = i.$$

$\mu(i)$ is the roommate of agent i under the matching μ.

$\mu(i) = i$ is interpreted as i staying alone / not being matched to anyone.

Core $\equiv$ Stability

A matching μ is in the *core* of (N, R) is there is no $N' \subset N$ and a matching μ' such that:

1. $\mu'(N') = N'$.
2. $\mu'(i)R_i\mu(i)$ for all $i \in N'$ and $\mu'(i)P_i\mu(i)$ for some $i \in N'$.

Note that if μ is in the core, then it is:

1. *Pareto efficient*: $\nexists$ another matching μ' such that $\mu'(i)R_i\mu(i)$ for all $i \in N$ and $\mu'(i)P_i\mu(i)$ for some $i \in N$.
2. *Stable:*
   (a) *Individually Rational*: $\mu(i)R_i i$ for all $i \in N$.
   (b) *Pairwise Stable*: There are no two agents $i, j \in N$ such that $jP_i\mu(i)$ and $iP_j\mu(j)$.

Proposition *Given* (N, R), *a matching is in the core if and only if it is stable.*

The Core may be Empty

Example: (Gale & Shapley (1962)) Let $N = \{1, 2, 3\}$ and

R_1	R_2	R_3
2	3	1
3	1	2
1	2	3

Further Literature on the Roommates Problem:

-Tan (1991) and Chung (2000) study conditions on the preference profile guaranteeing nonempty core.

For the case where n is even and jR_ii for all $i, j \in N$:

-Irving (1985) introduces an $O(n^2)$ algorithm that checks whether the core is empty or not, and in the latter case returns a matching in the core.

-Morrill (2008) introduces $O(n^2)$ algorithms that check whether a given matching is Pareto efficient or not, and in the latter case find a Pareto improving matching.

The Marriage Market:
Two-Sided One-to-One Matching

The Marriage Market Model

A *marriage market* (M, W, R) consists of finite (disjoint) sets of men M, and women W, and a preference profile $R = ((R_m)_{m \in M}, (R_w)_{w \in W})$ such that:

(a) $\forall m \in M$: R_m is a linear order over $W \cup \{m\}$, and

(b) $\forall w \in W$: R_w is a linear order over $M \cup \{w\}$.

w [m] is *acceptable* for m [w] if $wR_m m$ [$mR_w w$].

A *matching* is a function $\mu : M \cup W \to M \cup W$ such that:

1.(a) $\forall m \in M$, $\mu(m) \in W \cup \{m\}$

(b) $\forall w \in W$, $\mu(w) \in M \cup \{w\}$

2. $\forall i \in M \cup W$, $\mu(\mu(i)) = i$.

Note: The marriage market can be seen as a special case of the roommates problem.

Core $\equiv$ Stability

A matching μ is in the *core* of the marriage market (M, W, R) if there is no $M' \subset M$, $W' \subset W$, and a matching μ' s.t.:

1. $\mu'(M' \cup W') = M' \cup W'$.
2. $\forall i \in M' \cup W'$: $\mu'(i) R_i \mu(i)$ and $\exists i \in M' \cup W'$: $\mu'(i) P_i \mu(i)$.

If μ is in the core, then it is Pareto efficient and *stable:*

1. *Individually Rational*: $\mu(i) R_i i$ for all $i \in M \cup W$.
2. *Pairwise Stable*: There is no pair $(m, w) \in M \times W$ such that $w P_m \mu(m)$ and $m P_w \mu(w)$.

Proposition *Given a marriage market, a matching is in the core if and only if it is stable.*

Theorem *(Gale & Shapley (1962)) There exists a stable matching in every marriage market.*

The Men-Proposing Deferred Acceptance (DA) Algorithm

Step 1: Every man m proposes to his favorite acceptable woman. For each w, among the men who proposed to w, w places her favorite acceptable man tentatively in her waiting list, and rejects the others.

Step $k \geq 2$: Men rejected at step $k-1$ propose to their next best acceptable women. For each w, the favorite acceptable man, among the new proposers and the one (if any) already in the waiting list of w from step $k-1$, is placed on her new waiting list and the rest are rejected.

Algorithm terminates when every man is either on a waiting list or has been rejected by every acceptable woman.

At the end, women get matched to the man in their final waiting list, the others (women with empty waiting lists and the men who are not on any waiting list) are unmatched.

An Example

$M = \{1, 2, 3, 4, 5\}$, $W = \{a, b, c, d\}$,

R_1	R_2	R_3	R_4	R_5
a	d	d	a	a
b	b	c	d	b
c	c	a	c	d
d	a	b	b	5

R_a	R_b	R_c	R_d
2	3	5	1
3	1	4	4
1	2	1	5
4	4	2	2
5	5	3	3

Everybody finds everybody in the other side of the market acceptable except 5 who does not find c acceptable.

Outcomes of the men-proposing and women-proposing DA:

$$\mu_M = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ a & b & c & d & 5 \end{pmatrix} \qquad \mu_W = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ d & a & b & c & 5 \end{pmatrix}$$

Note that all men weakly prefer μ_M to μ_W, and all women weakly prefer μ_W to μ_M.

DA Outcome and Stability

Fix a marriage market (M, W, R).

Let μ_M and μ_W denote the outcomes of the men-proposing and women-proposing DA algorithms, respectively.

Define the binary relation $\geq_M$ on the set of matchings by

$$\mu \geq_M \nu \;\; \Leftrightarrow \;\; \forall m \in M : \mu(m) R_m \nu(m).$$

Note that $\geq_M$ is a partial order. Define $\geq_W$ similarly.

Proposition *(Gale & Shapley (1962))* μ_M *is stable and* $\mu_M \geq_M \nu$ *for any stable matching* ν*. Similarly,* μ_W *is stable and* $\mu_W \geq_W \nu$ *for any stable matching* ν*.*

Therefore, μ_M is called the *men-optimal* stable matching. Similarly, μ_W is called the *women-optimal* stable matching.

Opposing Interests of the Two Sides over Stable Matchings

Proposition *(Knuth (1976)) Given a marriage market* (M, W, R)*, for any two stable matchings* μ *and* ν*:*

$$\mu \geq_M \nu \Leftrightarrow \nu \geq_W \mu.$$

Corollary *Given a marriage market* (M, W, R)*,* μ_W *is the worst stable matching for men and* μ_M *is the worst stable matching for women.*

The Lattice Structure of Stable Matchings

Theorem *(Conway, Knuth (1976)) Fix a marriage market* (M, W, R) *and let* $\mathcal{S}$ *denote its stable matchings.*

Let $\mu, \nu \in \mathcal{S}$, *and define the function* $\lambda : M \cup W \to M \cup W$ *giving each man* m *his preferred match in* $\{\mu(m), \nu(m)\}$ *and each women* w *her less preferred match in* $\{\mu(w), \nu(w)\}$*:*

$$\lambda(m) = \begin{cases} \mu(m) & \text{if } \mu(m) R_m \nu(m) \\ \nu(m) & \text{otherwise.} \end{cases} \quad \lambda(w) = \begin{cases} \nu(w) & \text{if } \mu(w) R_m \nu(w) \\ \mu(w) & \text{otherwise.} \end{cases}$$

Similarly define λ' *that gives each man* m *his less preferred match in* $\{\mu(m), \nu(m)\}$ *and each women* w *her preferred match in* $\{\mu(w), \nu(w)\}$. *Then,*

1. λ *and* λ' *are stable matchings.*
2. $(\mathcal{S}, \geq_M)$ *and* $(\mathcal{S}, \geq_W)$ *are lattices, where* $\lambda = \mu \vee_M \nu = \mu \wedge_W \nu$ *and* $\lambda' = \mu \wedge_M \nu = \mu \vee_W \nu$.

Strategic Aspects

Fix (M, W). The *men-optimal stable mechanism* is the map that associates the men-optimal stable matching with every preference profile.

Theorem *(Dubins & Freedman (1981), Roth (1982))The men-optimal stable mechanism is strategyproof for men.*

As usual, the *women-optimal stable mechanism* can be defined analogously and the symmetric result holds.

Example The women side can manipulate the men-optimal stable mechanism. Let $M = \{1, 2\}$, $W = \{a, b\}$

$$\begin{array}{c|c} R_1 & R_2 \\ \hline a & b \\ b & a \end{array} \quad \begin{array}{c|c} R_a & R_b \\ \hline 2 & 1 \\ 1 & 2 \end{array} \mapsto \mu_M = (1a, 2b)$$

Let $R'_a \colon 2P'_a a P'_a 1$. Then, $(R'_a, R_{-a}) \mapsto \mu'_M = (1b, 2a)$.