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Topics in Advanced Theory I
GAME THEORY

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Matching with Contracts

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Hatfield and Milgrom (2005)

The Matching Model with Contracts

D: A finite set of doctors. H: A finite set of hospitals.

X: A finite set of contracts. Each contract $x \in X$ can be signed by the doctor hospital pair $(x_D, x_H) \in D \times H$.

Examples of Contracts:

In Gale and Shapley (1962), each doctor hospital pair can sign a single contract with exogenously fixed terms, i.e. $X \equiv D \times H$.

In Kelso and Crawford (1982), a contract between a doctor and a hospital also determines the wage, i.e. $X \equiv D \times H \times W$ for some finite set of wages $W \subset \mathbb{R}_+$.

More generally, a contract can specify any terms and conditions regarding the match (working hours, wage, etc.).

Preferences and Choices

Each doctor d has a strict preference $\succsim_d$ over the set of contracts involving her $\{x \in X : x_D = d\}$ and the null contract $\emptyset$ leaving her unmatched. For any subset of contracts $X' \subset X$, doctor d's choice out of X' is:

$$C_d(X') = \begin{cases} \emptyset & \text{if } \nexists x \in X' \text{ s.t. } x_D = d \text{ and } x \succ_d \emptyset \\ \{\max_{\succsim_d}\{x \in X' : x_D = d\}\} & \text{otherwise.} \end{cases}$$

Each hospital h has a strict preference $\succsim_h$ over the set of feasible contracts involving her, i.e. over all $X'' \subset X$ s.t.:

1. $\forall x \in X''$: $x_H = h$, and
2. $\forall x, x' \in X''$: $x_D = x'_D \Rightarrow x = x'$.

For any subset of contracts $X' \subset X$, hospital h's choice set $C_h(X')$ is her most-preferred set $X'' \subset X'$ satisfying (1) and (2) above.

Stability

For any $X' \subset X$, the chosen and rejected sets of contracts:

$$C_D(X') = \cup_{d \in D} C_d(X') \quad C_H(X') = \cup_{h \in H} C_h(X')$$

$$R_D(X') = X' \setminus C_D(X') \quad R_H(X') = X' \setminus C_H(X')$$

A set of contracts $X' \subset X$ is **stable** if:

1. *Individual Rationality:* $X' = C_D(X') = C_H(X')$.
2. There is no hospital h and a set of contracts $X'' \neq C_h(X')$ such that $X'' = C_h(X' \cup X'') \subset C_D(X' \cup X'')$.

Note that individual rationality also implies that each doctor signs at most one contract in X'.

Exercise: Prove Stability≡Core.

Characterization of Stability

Theorem *If* $(X_D, X_H) \subset X^2$ *is a solution to the system of equations:*

$$X_D = X \setminus R_H(X_H) \text{ and } X_H = X \setminus R_D(X_D) \qquad (1)$$

then $X_D \cap X_H$ *is a stable set of contracts. Conversely, for any stable set of contracts* X'*, there exists* (X_D, X_H) *satisfying Equation* (1) *such that* $X' = X_D \cap X_H$.

Substitutes

Definition Elements of X are **substitutes** for hospital h if for all subsets $X' \subset X'' \subset X$, we have $R_h(X') \subset R_h(X'')$.

Example Let $X \equiv D \times H$. Suppose that hospital h has a quota q_h. She also has a strict preference ranking over individual doctors in D and the null contract $\emptyset$. From a subset of contracts X', $C_h(X')$ is the best q_h contracts in X' involving hospital h if all these contracts are acceptable to h (and all acceptable contracts involving hospital h otherwise). Such choices are called *responsive*.

Relation to Gross Substitutes Kelso and Crawford (1982)

Suppose there is a single hospital h and let $C = C_h$. Let $X \equiv D \times W$ for some finite set of wages $W \subset \mathbb{R}_+$.

Suppose that the hospital never hires a doctor at the maximum wage $\bar{w} \in W$. For any wage vector $\mathbf{w} \in W^D$, define:

$$C(\mathbf{w}) \equiv C(\{(d, \mathbf{w}_d) : d \in D\})$$

Definition C satisfies *the demand theory substitutes (the Kelso-Crawford gross substitutes)* condition if $d \neq d'$, $(d', \mathbf{w}_{d'}) \in C(\mathbf{w})$, and $w'_d > \mathbf{w}_d$ imply $(d', \mathbf{w}_{d'}) \in C(\mathbf{w}_{-d}, w'_d)$.

Theorem *Let* $\hat{W}_d(X') = \min\{w | w = \bar{w}$ *or* $(d, w) \in X'\}$ *and suppose that* $C(X') = C(\hat{W}(X'))$. *Then,* C *satisfies the demand theory substitutes condition if and only if contracts are substitutes.*

The Generalized Gale-Shapley Algorithm

Define an order on $2^X \times 2^X$ by:

$$(X_D, X_H) \geq (X'_D, X'_H) \iff X_D \supset X'_D \text{ and } X_H \subset X'_H,$$

which makes it into a finite a lattice.

The algorithm is defined by iterated application of the following function $F : 2^X \times 2^X \to 2^X \times 2^X$

$$F_1(X') = X \setminus R_H(X') \text{ and } F_2(X') = X \setminus R_D(X')$$

$$F(X_D, X_H) = (F_1(X_H), F_2(F_1(X_H)))$$

Notes:

1. (X_D, X_H) satisfies Equation (1) if and only if it is a fixed point of F.
2. If hospital choices satisfy the substitutes condition, then F is isotone.

Existence and the Lattice Structure of Stable Allocations under the Substitutes Condition

Theorem *Assume that hospital choices satisfy the substitutes condition. Then,*

1. *The set of fixed points of F is a finite lattice including the highest and the smallest fixed points $(\bar{X}_D, \bar{X}_H)$ and $(\underline{X}_D, \underline{X}_H)$ respectively.*

2. *Starting at $(X, \emptyset)$, the generalized Gale-Shapley algorithm converges to the highest fixed point $(\bar{X}_D, \bar{X}_H)$.*

3. *Starting at $(\emptyset, X)$, the generalized Gale-Shapley algorithm converges to the smallest fixed point $(\underline{X}_D, \underline{X}_H)$.*

Then, $\bar{X}_D \cap \bar{X}_H$ is the best stable allocation for the doctors and the worst one for the hospitals. Similarly for $\underline{X}_D \cap \underline{X}_H$. (See Exercises for a stronger result.)

Example

Let $(X_D(t), X_H(t)) = F^t(X, \emptyset)$ for $t \geq 0$. Interpretation:

$X_H(t)$: Cumulative set of contracts offered to hospitals by the doctors in rounds $1, \ldots, t$.

$R_H(X_H(t))$: Accumulated set of rejected offers by t.

$X_D(t)$: Accumulated set of offers not rejected by t.

Example: $D = \{d_1, d_2\}$, $H = \{h_1, h_2\}$, $X = D \times H$ and:

$$h_1 \succ_{d_i} h_2 \succ_{d_i} \emptyset \text{ for } i = 1, 2$$

$$\{d_1\} \succ_{h_1} \{d_2\} \succ_{h_1} \emptyset, \text{ and } \{d_1, d_2\} \succ_{h_2} \{d_1\} \succ_{h_2} \{d_2\} \succ_{h_2} \emptyset,$$

$(X_D(0), X_H(0)) = (X, \emptyset)$.
$(X_D(1), X_H(1)) = (X, \{(d_1, h_1), (d_2, h_1)\})$
$(X_D(2), X_H(2)) = (X \setminus \{d_2, h_1\}, \{(d_1, h_1), (d_2, h_2)\})$
$(X_D(3), X_H(3)) = (X_D(2), X_H(2)) = (\bar{X}_D, \bar{X}_H)$.

The doctor-optimal stable allocation $\bar{X}_D \cap \bar{X}_H = \{(d_1, h_1), (d_2, h_2)\}$

Law of Aggregate Demand, Rural Hospitals, and Doctors' Incentives

Definition The choices of hospital h satisfy the *law of aggregate demand* if $X' \subset X'' \subset X \Rightarrow |C_h(X')| \leq |C_h(X'')|$.

Responsive choices satisfy the law of aggregate demand.

Theorem *Assume that hospital choices satisfy substitutes and the law of aggregate demand. Then,*

1. *(Rural Hospitals Theorem) The number of contracts signed by each hospital and doctor is the same for all stable allocations.*

2. *The mechanism selecting the doctor-optimal stable allocation is strategy-proof for the doctors.*

Hatfield and Kojima (2008) show that the doctor-optimal stable mechanism is weakly group strategy-proof for the doctors under weaker conditions on hospital preferences.